Models of the TS

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Asset Pricing with Time-Varying Expected Returns

Consider a situation in which in each period k state of nature can occur and each state has a probability $\pi(k)$, in the absence of arbitrage opportunities the price of an asset i at time t can be written as follows:

$$P_{i,t} = \sum_{s=1}^{k} \pi_{t+1}(s) m_{t+1}(s) X_{i,t+1}(s)$$

where $m_{t+1}(s)$ is the discounting weight attributed to future pay-offs, which (as the probability π) is independent from the asset i, $X_{i,t+1}(s)$ are the payoffs of the assets (in case of stocks we have $X_{i,t+1} = P_{t+1} + D_{t+1}$, in case of zero coupon bonds, $X_{i,t+1} = P_{t+1}$), and therefore returns on assets are defined as $1 + R_{s,t+1} = \frac{X_{i,t+1}}{P_{i,t}}$.

Asset Pricing with Time-Varying Expected Returns

For the safe asset, whose payoffs do not depend on the state of nature, we have:

$$\begin{aligned} P_{s,t} &= X_{i,t+1}\sum_{s=1}^{k}\pi_{t+1}(s)m_{t+1}(s) \\ 1+R_{s,t+1} &= \frac{1}{\sum_{j=1}^{m}\pi_{t+1}(s)m_{t+1}(s)} \\ 1+R_{s,t+1} &= \frac{1}{E_t(m_{t+1})} \end{aligned}$$

Asset Pricing with Time-Varying Expected Returns

consider now a risky asset :

$$\begin{aligned} E_t\left(m_{t+1}\left(1+R_{i,t+1}\right)\right) &= 1 \\ Cov\left(m_{t+1}R_{i,t+1}\right) &= 1-E_t\left(m_{t+1}\right)E_t\left(1+R_{i,t+1}\right) \\ E_t\left(1+R_{i,t+1}\right) &= -\frac{Cov\left(m_{t+1}R_{i,t+1}\right)}{E_t\left(m_{t+1}\right)}+\left(1+R_{s,t+1}\right) \end{aligned}$$

Turning now to excess returns we can write:

$$E_t\left(R_{i,t+1}-R_{s,t+1}\right)=-\left(1+R_{s,t+1}\right)cov\left(m_{t+1}R_{i,t+1}\right)$$

Assets whose returns are low when the stochastic discount factor is high (i.e. when agents values payoffs more) require an higher risk premium, i.e. an higher excess return on the risk-free rate.

Physical and risk-neutral probabilities

Consider an optimal portfolio equilibrium when an investor can choose between investing at period 0 in a risky asset and a in risk-free asset:

$$\begin{aligned}
E\left[u'\left(Y_1\right)\left(1+r_r\right)\right] &= E\left[u'\left(Y_1\right)\left(1+r_{rf}\right)\right] \\
&= \left(1+r_{rf}\right)E\left[u'\left(Y_1\right)\right] \\
\frac{E\left[u'\left(Y_1\right)\left(1+r_r\right)\right]}{E\left[u'\left(Y_1\right)\right]} &= \left(1+r_{rf}\right) \\
E\left[\bar{m}\left(1+r_r\right)\right] &= \left(1+r_{rf}\right) \\
\bar{m} &= \frac{u'\left(Y_1\right)}{E\left[u'\left(Y_1\right)\right]} = m\left(1+r_{rf}\right)
\end{aligned}$$

$\bar{m}$ is known as the "pricing kernel" .

Physical and risk-neutral probabilities

Consider a simplified scenario in which there are only two states of the world (g,b)

$$
\begin{aligned}
E\left[\bar{m}\left(1+r_r\right)\right] &= \left(1+r_{rf}\right) \\
\pi^g \bar{m}^g\left(1+r_r^g\right)+\pi^b \bar{m}^b\left(1+r_r^b\right) &= \left(1+r_{rf}\right) \\
\bar{m}^g &= \frac{u'\left(Y_1^g\right)}{E\left[u'\left(Y_1\right)\right]},\ \bar{m}^b=\frac{u'\left(Y_1^b\right)}{E\left[u'\left(Y_1\right)\right]} \\
E\left[u'\left(Y_1\right)\right] &= \pi^g u'\left(Y_1^g\right)+\pi^b u'\left(Y_1^b\right)
\end{aligned}
$$

