Econometric Methods for Finance and Macro

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Phd

VAR and CVAR

- Statistical Representation
- Identification and Hypothesis-Testing with multiple cointegrating vectors
- Using VAR: Alternative Representation
- Identification of Shocks: Statistical and Narrative
- Description of VAR models
- Cointegration and Multivariate trend-cycle decompositions
- Global VARs
- Cointegrated VAR and Present Value Models
- VAR-based model evaluation in Macroeconomics

Statistical Representation

Consider the multivariate vector autoregressive representation (VAR) for the vector of, possibly non-stationary, m-variables $\mathbf{y}$:

$$\mathbf{y}_t = \mathbf{A}_0 + \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{u}_t. \tag{1}$$

This representation is general in that a VAR of any order can be always re-parameterised as a VAR of order 1. By taking first differences:

$$\begin{aligned} \Delta \mathbf{y}_t &= \mathbf{A}_0 + \mathbf{\Pi} \mathbf{y}_{t-1} + \mathbf{u}_t. \\ \mathbf{\Pi} &= (\mathbf{A}_1 - \mathbf{I}) \end{aligned}$$

Statistical Representation

Clearly the long-run properties of the system are described by the properties of the matrix Π. There are three cases of interest:

1. rank $(\Pi) = 0$. The system is non-stationary, with no cointegration between the variables considered. This is the only case in which non-stationarity is correctly removed simply by taking the first differences of the variables;

2. rank $(\Pi) = m$, full. The system is stationary;

3. rank $(\Pi) = k < m$. The system is non-stationary but there are k cointegrating relationships among the considered variables. In this case $\mathbf{\Pi} = \alpha\beta'$, where α is an $(m \times k)$ matrix of weights and β is an $(m \times k)$ matrix of parameters determining the cointegrating relationships.

Therefore, the rank of Π is crucial in determining the number of cointegrating vectors (Johansen Procedure).

Statistical Representation

A cointegrated VAR can be therefore represented as:

$$\begin{aligned} \Delta \mathbf{y}_t &= \mathbf{A}_0 + \mathbf{\Pi} \mathbf{y}_{t-1} + \mathbf{u}_t. \\ \mathbf{\Pi} &= \alpha \beta' \end{aligned}$$

where α capture the speed of adjustment of the system with respect to deviation from long-run equilibria and β determine the long-run equilibria.

An Illustrative Example

Consider the VAR representation for the two variables, x and y:

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}. \qquad (2)$$

System (2) can be reparameterized as follows in terms of the VECM representation:

$$\begin{pmatrix} \Delta y_t \\ \Delta x_t \end{pmatrix} = \begin{pmatrix} a_{11}-1 & a_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}, \qquad (3)$$

from which, clearly,

$$\Pi = \begin{pmatrix} a_{11}-1 & a_{12} \\ 0 & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} a_{11}-1 \\ 0 \end{pmatrix}, \quad \beta' = \begin{pmatrix} 1 & -\frac{a_{12}}{1-a_{11}} \end{pmatrix}.$$

Identification of Cointegrating Vectors

In the case of existence of multiple cointegrating vectors, an interesting identification problem arises: α and β are only determined up to the space spanned by them. Thus, for any non-singular matrix ξ conformable by product:

$$\Pi = \alpha\beta' = \alpha\xi^{-1}\xi\beta'.$$

In other words β and $\beta'\xi$ are two observationally equivalent bases of the cointegrating space. Identification can be achieved by imposing a sufficient number of restrictions on parameters such that the matrix satisfying such constraints in the cointegrating space is unique. Such a criterion is derived by Johansen (1992) and discussed in the works of Johansen and Juselius.

