The Empirical Application of Finance

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Chapter 4

Factor Models and Reduction in Dimensionality

- The traditional approach to asset allocation among N risky assets requires the estimation of their future distribution $\mathbf{r} \sim \mathcal{D}(\mu, \Sigma)$
- One of the most relevant problems in the implementation of the traditional approach to portfolio allocation is dimensionality.
- The implementation of asset allocation and risk measurement among n assets requires the estimation of a very large number of parameters: $\frac{n(n+1)}{2} + n$.
- Factor models allow to simplify the structure of the model and to reduce the number of parameters to be estimated.
- These models have been also successfully employed to parsimoniously characterize the cross-section of average one-period (often monthly) returns

Factor Models:Time-Series Representation

The statistical distribution of N assets (i=1...n) is conditioned on a vector of K factors **f** (where N is large and K is small)

$$\begin{aligned}
r^i_{t,t+k} &= \gamma^i_0 + \gamma^{i\prime}_1 \mathbf{f}_{t,t+k} + v^i_{t,t+k} \\
\mathbf{f}_{t,t+k} &= \mu^f + \mathbf{H}^f \epsilon_{t,t+k} \\
\Sigma^f &= \mathbf{H}^f \mathbf{H}^{f\prime}. \\
\mathbf{E}\left(v^i_{t,t+k}, v^j_{t,t+k}\right) &= 0 \\
\mathbf{E}\left(v^i_{t,t+k}, \epsilon^j_{t,t+k}\right) &= 0 \\
\epsilon_{t+k} &\sim \mathcal{D}\left(\mathbf{0}, \mathbf{I}\right)
\end{aligned}$$

Factor Models: Cross-Sectional representation

The multifactor model has the following cross-sectional representation for the (Nx1) vector of returns at time t

$$
\begin{aligned}
\underset{(Nx1)}{\mathbf{r}_{t,t+k}} &= \underset{(Nx1)}{\alpha} + \underset{(NxK)}{B}\,\underset{(Kx1)}{\mathbf{f}_{t,t+k}} + \underset{(Nx1)}{\mathbf{v}_t} \\
\underset{(Kx1)}{\mathbf{f}_{t,t+k}} &= \underset{(Kx1)}{\mu^f} + \underset{(KxK)}{\mathbf{H}^f}\,\underset{(Kx1)}{\epsilon^f} \\
\Sigma^v &= \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ .. & .. & .. & .. \\ 0 & 0 & 0 & \sigma_n \end{bmatrix} \\
\Sigma^f &= \mathbf{H}^f \mathbf{H}^{f\prime}.
\end{aligned}
$$

Factor Models as Parsimoniuos Representation

- Using the joint distribution of returns to estimate the variance covariance matrix (the CER model) requires the estimation of n+n(n+1)/2 parameters;
- using a structure of k factors requires the estimation of (2n+nk)+ (k+k(k+1)/2) parameters.
- Think for example of an asset allocation problem with 30 assets and 4 factors. The CER would require the estimation of 505 parameters, the factor model would reduce that number to 194.

Factor-based Portfolios and Factor Exposures

Define the returns of an optimal portfolio obtained by combining n assets as $r_{t+1}^{p} = \sum_{i=1}^{N} w_i r_{t+1}^{i}$

$$r_{t+1}^{p} = \alpha_1 + \beta_{f^1} f_{t+1}^{1} + \beta_{f^2} f_{t+1}^{2} + \cdots + \beta_{f^k} f_{t+1}^{k} + v_{t+1}$$

$$\begin{aligned} Var\left(r_{t+1}^{p}\right) &= Cov\left(r_{t+1}^{p}, r_{t+1}^{p}\right) \\ &= \beta_{f^1} Cov\left(f_{t+1}^{1}, r_{t+1}^{p}\right) + \ldots \beta_{f^k} Cov\left(f_{t+1}^{k}, r_{t+1}^{p}\right) \\ &+ Cov\left(v_{t+1}, r_{t+1}^{p}\right) \end{aligned}$$

The factor exposure can then be computed as the share of total variance portfolio attributable to each factor:

$$EXP_{f^i}^{p} = \frac{\beta_{f^i} Cov\left(f_{t+1}^{i}, r_{t+1}^{p}\right)}{Var\left(r_{t+1}^{p}\right)}$$

"Smart beta" strategies can be implemented through alternative weighting methods that emphasize the exposures to specific factors.