Physical and risk-neutral probabilities

We have

$$
\begin{aligned}
E\left[\bar{m}\right] &= 1 \\
\pi^g \bar{m}^g + \pi^b \bar{m}^b &= 1 \\
\pi^g \frac{u'\left(Y_1^g\right)}{E\left[u'\left(Y_1\right)\right]} + \pi^b \frac{u'\left(Y_1^b\right)}{E\left[u'\left(Y_1\right)\right]} &= 1 \\
E\left[u'\left(Y_1\right)\right] &= \pi^g u'\left(Y_1^g\right) + \pi^b u'\left(Y_1^b\right)
\end{aligned}
$$

Physical and risk-neutral probabilities

The quantity $\pi^i \bar{m}^i = q^i$ is naturally intepreted as a new probability measure of the state of the world i. The expected return for every risky asset under this new probability measure is the risk free rate:

$$\begin{aligned} \pi^g \bar{m}^g \left(1 + r_r^g\right) + \pi^b \bar{m}^b \left(1 + r_r^b\right) &= \left(1 + r_{rf}\right) \\ q^g \left(1 + r_r^g\right) + q^b \left(1 + r_r^b\right) &= \left(1 + r_{rf}\right) \end{aligned}$$

Note that $\frac{\pi^i \bar{m}^i}{\left(1+r_{rf}\right)}$ is the price that an investor is willing to pay for a security that generates a marginal increase of wealth in the state of the world i and no change of wealth in other states of the world.

Bond Returns:YTM, Duration and HPR

Cash-flows from different type of bonds:

	$t+1$	$t+2$	$t+3$	...	T
general	CF_{t+1}	CF_{t+2}	CF_{t+3}	...	CF_T
coupon bond	C	C	C	...	$1+C$
1-period zero	1	0	0	...	0
2-period zero	0	1	0	...	0
...				...	
(T–t) -period zero	0	0	0	...	1

ZC Bonds

Define the relationship between price and yield to maturity of a zero-coupon bond as follows:

$$P_{t,T} = \frac{1}{\left(1 + Y_{t,T}\right)^{T-t}}, \tag{1}$$

where $P_{t,T}$ is the price at time t of a bond maturing at time T, and $Y_{t,T}$ is yield to maturity. Taking logs we have the following relationship:

$$p_{t,T} = -\left(T - t\right) y_{t,T}, \tag{2}$$

which clearly illustrates that the elasticity of the yield to maturity to the price of a zero-coupon bond is the maturity of the security.

ZC Bonds

Price and YTM of zero-coupon bonds							
Mat	1	2	3	5	7	10	20
$P_{t,T}$	0.9524	0.9070	0.8638	0.7835	0.7106	0.6139	0.3769
$Y_{t,T}$	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
$p_{t,T}$	−0.0487	−0.0976	−0.1464	−0.2439	−0.3416	−0.4879	−0.9757
$y_{t,T}$	0.0488	0.0488	0.0488	0.0488	0.0488	0.0488	0.0488

ZC Bonds

The one-period uncertain holding-period return on a bond maturing at time T, $r^T_{t,t+1}$, is then defined as follows:

$$r^T_{t,t+1} \equiv p_{t+1,T} - p_{t,T} = -\,(T-t-1)\,y_{t+1,T} + (T-t)\,y_{t,T} \quad (3)$$
$$= y_{t,T} - (T-t-1)\,(y_{t+1,T} - y_{t,T})\,,$$
$$= (T-t)\,y_{t,T} - (T-t-1)\,y_{t+1,T}, \quad (4)$$

which means that yields and returns differ by the a scaled measure of the change between the yield at time $t+1$, $y_{t+1,T}$, and the yield at time t, $y_{t,T}$. Think of a situation in which the one-year YTM stands at 4.1 per cent while the 30-year YTM stands at 7 per cent. If the YTM of the thirty year bonds goes up to 7.1 per cent in the following period, then the period returns from the two bonds is the same.