Alternative Representation

Consider a vector $\mathbf{y}_t$ containing two variables x_t and z_t cointegrated with an equilibrium error $S_t = x_t - \beta z_t$. The Johansen representation for such system will be:

$$\begin{pmatrix} \Delta x_t \\ \Delta z_t \end{pmatrix} = \mathbf{\Pi}_1 \begin{pmatrix} \Delta x_{t-1} \\ \Delta z_{t-1} \end{pmatrix} + \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix} \begin{pmatrix} 1 & -\beta \end{pmatrix} \begin{pmatrix} x_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}.$$

$$\begin{pmatrix} \Delta x_t \\ \Delta z_t \end{pmatrix} = \mathbf{\Pi}_1 \begin{pmatrix} \Delta x_{t-1} \\ \Delta z_{t-1} \end{pmatrix} + \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix} S_{t-1} + \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}$$

Define a matrix M such that

$$M \begin{pmatrix} \Delta x_t \\ \Delta z_t \end{pmatrix} = \begin{pmatrix} \Delta x_t \\ \Delta S_t \end{pmatrix}$$

so

$$M = \begin{pmatrix} 1 & 0 \\ 1 & -\beta \end{pmatrix}$$

Alternative Representation

then we have:

$$
\begin{aligned}
M\begin{pmatrix}\Delta x_t\\ \Delta z_t\end{pmatrix} &= M\mathbf{\Pi}_1\begin{pmatrix}\Delta x_{t-1}\\ \Delta z_{t-1}\end{pmatrix}+M\begin{pmatrix}\alpha_{11}\\ \alpha_{21}\end{pmatrix}S_{t-1}+M\begin{pmatrix}v_{1t}\\ v_{2t}\end{pmatrix}\\
\begin{pmatrix}\Delta x_t\\ \Delta S_t\end{pmatrix} &= M\mathbf{\Pi}_1M^{-1}\begin{pmatrix}\Delta x_{t-1}\\ \Delta S_{t-1}\end{pmatrix}+M\begin{pmatrix}\alpha_{11}\\ \alpha_{21}\end{pmatrix}S_{t-1}+M\begin{pmatrix}v_{1t}\\ v_{2t}\end{pmatrix}
\end{aligned}
$$

The system can be rearranged so that it describes levels rather than differences of S_t.
The result is a second order VAR as follows:

$$
\begin{pmatrix}\Delta x_t\\ S_t\end{pmatrix}=G_1\begin{pmatrix}\Delta x_{t-1}\\ S_{t-1}\end{pmatrix}+G_2\begin{pmatrix}\Delta x_{t-2}\\ S_{t-2}\end{pmatrix}+M\begin{pmatrix}v_{1t}\\ v_{2t}\end{pmatrix}
$$

Using VAR Models

A cointegrated VAR, after the identification of the number and shape of cointegrating vector(s), provides a statistical model of the joint distribution of the variables of interests:

$$\begin{aligned} \Delta \mathbf{y}_t &= \alpha\beta' \mathbf{y}_{t-1} + \mathbf{u}_t \\ \mathbf{u}_t &\sim N\left(0, \sum\right) \end{aligned} \tag{4}$$

where $\mathbf{y}_t$ is a vector of length N containing the modelled variables.

- The reduced form specification (4) can be adopted directly for forecasting purposes;
- Some further identification choice must be made to recover the "structural" shocks from the VAR.

Identification of Structural Shocks

From VAR innovations to structural shocks

$$\mathbf{A}\mathbf{u}_t = \mathbf{B}\upsilon_t,$$
$$\upsilon_t \sim N(0, I)$$

from which we can derive the relation between the variance-covariance matrices of $\mathbf{u}_t$ (observed) and ν_t (unobserved) as follows:

$$E\left(\mathbf{u}_t\mathbf{u}_t'\right) = \mathbf{A}^{-1}\mathbf{B}E\left(\upsilon_t\upsilon_t'\right)\mathbf{B}'\mathbf{A}^{-1}.$$

Substituting population moments with sample moments we have:

$$\widehat{\sum} = \widehat{\mathbf{A}}^{-1}\mathbf{B}\mathbf{I}\widehat{\mathbf{B}}'\widehat{\mathbf{A}}^{-1}, \tag{5}$$

$\widehat{\sum}$ contains $n(n+1)/2$ different elements, which is the maximum number of identifiable parameters in matrices $\mathbf{A}$ and $\mathbf{B}$. Necessary condition for identification is that the maximum number of parameters contained in the two matrices equals $n(n+1)/2$, For such a condition also to be sufficient for identification no equation in (5) should be a linear combination of the other equations in the system (see Amisano and Giannini 1996, Hamilton 1994).