A single factor model:The CAPM

The time series representation

$$\begin{aligned}
\left(r_t^i - r_t^{rf}\right) &= \beta_{0,i} + \beta_{1,i}\left(r_t^m - r_t^{rf}\right) + u_{i,t} \\
\left(r_t^m - r_t^{rf}\right) &= \mu_m + u_{m,t} \\
u_{i,t} &\sim n.i.d.\left(0, \sigma_i^2\right) \\
\begin{pmatrix} u_{i,t} \\ u_{m,t} \end{pmatrix} &\sim n.i.d.\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{ii} & 0 \\ 0 & \sigma_{mm} \end{pmatrix}\right]
\end{aligned}$$

The CAPM: cross-section representation

Considering the cross-section representation for the returns :

$$
\begin{aligned}
\mathbf{r}_t &= \beta_{\mathbf{0}} + \beta_{\mathbf{1}} r_t^m + \mathbf{u}_t \\
r_t^m &= E\left(r^m\right) + \sigma_m \mathbf{u}_{m,t} \\
\Sigma &= \beta_{\mathbf{1}} \beta_{\mathbf{1}}' \sigma_m^2 + \Sigma_u \\
\mu &= \beta_0 + \beta_1 E\left(r^m\right)
\end{aligned}
$$

and μ, Σ can be obtained with the estimation of 3n+2 parameters only.

Asset Allocation with the CER and the CAPM in R

The R programme **CAPM.r** allows you to

- upload a data set of US stock market returns
- perform descriptive and graphical analysis
- implement optimal portfolio allocation with the CER model
- implement optimal portfolio allocation with the CAPM model
- compare the results

Asset Allocation with the CER and the CAPM in R

In the application portfolio optimization is aimed at computing the minimum variance portfolio:

$$\min_{\mathbf{w}}(\mathbf{w}'\Sigma\mathbf{w})$$

subject to

$$\mathbf{w}'\mathbf{e} = 1$$

with solution:

$$\mathbf{w} = \frac{\Sigma^{-1}\mathbf{e}}{\mathbf{e}'\Sigma^{-1}\mathbf{e}}$$

Asset Allocation with the CER and the CAPM in R

Two alternative approaches are followed to derive optimal portfolio weights

- a standard CER
- a CAPM
- the CAPM model is validated by comparing the correlation matrix of returns with the correlation matrix of their estimated idiosyncratic components
- results are compared

Validating Factor Models

- The diagonality of the variance-covariance matrix of the residuals coming from projecting asset returns on factors is a necessary–and testable–requirement for the validity of any factor model.
- Further validation is based on testing restrictions on their coefficients

Validation by testing restrictions

Consider once again the time-series representation of a factor model

$$r_{t+1}^{i} = \alpha_1 + \beta_i^{f^1} f_{t+1}^{1} + \beta_i^{f^2} f_{t+1}^{2} + \cdots + \beta_i^{f^k} f_{t+1}^{k} + v_{t+1} \qquad (1)$$

After having estimated N equations for the N assets you have available the following k vectors of coefficients, each of length N: $\boldsymbol{\beta}^{f1}$, $\boldsymbol{\beta}^{f2}$,..., $\boldsymbol{\beta}^{fk}$. Using the sample of t observations on the returns of the N assets you can compute the vector of length N of average sample returns for the assets: $E(\mathbf{r})$.

Validation by testing restrictions

You can now run the affine expected return-beta cross-sectional regression is:

$$E(\mathbf{r}) = \gamma_0 + \gamma_1 \boldsymbol{\beta}_{f1} + \gamma_2 \boldsymbol{\beta}_{f2} + \cdots + \gamma_k \boldsymbol{\beta}_{fk} + \mathbf{u}$$

A two-step test (FamaMacBeth) for the validity of any factor model can be run by considering the following null hypothesis:

$$\hat{\gamma}_0 = \bar{r}^f, \; \hat{\gamma}_i = E\left(f^i\right)$$

Fama-MacBeth

- care must be exercise in the test as the variance-covariance matrix of the residuals in the cross-sectional regression will not be diagonal and corrections for heteroscedasticity should be implemented.
- note also that, if both test assets and factors are excess returns, the validity of the model can be simply tested by evaluating the null that all intercepts in the time-series model for excess returns are zero.
- this null is inevitably rejected. Two industries have emerged (i) the factors "zoo", thath looks for omitted factors (ii) the performance evaluation industry that classifies fund manager performance according to their

Which Factors ?