Modelling the Term Structure

Apply the no arbitrage condition to a one-period bond (the safe asset) and a T-period bond:

$$\begin{aligned} E_t\left(r^T_{t,t+1} - r^1_{t,t+1}\right) &= E_t\left(r^T_{t,t+1} - y_{t,t+1}\right) = \phi^T_{t,t+1} \\ E_t\left(r^T_{t,t+1}\right) &= y_{t,t+1} + \phi^T_{t,t+1} \end{aligned}$$

Solving forward the difference equation $p_{t,T} = p_{t+1,T} - r^T_{t,t+1}$, we have :

$$\begin{aligned} y_{t,T} &= \frac{1}{(T-t)}\sum_{i=0}^{n-1} E_t\left(r^T_{t+i,t+i+1}\right) \\ &= \frac{1}{(T-t)}\sum_{i=0}^{n-1} E_t\left(y_{t+i,t+i+1} + \phi^T_{t+i,t+i+1}\right) \end{aligned}$$

Modelling the Term Structure

The model clearly shows that Bond yields are driven by two unobservable factors

- Expectations of future monetary policy (risk free) rates over the residual life of the bonds
- Compensation for risk (risk premia)

Modelling the Term Structure

By subtracting from the left-hand side and the right-hand side of the equation above the one-period rate, we have the following equation for the term spread:

$$y_{t,T} - y_{t,t+1} = \sum_{i=1}^{T-t-1} \left(1 - \frac{i}{T-t}\right) E_t \Delta y_{t+i,t+i+1} + \frac{1}{T-t} \sum_{i=1}^{T-t-1} \phi_{t+i,t+i+1}^{T} \quad (5)$$

The model clearly shows that if yields at long maturities are cointegrated with short-term rates

- there is a common stochastic trend affecting the entire term structure which is the trend in short-term rates
- Compensation for risk (risk premia) are stationary and therefore cyclical

Forward Rates

Forward rates are returns on an investment at time t, made in the future at time t' with maturity at time T. The return on this strategy is equivalent to the return on a strategy that buys at time t zero coupon with maturity T and sells at time t the same amount of bonds with maturity t'.
The price of the investment strategy is
$\left(-\left(T-t\right)y_{t,T}+\left(t'-t\right)y_{t,t'}\right)$and using the usual formula that links prices to returns we have :

$$f_{t,t',T}=\frac{\left(T-t\right)y_{t,T}-\left(t'-t\right)y_{t,t'}}{T-t'} \tag{6}$$

Applying the general formula to specific maturities we have :

$$f_{t,t+1,t+2}=2y_{t,t+2}-y_{t,t+1} \tag{7}$$

$$f_{t,t+2,t+3}=3y_{t,t+3}-2y_{t,t+2} \tag{8}$$

Forward Rates

Using all these equations we have:

$$y_{t,t+n} = \frac{1}{n}\left(y_{t,t+1} + f_{t,t+1,t+2} + f_{t,t+2,t+3} + ... + f_{t,t+n-1,t+n}\right) \quad (11)$$

$$y_{t,t+n} = \frac{1}{n}\sum_{i=0}^{n-1} E_t\left(y_{t+i,t+i+1} + \phi^T_{t+i,t+i+1}\right)$$

$$f_{t,t+i,t+i+1} = E_t\left(y_{t+i,t+i+1} + \phi^T_{t+i,t+i+1}\right)$$

Forward Rates

Think of using forward rates to assess the impact of monetary policy. Let us analyze a potential movement of spot and forward rates around a shift in the central bank target rate.

Before CB intervention

1-year spot and forward rates					
maturity	i=1	i=2	i=3	i=4	i=5
$y_{t,t+i}$	0.05	0.05	0.05	0.05	0.05
$f_{t,t+i,t+i+1}$	0.05	0.05	0.05	0.05	

Forward Rates

After CB intervention :

1-year spot and forward rates					
maturità	i=1	i=2	i=3	i=4	i=5
$y_{t,t+i}$	0.06	0.06	0.05	0.045	0.04
$f_{t,t+i,t+i+1}$	0.06	0.03	0.03	0.02	

Please remember that the interpretion of future forward as expected rates requires some assumption on the risk premium.