Description of VAR Models

Given an identified and estimated structural VAR

$$\begin{aligned} \mathbf{y}_t &= \mathbf{C}_1 \mathbf{y}_{t-1} + \mathbf{u}_t, \\ \mathbf{A}\mathbf{u}_t &= \mathbf{B}\upsilon_t, \end{aligned}$$

we can re-write it as:

$$\mathbf{y}_t = \mathbf{C}_1 \mathbf{y}_{t-1} + \mathbf{A}^{-1}\mathbf{B}\upsilon_t,$$

Description of VAR Models

By recursive substitution we obtain the moving average representation for our VAR process in terms of the structural shocks:

$$\begin{aligned} \mathbf{y}_t &= \mathbf{C}(L)\,\mathbf{v}_t, \qquad (6)\\ \mathbf{y}_t &= \mathbf{C}_0\mathbf{v}_t + \mathbf{C}_1\mathbf{v}_{t-1} + ... + \mathbf{C}_s\mathbf{v}_{t-s}, \\ \mathbf{C}(L) &= \mathbf{C}_1^L\mathbf{C}_0, \\ \mathbf{C}_0 &= \mathbf{A}^{-1}\mathbf{B}. \end{aligned}$$

The MA representation can be used to derive

- Impulse Response functions
- Forecasting Error Variance Decompositions
- Historical Decompositions

Identification Strategies

- Choleski Decomposition
- Structural VAR with Long-run restrictions
- Permanent Temporary Decompositions
- Sign Restrictions
- GIRF
- Non-VAR based identification
  - Narrative identification
  - identification from financial variables

Global VARs

The Global VAR (GVAR) approach advanced in Pesaran, Schuermann and Weiner (2004, PSW) provides a flexible reduced-form framework capable of accommodating a time-varying co-movement across domestic variables and their foreign counterpart. The general specification of a GVAR can be described as follows:

$$\mathbf{x}_{it} = B_{id}\mathbf{d}_t + B_{i1}\mathbf{x}_{it-1} + B^*_{i0}\mathbf{x}^*_{it} + B^*_{i1}\mathbf{x}^*_{it-1} + \mathbf{u}_{it}$$

where $\mathbf{x}_{it}$ is a vector of domestic variables, $\mathbf{d}_t$ is a vector of deterministic elements as well as observed common exogenous variables, $\mathbf{x}^*_{it}$ is a vector of foreign variables specific to country i. In general $\mathbf{x}^*_{it} = \sum_{j \neq i} w_{ji}\mathbf{x}_{jt}$ where w_{ji} is the share of country j in the trade (exports plus imports) of country i. Finally $\mathbf{u}_{it}$ is a vector of country-specific idiosyncratic shocks.

Cointegration and Present Value Models

CS consider the case of the risk free rate and a very long term bond.

$$y_{t,T} = y^*_{t,T} + E[\Phi_T \mid I_t] = (1-\gamma)\sum_{j=0}^{T-t-1} \gamma^j E[r_{t+j} \mid I_t] + E[\Phi_T \mid I_t]$$

Subtracting the risk-free rate from both sides of this equation we have:

$$\begin{aligned} S_{t,T} = y_{t,T} - r_t &= \sum_{j=1}^{T-1} \gamma^j E[\Delta r_{t+j} \mid I_t] + E[\Phi_T \mid I_t] \\ &= S^*_{t,T} + E[\Phi_T \mid I_t] \end{aligned}$$

Cointegration and Present Value Models

- A necessary condition for the ET to hold is stationarity of the spread. This implies that, if yields are non-stationary, they should be cointegrated with a cointegrating vector (1,-1).

- However, the necessary and sufficient conditions for the validity of the ET impose restrictions both on the long-run and the short run dynamics.