Many different set of factors have been considered in the literature :

- Fundamental Factors
  - Fama-French five factors with observable characteristics and estimated betas (MKT, SMB, HML, RMW, CMA and momentum MOM
  - BARRA factors with known time-invariant betas and unobservable factor realizations estimated by cross-sectional regressions.
- Macroeconomic Factors (inflation, growth and uncertainty)
- Statistical Factors (for example principal components)

Factor Models and Predictability

Factor models are commonly used to characterize parsimoniously the predictive distribution of asset returns. Specifically, multi-factor models in which k factors characterize in a lower parametric dimension the distribution of n asset returns, have the following general form:

$$r_{i,t+1} = \alpha_i + \beta_i' \mathbf{f}_{t+1} + v_{i,t+1}, \tag{2}$$

$$\mathbf{f}_{t+1} = E\left(\mathbf{f}_{t+1} \mid I_t\right) + \epsilon_{t+1} \qquad \text{with } \epsilon_{t+1} \sim \mathcal{D}\left(\mathbf{0}, \mathbf{\Sigma}\right) \tag{3}$$

where $Cov\left(v_{i,t+1}, v_{j,t+1}\right) = 0$ for $i \neq j$, $\mathbf{f}_{t+1}$ is a k-dimensional vector of factors at time $t+1$, $r_{i,t+1}$ is the return on the i-th of the n assets at time $t+1$, and the vector β_i' contains the loadings for asset i on the k factors. Equation (2) specifies the conditional distribution of returns on factors, while equation (3) specifies the predictive distribution for factors at time $t+1$ conditioning on information available at time t. A baseline specification for this system assumes away factors predictability thus implying that conditional expectations of factors have no variance (i.e., $E\left(\mathbf{f}_{t+1} \mid I_t\right) = \mu$).

Factor Models with Predictability

Consider an asset i and denote its log one-period return by $r_{i,t}$. We define the log price of this asset as:

$$\ln P_{i,t} = \ln P_{i,t-1} + \mathbf{r}_{i,t} \ , \tag{4}$$

i.e., prices of any asset are cumulative returns. The analogous of the (log) price for an asset can be constructed for any given factor. We define as factor (log) price the cumulative returns of a portfolio investing in standard factors (e.g., the aggregate market return). The generic prices associated to factors with a log period returns of $\mathbf{f}_t$ evolve according to the following process:

$$\ln \mathbf{F}_t = \ln \mathbf{F}_{t-1} + \mathbf{f}_t \ . \tag{5}$$

If returns to test assets and factors are stationary, then portfolio prices and factor-prices are non-stationary.

Factor Prices and Asset Prices

If factor prices are the non-stationary variables that drive the non-stationary dynamics of portfolio prices, then a linear combination of prices and risk drivers should be stationary, i.e., asset and factor prices should be cointegrated.
Consider the following model describing the exposure of a given portfolio price $P_{i,t}$ to factor prices $\mathbf{F}_t$:

$$\ln P_{i,t} = \alpha_{0,i} + \alpha_{1,i}t + \beta_i' \ln \mathbf{F}_t + u_{i,t} \ .$$

The estimation of such regression delivers stationary residuals $u_{i,t}$ anytime the chosen set of factor prices captures the stochastic trend that determines the long-run dynamics of prices.

From Prices to Returns

Formally, we define the residual from the long-run cointegrating relationship as:

$$u_{i,t} = ECT_{i,t} \equiv \ln P_{i,t} - \hat{\alpha}_{0,i} - \hat{\alpha}_{1,i} t - \hat{\beta}_i' \ln \mathbf{F}_t \ . \tag{6}$$

For expository purposes, it is useful to specify the error term $u_{i,t}$ as an AR(1) process. In sum, we model the joint distribution of asset prices, factor prices, asset returns and factors as follows:

$$\begin{aligned} \ln P_{i,t+1} &= \alpha_{0,i} + \alpha_{1,i} t + \beta_i' \ln \mathbf{F}_{t+1} + u_{i,t+1} \\ u_{i,t+1} &= \rho_i u_{i,t} + v_{i,t+1} \\ \mathbf{f}_{t+1} &= E\left(\mathbf{f}_{t+1} \mid I_t\right) + \epsilon_{t+1} \\ \ln P_{i,t} &= \ln P_{i,t-1} + r_{i,t} \\ \ln \mathbf{F}_t &= \ln \mathbf{F}_{t-1} + \mathbf{f}_t \end{aligned} \tag{7}$$

where $\epsilon_{t+1} \sim \mathcal{D}\left(\mathbf{0}, \mathbf{\Sigma}\right)$, $u_{i,t+1}$ and $v_{i,t+1}$ have zero mean and variance $\sigma_{u,i}^2$ and $\sigma_{v,i}^2$, respectively, and $Cov\left(v_{i,t+1}, v_{j,t+1}\right) = 0$ for $i \neq j$.