Instantaneous Forward Rates

Define the instanteous froward as the forwad rate on the contract with infinitesimal maturity:

$$f_{t,t'} = \lim_{T \to t'} f_{t,t',T} \tag{12}$$

given the sequence of forward rates you can define forward rate at any settlement date as follows :

$$f_{t,t',T} = \frac{\int_{\tau=t'}^{T} f_{\tau t} d\tau}{(T - t')}$$

Instantaneous Forward Rates

As a consequence the relationship between spot and forward rate is written as:

$$y_{t,T} = \frac{\int_{\tau=t}^{T} f_{\tau t} d\tau}{(T-t)}$$

and therefore

$$f_{t,T} = y_{t,T} + (T-t)\frac{\partial y_{t,T}}{\partial T} \qquad (13)$$

so instantaneous forward rates and spot rates coincide at the very short and very long-end of the term structure, forward rates are above spot rates when the yield curve slopes positevely and forward rates are below spot rates when the yield curve slopes negatively.

The relevance of the RP

To assess the importance of RP, different implications of the Expectations theory can be brought to the data:

$$\begin{aligned} y_{t,T} - (T-t-1) E_t \left(y_{t+1,T} - y_{t,T} \right) &= y_{t,t+1} + \phi_{t,T}. \\ f_{t,t+i,t+i+1} &= E_t \left(y_{t+i,t+i+1} + \phi^T_{t+i,t+i+1} \right) \\ y_{t,t+n} - y_{t,t+1} &= \sum_{i=1}^{n-1} \left(1 - \frac{1}{n} \right) E_t \Delta y_{t+i,t+i+1} \\ &\quad + \frac{1}{n} \sum_{i=1}^{n-1} \phi^T_{t+i,t+i+1} \end{aligned}$$

$$y_{t,T} = \frac{1}{(T-t)} \sum_{i=0}^{n-1} E_t \left(y_{t+i,t+i+1} \right) + RP_{t,T}$$

Single Equation based evidence

- Estimate the following model :

$$y_{t+1,T} - y_{t,T} = \beta_0 + \beta_1 \frac{1}{T-t-1}\left(y_{t,T} - y_{t,t+1}\right) + u_{t+1}$$

to test $\beta_0 = 0, \beta_1 = 1$.

- Estimate the following model :

$$\begin{aligned} S^*_{t,n} &= \beta_0 + \beta_1 S_{t,n} + u_t \\ S_{t,n} &= y_{t,t+n} - y_{t,t+1} \\ S^*_{t,n} &= \sum_{i=1}^{n-1}\left(1 - \frac{1}{n}\right)\Delta y_{t+i,t+i+1} \end{aligned}$$

to test $\beta_0 = 0, \beta_1 = 1$.

Single Equation based evidence

- Estimate the following model :

$$\left(y_{t+i,t+i+1} - y_{t,t+1}\right) = \beta_0 + \beta_1 \left(f_{t,t+i,t+i+1} - y_{t,t+1}\right) + u_{t+i+1}$$

to test $\beta_0 = 0, \beta_1 = 1$.

- Estimate the following model :

$$y_{t,T} - (T - t - 1)\left(y_{t+1,T} - y_{t,T}\right) - y_{t,t+1} = \beta_0 + \beta_1 \left(f_{t,t+i,t+i+1} - y_{t,t+1}\right) + u_{t+i+1}$$

to test $\beta_1 = 0$.

Single Equation based evidence

The empirical evidence shows that:

- i) high yields spreads fare poorly in predicting increases in long rates(see Campbell, 1995)
- ii) the change in yields does not move one-to-one with the forward spot spread (see Fama and Bliss,1986)
- iii) period excess returns on long-term bond are predictable using the information in the forward-spot spread (see Cochrane,1999,Cochrane-Piazzesi 2005)