Assuming that $y_{t,T}$ and r_t are cointegrated with a cointegrating vector (1,-1), CS construct a bivariate stationary VAR in the first difference of the short-term rate and the spread :

$$\begin{aligned} \Delta r_t &= a(L)\Delta r_{t-1} + b(L)S_{t-1} + u_{1t} \\ S_t &= c(L)\Delta r_{t-1} + d(L)S_{t-1} + u_{2t} \end{aligned} \tag{7}$$

Cointegration and Present Value Models

Stack the VAR as:

$$\begin{bmatrix} \Delta r_t \\ . \\ . \\ \Delta r_{t-p+1} \\ S_t \\ . \\ . \\ S_{t-p+1} \end{bmatrix} = \begin{bmatrix} a_1 & . & . & a_p & b_1 & . & . & b_p \\ 1 & . & . & 0 & 0 & . & . & 0 \\ 0 & . & . & 0 & 0 & . & . & 0 \\ 0 & . & 1 & 0 & 0 & . & . & 0 \\ c_1 & . & . & c_p & d_1 & . & . & d_p \\ 0 & . & . & 0 & 1 & . & . & 0 \\ 0 & . & . & 0 & 0 & . & . & 0 \\ 0 & . & . & 0 & 0 & . & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta r_{t-1} \\ . \\ . \\ \Delta r_{t-p} \\ S_{t-1} \\ . \\ . \\ S_{t-p} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ . \\ . \\ 0 \\ u_{2t} \\ . \\ . \\ 0 \end{bmatrix} \tag{8}$$

Cointegration and Present Value Models

This can be written more succinctly as:

$$z_t = Az_{t-1} + v_t \tag{9}$$

The ET null puts a set of restrictions which can be written as :

$$g'z_t = \sum_{j=1}^{T-1} \gamma^j h' A^{j\prime} z_t \tag{10}$$

where $g\prime$ and $h\prime$ are selector vectors for S and Δr correspondingly (i.e. row vectors with 2p elements, all of which are zero except for the p+1st element of $g\prime$ and the first element of $h\prime$ which are unity).

Cointegration and Present Value Models

Since the above expression has to hold for general z_t, and, for large T, the sum converges under the null of the validity of the ET, it must be the case that:

$$g' = h'\gamma A(I - \gamma A)^{-1} \tag{11}$$

which implies:

$$g'(I - \gamma A) = h'\gamma A \tag{12}$$

and we have the following constraints on the individual coefficients of VAR :

$$\{c_i = -a_i, \forall i\}, \{d_1 = -b_1 + 1/\gamma\}, \{d_i = -b_i, \forall i \neq 1\} \tag{13}$$

The above restrictions are testable with a Wald test. By doing so using US data between the fifties and the eighties Campbell and Shiller (1987) rejected the null of the ET.

Cointegration and Present Value Models

However, when CS construct a theoretical spread $S^*_{t,T}$, by imposing the (rejected) ET restrictions on the VAR they find that, despite the statistical rejection of the ET, $S^*_{t,T}$ and $S_{t,T}$ are strongly correlated.

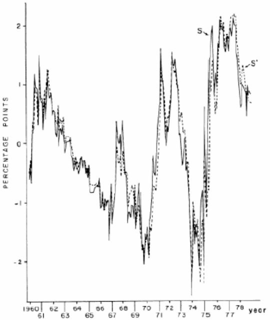

FIG. 1.—Term structure: deviations from means of long-short spread S_t and theoretical spread S'_t.

Cointegration and Present Value Models

Things look very different for the stock market. In this case we have:

$$(p_t - d_t)^* = \sum_{j=1}^{m} \rho^{j-1} (\Delta d_{t+j})$$

and the variable $(p_t - d_t)^*$ can be obtained by imposing the appropriate cross-equation restrictions on a bivariate VAR for the dividend-yield and dividend growth.

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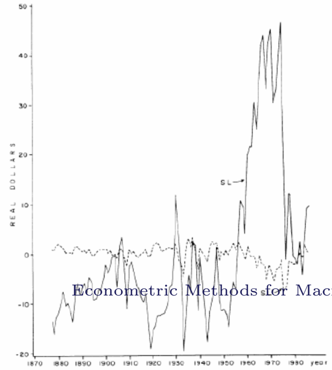

FIG. 2.—Stock market: deviations from means of actual spread (SL_t = Price$_t$ − θ · Dividend$_{t-1}$) and theoretical spread SL'_t, θ = 12.195.