Factor Models with Predictability

By taking first differences of our model we obtain a novel specification for returns and factors, where asset returns relate to factors *plus* the ECT:

$$r_{i,t+1} = \alpha_{1,i} + \beta_i' \mathbf{f}_{t+1} + \underbrace{(\rho_i - 1)}_{\delta_i} \underbrace{u_{i,t}}_{\equiv ECT_{i,t}} + v_{i,t+1}. \tag{8}$$

The above equation represents the Factor Error Correction Model (FECM).

Factor Models with Predictability

The standard cross sectional representation of 1-period ahead returns becomes now

$$\begin{aligned}
\underset{(Nx1)}{\mathbf{r}_{t,t+1}} &= \underset{(Nx1)}{\alpha} + \underset{(NxK)}{B}\underset{(Kx1)}{\mathbf{f}_{t,t+1}} + \underset{(NxN)}{\Gamma}\underset{(Nx1)}{\mathbf{u}_t} + \underset{(Nx1)}{\mathbf{v}_t} \qquad (9) \\
\underset{(Kx1)}{\mathbf{f}_{t,t+1}} &= \underset{(Kx1)}{\mu^f} + \underset{(KxK)}{\mathbf{H}^f}\underset{(Kx1)}{\epsilon^f} \\
\Sigma^v &= \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ .. & .. & .. & .. \\ 0 & 0 & 0 & \sigma_n \end{bmatrix} \\
\Sigma^f &= \mathbf{H}^f \mathbf{H}^{f\prime}.
\end{aligned}$$

Predictability emerges as the conditional expectations of one-period ahead expected returns is time varying, the relevant conditional variance-covariance matrix of predicted asset returns also changes as the variance of the one-period ahead predictive error is different for the variance of asset returns.

Predicted Returns with the standard factor model

- Standard Factor models have the general form:

$$
\begin{aligned}
r_{i,t+1} &= \alpha_i + \beta_i' \mathbf{f}_{t+1} + v_{i,t+1}. \\
\mathbf{f}_{t+1} &= \mu + \epsilon_{t+1}
\end{aligned}
$$

- predicted returns will then be constant and equal to:

$$
r_{i,t+1}^{p} = \alpha_i + \beta_i' \mu
$$

Predicted Returns with the FECM model

- FECM models have the general form:

$$\begin{aligned} r_{i,t+1} &= \alpha_{1,i} + \beta_i' \mathbf{f}_{t+1} + \underbrace{(\rho_i - 1)}_{\delta_i} \underbrace{u_{i,t}}_{\equiv ECT_{i,t}} + v_{i,t+1} \\ \mathbf{f}_{t+1} &= \mu + \epsilon_{t+1} \end{aligned}$$

- predicted returns will then be time varying and equal to:

$$r^p_{i,t+1} = \alpha_i + \beta_i' \mu + (\rho_i - 1) u_{i,t}$$

Multi-horizon predicted returns with the FECM model

Consider now horizons beyond the one-period with the FECM

$$\begin{aligned} r^p_{i,t+2} &= \alpha_i + \beta_i'\mu + \rho_i(\rho_i - 1)u_{i,t} \\ &... \\ r^p_{i,t+n} &= \alpha_i + \beta_i'\mu + \rho_i^{n-1}(\rho_i - 1)u_{i,t} \end{aligned}$$

and, for large n, the effect of the disequilibrium on the prediction naturally disappears.

What's next?

- ECT, Risk management and Portfolio timing
  - Modeling the relation between risk drivers and asset prices contributes to the description of the dynamics of returns
  - The predictive distribution of returns at time $t+1$ is centered on the ECT observed at time t.
- ECT and time-varying alphas
  - Recall

$$r_{i,t+1} = \alpha_{1,i} + \beta_i' \mathbf{f}_{t+1} + \delta_i ECT_{i,t} + \ v_{i,t+1}.$$

  - The test that the intercept is zero becomes a test that $\widehat{\alpha}_{1,i} + \widehat{\delta}_i ECT_{i,t} = 0$.

An illustration with R

The R code **FECM_CAPM.r**

- considers the assets in the previous asset allocation example,
- runs the long runs regressions of asset prices and factor prices,
- and illustrates how the CAPM can be modified to derive a factor model with returns predictability