Debt Instruments

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1 Introduction

Government debt instruments play a pivotal role in fiscal macroeconomics, serving as tools for financing public expenditure and managing liquidity in the economy. This chapter introduces the variety of bonds that governments can issue, each with unique features and pricing mechanisms.

Specifically, we distinguish between the following types of bonds:

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- **Nominal Bonds:** The value and payments of these bonds are fixed in nominal terms, without adjustment for inflation.

    - **Zero-Coupon Bonds:** These bonds do not pay periodic interest; instead, they are issued at a discount to their face value and redeemed at par upon maturity.

    - **Coupon Bonds:** These bonds pay periodic interest, known as coupons, until maturity, when the principal is repaid.

- **Inflation-Indexed Bonds:** These bonds adjust their principal and/or interest payments based on inflation, providing a hedge against price level changes.

The primary focus of this chapter is to illustrate how these different types of bonds are priced in financial markets. We will explore the theoretical frameworks and practical methods used to determine their value and compute yields. Additionally, we will examine how yields across different maturities can be identified and analyzed, providing insights into the term structure of interest rates.

2 Bond Returns: Yields-to-Maturity and Holding Period Returns

We turn now to bonds. We distinguish between two types of bonds: those paying a coupon each given period and those that do not pay a coupon but just reimburse the entire capital upon maturity (zero-coupon bonds). Cash-flows from different type of bonds:

	$t+1$	$t+2$	$t+3$	$\ldots$	T
General	CF_{t+1}	CF_{t+2}	CF_{t+3}	$\ldots$	CF_T
Coupon bond	C	C	C	$\ldots$	$1+C$
1-period zero	1	0	0	$\ldots$	0
2-period zero	0	1	0	$\ldots$	0
$\vdots$				$\ldots$	
$(T-t)$-period zero	0	0	0	$\ldots$	1

2.1 Zero-Coupon Bonds

Define the relationship between price and yield to maturity of a zero-coupon bond as follows:

$$P_{t,T} = \frac{1}{(1 + Y_{t,T})^{T-t}}, \tag{1}$$

where $P_{t,T}$ is the price at time t of a bond maturing at time T, and $Y_{t,T}$ is yield to maturity. Taking logs of the left and the right-hand sides of the expression for $P_{t,T}$, and defining the continuously compounded *yield, $y_{t,T}$,* as $\log(1 + Y_{t,T})$, we have the following relationship:

$$p_{t,T} = -(T - t)\, y_{t,T}, \tag{2}$$

which clearly illustrates that the elasticity of the yield to maturity to the price of a zero-coupon bond is the maturity of the security. Therefore, the duration of the bond equals maturity as no coupons are paid.

Price and YTM of zero-coupon bonds							
Mat	1	2	3	5	7	10	20
$P_{t,T}$	0.9524	0.9070	0.8638	0.7835	0.7106	0.6139	0.3769
$Y_{t,T}$	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
$p_{t,T}$	−0.0487	−0.0976	−0.1464	−0.2439	−0.3416	−0.4879	−0.9757
$y_{t,T}$	0.0488	0.0488	0.0488	0.0488	0.0488	0.0488	0.0488

The one-period uncertain holding-period return on a bond maturing at time T, $r^T_{t,t+1}$, is then defined as follows:

$$\begin{aligned}
r^T_{t,t+1} &\equiv p_{t+1,T} - p_{t,T} = -(T - t - 1)\, y_{t+1,T} + (T - t)\, y_{t,T} \tag{3}\\
&= y_{t,T} - (T - t - 1)\,(y_{t+1,T} - y_{t,T}),\\
&= (T - t)\, y_{t,T} - (T - t - 1)\, y_{t+1,T}, \tag{4}
\end{aligned}$$

which means that yields and returns differ by a scaled measure of the change between the yield at time $t + 1$, $y_{t+1,T}$, and the yield at time t, $y_{t,T}$. Think of a situation in which the one-year YTM stands at 4.1 per cent while the 30-year YTM stands at 7 per cent. If the YTM of the thirty year bonds goes up to 7.1 per cent in the following period, then the period returns from the two bonds is the same.