Cointegration and Predictability of Bond Returns

- Bond prices are drifting in the data
- Cointegration in modelling bond returns has been introduced by Campbell-Shiller (1987)

$$y_{t,T} = (1-\gamma)\sum_{j=1}^{T-t-1}\gamma^j E[r_{t+j} \mid I_t] + RP_{t,t+j}$$

$$S_{t,T} = \sum_{j=1}^{T-t-1}\gamma^j E[\Delta r_{t+j} \mid I_t] + RP_{t,t+j}$$

$$S_{t,T} = y_{t,T} - r_t$$

If short term rates are non-stationary and the $RP_{t,t+j}$ are stationary, then the entire term structure share their stochastic trends

VAR models of the term structure

The statistical model for the TS is a bivariate stationary VAR in the first difference of the short-term rate and the spread :

$$\Delta y_t^{(1)} = a(L)\Delta y_{t-1}^{(1)} + b(L)S_{t-1} + u_{1t}$$
$$S_t = c(L)\Delta y_{t-1}^{(1)} + d(L)S_{t-1} + u_{2t}$$

- The level of the short-term rate is left undetermined.
- The model is stationary but it leaves the level of yields undetermined.

Factor models of the term structure

Represent the dynamics of the term structure in a state-space framework:

$$y_{t,t+n} = \frac{-1}{n}\left(A_n + B_n' X_t\right) + \varepsilon_{t,t+n} \qquad \varepsilon_t \sim i.i.d.N(0, \sigma^2 I) \tag{14}$$

$$X_t = \mu + \Phi X_{t-1} + v_t \qquad v_t \sim i.i.d.N(0, \Omega) \tag{15}$$

- parsimonious representation
- it assumes stationarity
- no arbitrage generates cross-equation restrictions

Factor Models of the Term Structure

Gurkanyak et al. estimate the following interpolant at each point in time, by non-linear least squares, on the cross-section of yields:

$$\begin{aligned} y_{t,t+k} &= L_t + SL_t \frac{1-\exp\left(-\frac{k}{\tau_1}\right)}{\frac{k}{\tau_1}} + C_t^1 \left(\frac{1-\exp\left(-\frac{k}{\tau_1}\right)}{\frac{k}{\tau_1}} - \exp\left(-\frac{k}{\tau_1}\right) \right) \qquad (16) \\ &+ C_t^2 \left(\frac{1-\exp\left(-\frac{k}{\tau_2}\right)}{\frac{k}{\tau_2}} - \exp\left(-\frac{k}{\tau_2}\right) \right) \end{aligned}$$

which is an extension originally proposed by Svensson(1994) on the original parameterization adopted by Nelson and Siegel (1987) that sets $C_t^2 = 0$.

Nelson-Siegel, Diebold-Li models of the Term Structure

Forward rates are easily derived as

$$f_{tk} = L_t + SL_t \exp\left(-\frac{k}{\tau_1}\right) + C_t^1 \frac{k}{\tau_1} \exp\left(-\frac{k}{\tau_1}\right) + C_t^2 \frac{k}{\tau_2} \exp\left(-\frac{k}{\tau_2}\right) \quad (17)$$

When maturity k goes to zero forward and spot rates coincide at $L_t + SL_t$, and when maturity goes to infinite forward and spot coincide at L_t. Terms in C_t^1 and C_t^2 describes two humps starting at zero at different starting points and ending at zero.

Nelson-Siegel, Diebold-Li models of the Term Structure

- L_t, SL_t, C_t^1, C_t^2, which are estimated as parameters in a cross-section of yields, can be interpreted as latent factors.
- L_t has a loading that does not decay to zero in the limit, while the loading on all the other parameters do so, therefore this parameter can be interpreted as the long-term factor, the level of the term-structure.
- The loading on SL_t is a function that starts at 1 and decays monotonically towards zero; it may be viewed a short-term factor, the slope of the term structure. In fact, $r_t^{rf} = L_t + SL_t$ is the limit when k goes to zero of the spot and the forward interpolant. We naturally interpret r_t^{rf} as the risk-free rate.
- C_t are medium term factor, in the sense that their loading start at zero, increase and then decay to zero (at different speed). Such factors capture the curvature of the yield curve.

A general state-space representation

To generalize the NS approach we can put the dynamics of the term structure in a state-space framework:

$$y_{t,t+n} = \frac{-1}{n}\left(A_n + B_n' X_t\right) + \varepsilon_{t,t+n} \qquad \varepsilon_t \sim i.i.d.N(0, \sigma^2 I) \tag{18}$$

$$X_t = \mu + \Phi X_{t-1} + v_t \qquad v_t \sim i.i.d.N(0, \Omega) \tag{19}$$

In the case of original NS we have

$$B_n' = \left[-n \;\; , - \left(\frac{1 - e^{-\lambda n}}{\lambda} \right) \;\; , - \left(\frac{1 - e^{-\lambda n}}{\lambda} - n e^{-\lambda n} \right) \right] \text{ and } \; A_n = 0$$

ATS Models

We present the Adrian et al(2013) approach, in which factors are first obtained by extracting the first K (K=5) principal components for the entire term structure.
A VAR will then be appropriately specified for the stationary state variables, X_t

$$X_{t+1} = \mu + \Phi X_t + v_{t+1}, \tag{20}$$

$$v_{t+1} | \{X_s\}_{s=0}^{t} \sim \mathcal{N}(0, \Sigma), \tag{21}$$

The variables in X_t determine the market price of risk, λ_t in the following affine form:

$$\lambda_t = \Sigma^{-1/2}(\lambda_0 + \lambda_1 X_t), \tag{22}$$

The assumption of no-arbitrage implies that there exists a pricing kernel M_t such that:

$$P_t^{(n)} = E_t\left(M_{t+1} P_{t+1}^{(n-1)}\right), \tag{23}$$

for every $n > 0$ and $t \geq 0$. Where $P_t^{(n)} = exp\left[-nR_t^{(n)}\right]$ is the price of a zero coupon bond with maturity n.

ATS Models

assume that the pricing kernel is exponentially affine, i.e.,

$$m_{t+1} = -R_t^1 - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-1/2}v_{t+1}, \tag{24}$$

where $R_t^1 = log(P_t^{(1)}) = p_t^{(1)}$ is the continuously compounded risk-free rate, and $m_t = \log M_t$. The excess log returns are given by:

$$rx_{t+1}^{(n-1)} = p_{t+1}^{(n-1)} - p_t^{(n)} - R_t^1, , \tag{25}$$

where $p_t^{(n)} = \log P_t^{(n)}$. From (24) and (23) it follows that

$$1 = E_t\left[\exp\left(rx_{t+1}^{(n-1)} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-1/2}v_{t+1}\right)\right]. \tag{26}$$

ATS Models

Assuming that $\left(rx_{t+1}^{(n-1)}, v_{t+1}'\right)$ are jointly normally distributed, we obtain:

$$\begin{aligned} 0 &= \log\left(E_t\left[exp\left(rx_{t+1}^{(n-1)} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-1/2}v_{t+1}\right)\right]\right) \\ &= E_t\left(rx_{t+1}^{(n-1)}\right) + \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right) - \frac{1}{2}\lambda_t'\lambda_t + \frac{1}{2}\lambda_t'\lambda_t \\ &- Cov_t\left[rx_{t+1}^{(n-1)}, v_{t+1}'\Sigma^{-1/2}\lambda_t\right]. \end{aligned} \tag{27}$$

We can then write

$$E_t\left(rx_{t+1}^{(n-1)}\right) = Cov_t\left[rx_{t+1}^{(n-1)}, v_{t+1}'\Sigma^{-1/2}\lambda_t\right] - \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right), \tag{28}$$

and denote $\beta_t^{(n-1)'}$ as:

$$\beta_t^{(n-1)'} = \text{Cov}_t\left[rx_{t+1}^{(n-1)}, v_{t+1}'\right]\Sigma^{-1}, \tag{29}$$

ATS Models

By substituting from (45) into (28), using (22), we have:

$$E_t\left(rx_{t+1}^{(n-1)}\right)=\beta_t^{(n-1)'}(\lambda_0+\lambda_1 X_t)-\frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right), \tag{30}$$

The unexpected excess return can be decomposed in a component that is correlated with v_{t+1} and another component which is orthogonal:

$$rx_{t+1}^{(n-1)}-E_t\left(rx_{t+1}^{(n-1)}\right)=\beta_t^{(n-1)'}v_{t+1}+e_{t+1}^{(n-1)}, \tag{31}$$

Under the assumption that the return pricing error are i.i.d. with variance σ^2 and that β_t is constant, the generating process for log excess returns becomes:

$$p_{t+1}^{(n-1)}-p_t^{(n)}-R_t^1=\underbrace{\beta^{(n-1)'}(\lambda_0+\lambda_1 X_t)}_{\text{Expected Return}}-\underbrace{\frac{1}{2}\left(\beta^{(n-1)'}\Sigma\beta^{(n-1)}+\sigma^2\right)}_{\text{Convexity Correction}}+\underbrace{\beta^{(n-1)'}v_{t+1}}_{\text{Return Innovation}}+e_{t+1}^{(n-1)}, \tag{32}$$

ATS Models

Stacking (32) across N maturities and T time-periods we have the following matrix-form representation:

$$rx = \beta'(\lambda_0 \iota_T' + \lambda_1 X_-) - \frac{1}{2}(B^* \text{vec}(\Sigma) + \sigma^2 \iota_N)\iota_T' + \beta' V + E, \qquad (33)$$

rx is a NxT matrix of excess returns.
$\beta = [\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(N)}]$ is a KxN matrix of factor loadings.
ι_T and ι_N are Tx1 and Nx1 vectors of ones.
$X_- = [X_0, X_1, \ldots, X_{T-1}]$ is a KxT matrix of lagged pricing factors.
$B^* = [\text{vec}(\beta^{(1)}\beta^{(1)\prime}) \ldots \text{vec}(\beta^{(N)}\beta^{(N)\prime})]$ is an $N?K^2$ matrix.

V is a KxT matrix, and E is an NxT matrix

Estimation of the lambdas in ATS Models

1. Construct the pricing factors (X) from principal component analysis (PCA) of the cyclical components of yields derived in the first step. Estimate VAR (20) using OLS, decomposing the pricing factors into predictable components and factor innovations $\hat{V}$.

2. Regress excess returns on a constant, lagged pricing factors and contemporaneous pricing factor innovations according to

$$rX = a\iota_T' + \beta'\hat{V} + cX_{-} + E, \tag{34}$$

B^* can be computed from β and σ^2 from the residuals E.

3. From Eq (33), we can see $a = \beta'\lambda_0 - \frac{1}{2}(B^*\text{vec}(\Sigma) + \sigma^2\iota_N)$ and $c = \beta'\lambda_1$. From these,the estimators are obtained

$$\hat{\lambda}_0 = \left(\hat{\beta}\hat{\beta}'\right)^{-1}\left(\hat{a} + \frac{1}{2}\left((\hat{B}^*\text{vec}(\hat{\Sigma}) + \hat{\sigma}^2\iota_N\right)\right), \tag{35}$$

$$\hat{\lambda}_1 = \left(\hat{\beta}\hat{\beta}'\right)^{-1}\hat{\beta}\hat{c}. \tag{36}$$

ATSM: from returns to yields

Given the general expression for excess returns:

$$p_{t+1}^{(n-1)} - p_t^{(n)} - R_t^1 = \underbrace{\beta^{(n-1)'}(\lambda_0 + \lambda_1 X_t)}_{\text{Expected Return}} - \underbrace{\frac{1}{2}\left(\beta^{(n-1)'}\Sigma\beta^{(n-1)} + \sigma^2\right)}_{\text{Convexity Correction}} + \underbrace{\beta^{(n-1)'}v_{t+1}}_{\text{Return Innovation}} + e_{t+1}^{(n-1)},$$

The price of bonds at all maturities can expressed as:

$$p_t^n = A_n + B_n' X_t + u_t^n, \quad (37)$$

where, as a consequence of no-arbitrage, recursive restrictions apply to A_n and B_n. The risk-free rate R_t^1 is expressed as a linear function of the underlying factors -

$$R_t^1 = \delta_0 + \delta_1' X_t + \epsilon_t, \quad (38)$$

so $A_1 = -\delta_0$ and $B_1 = -\delta_1$ We then have:

$$A_n = A_{n-1} + B_{n-1}'(-\lambda_0) + \frac{1}{2}(\beta^{(n-1)'}\Sigma\beta^{(n-1)} + \sigma^2) - \delta_0, \quad (39)$$

$$B_n' = B_{n-1}'(\Phi - \lambda_1) - \delta_1', \quad (40)$$

$$\beta^{(n)} = B_n', \quad (41)$$

ATSM: understanding the restrictions

For the one-period bond R_t^1 we have

$$R_t^1 = \delta_0 + \delta_1' X_t + e_t^{(1)}, \tag{42}$$

Where parameters $\hat{\delta}_0$ and $\hat{\delta}_1$ can be estimated by projecting R_t^1 on the factors X_t.
For the two-period bond, we have

$$\begin{aligned} p_{t+1}^{(1)} - p_t^{(2)} - R_t^1 &= \beta^{(1)'}(\lambda_0 + \lambda_1 X_t) - \frac{1}{2}(\beta^{(1)'}\Sigma\beta^{(1)} + \sigma^2) + \beta^{(1)'}v_{t+1} + e_{t+1}^{(1)}, \\ p_{t+1}^{(1)} &= -\delta_0 - \delta_1'(\mu + \Phi X_t + v_{t+1}) \end{aligned}$$

By using the second equation in the first one, we have:

$$\begin{aligned} p_t^{(2)} &= -\delta_0 - \delta_1' X_t - e_t^{(1)} - \delta_0 - \delta_1'(\mu + \Phi X_t + v_{t+1}) \\ &- \beta^{(1)'}(\lambda_0 + \lambda_1 X_t) + \frac{1}{2}\left(\beta^{(1)'}\Sigma\beta^{(1)} - \sigma^2\right) - \beta^{(1)'}v_{t+1} - e_{t+1}^{(1)} \end{aligned}$$

which illustrates that

$$A_2 = A_1 + B_1'(-\lambda_0) + \frac{1}{2}(\beta^{(n-1)'}\Sigma\beta^{(n-1)} + \sigma^2) - \delta_0, \tag{43}$$

$$B_2' = B_1'(\Phi - \lambda_1) - \delta_1', \tag{44}$$

$$\beta^{(1)} = B_1', \beta^{(2)} = B_2' \tag{45}$$

ATSM: Measuring the Term Premia

Model simulation in two scenarios, a baseline with all parameters set are their fitted values and an alternative one in which the market price of risk is set to zero, i.e. $\lambda_0 = \lambda_1 = 0$, allows to compute term premia as the differences between the model implied yields and the risk neutral yields.

The problem with standard ATSM models

- The data tell us that yields are drifting, but excess returns are cyclical.
- In standard ATSM models a few factors, modelled by a Vector Autoregressive Process, are the common drivers of the dynamics of both the expected path of future one-period rates and the term premia. As yields drift, factors exhibit a high level of persistence. When these factors are modeled using a VAR (Vector Autoregression), the forecast of future one-period rates gradually converges to the mean of the sample used for estimation.
- Standard ATSM models tend to generate term premia that are a-cyclical and parallel to the secular trend in yields.
- These features of the term-premia are a by-product of the specification strategy for the dynamics of yields and excess-returns that adopts a common autoregressive factor structure for them.

A potential solution

Specify a model that takes into account trends and cycles:

$$r_t^{(1)} = r_t^{*,(1)} + u_t^{(1)} \tag{46}$$

$$r_t^{*,(1)} = \gamma_1 MY_t + \gamma_2 \Delta y_t^{pot} + \gamma_3 \pi_t^{LR} \tag{47}$$

$$r_t^{(n)} = r_t^{*,(1)} + u_t^{(n)} \tag{48}$$

$$p_t^{(n)} = p_t^{*,(n)} + A_n + B_n' X_t + \varepsilon_t^{(n)}, \tag{49}$$

$$X_t = \mu + \Phi X_{t-1} + v_{t+1} \tag{50}$$

in which the factors X_t are extracted from the cyclical components of yields, u_t^n, after the completion of the first stage of estimation. The standard recursive restrictions apply to A_n and B_n