2.1.1 A model of the term structure

Apply the no arbitrage condition to a one-period bond (the safe asset) and a T-period bond:

$$E_t \left(r_{t,t+1}^T - r_{t,t+1}^1 \right) = E_t \left(r_{t,t+1}^T - y_{t,t+1} \right) = \phi_{t,t+1}^T$$
$$E_t \left(r_{t,t+1}^T \right) = y_{t,t+1} + \phi_{t,t+1}^T$$

Solving forward the difference equation $p_{t,T} = p_{t+1,T} - r_{t,t+1}^T$, we have :

$$y_{t,T} = \frac{1}{(T-t)} \sum_{i=0}^{n-1} E_t \left(r_{t+i,t+i+1}^T \right)$$
$$= \frac{1}{(T-t)} \sum_{i=0}^{n-1} E_t \left(y_{t+i,t+i+1} + \phi_{t+i,t+i+1}^T \right)$$

The model clearly shows that Bond yields are driven by two unobservable factors

- Expectations of future monetary policy (risk free) rates over the residual life of the bonds

- Compensation for risk (risk premia)

2.2 Coupon Bonds

The relationship between price and yield to maturity of a constant coupon (C) bond is given by:
$$P_{t,T}^c = \frac{C}{\left(1 + Y_{t,T}^c \right)} + \frac{C}{\left(1 + Y_{t,T}^c \right)^2} + \dots + \frac{1+C}{\left(1 + Y_{t,T} \right)^{T-t}}.$$

When the bond is selling at par, the yield to maturity is equal to the coupon rate. To measure the length of time that a bondholder has invested money for we need to introduce the concept of duration:

$$D_{t,T}^c = \frac{\frac{C}{\left(1 + Y_{t,T}^c \right)} + 2\frac{C}{\left(1 + Y_{t,T}^c \right)^2} + \dots + (T-t)\frac{1+C}{\left(1 + Y_{t,T} \right)^{T-t}}}{P_{t,T}^c}$$
$$= \frac{C \sum_{i=1}^{T-t} \frac{i}{\left(1 + Y_{t,T}^c \right)^i} + \frac{(T-t)}{\left(1 + Y_{t,T} \right)^{T-t}}}{P_{t,T}^c}.$$

It can be shown [1] that in case of a coupon bond the period holding return can be approximated by extending the formula for zero-coupon bonds(in which case duration is

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[1]see Appendix

equal to maturity) as follows:

$$r_{t+1}^c = D_{t,T}^c y_{t,T}^c - \left(D_{t,T}^c - 1\right) y_{t+1,T}^c,$$

The formula can be made operational, given the information available in yields to maturity only, by approximating duration as follows:

$$D_{t,T}^c = \frac{1 - \left(1 + Y_{t,T}^c\right)^{-(T-t)}}{1 - \left(1 + Y_{t,T}^c\right)^{-1}}$$

In the case of long-dated coupon bonds the model for the term structure becomes:

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\begin{aligned}
y_{t,T} &= y_{t,T}^* + E[\Phi_T \mid I_t] = \frac{1 - \gamma}{1 - \gamma^{T-t}} \sum_{j=0}^{T-t-1} \gamma^j E[r_{t+j} \mid I_t] + E[\Phi_T \mid I_t] \\
\gamma &= 1/(1 + \bar{y})
\end{aligned}
$$

3 Nominal and Real Bonds

The most common form of government bonds are nominal bonds that pay fixed coupons and principal.

Inflation-indexed bonds, which in the U.S. are known as Treasury Inflation Protected Securities (TIPS), are bonds whose coupons and principal adjust automatically with the evolution of a consumer price index.

They aim to pay investors a fixed inflation-adjusted coupon and principal, in other words they are real bonds and their yields are typically considered the best proxy for the term structure of real interest rates in the economy. Investors holding either inflation-indexed or nominal government bonds are exposed to the risk of changing real interest rates.

In addition to real interest rate risk, nominal government bonds expose investors to inflation risk while real bonds do not. When future inflation is uncertain, the coupons and principal of nominal bonds can suffer from the eroding effects of inflationary surprises.

Finally, both the nominal and real bond are theoretically affected by a premium for liquidity risk. Liquidity risk, is, the risk of having to sell (or buy) a bond in a thin market and, thus, at an unfair price and with higher transaction costs. At time t the yields to maturity of nominal and real bonds maturing at T can be written as follows:

$$Y_{t,T}^n = rr_{t,T} + E_t\pi_{t,T} + RP_t^{rr} + RP_t^{\pi}$$
$$Y_{t,T}^r = rr_{t,T} + RP_t^{rr} + RP_t^{liq}$$

the difference in the yield to maturity, usually referred to as the breakeven inflation rate $B_{t,T}$, can be written as:

$$B_{t,T} = E_t\pi_{t,T} + RP_t^{\pi} - RP_t^{liq}$$

3.1 The Case of 10-year BTP and BTP Italia

BTP is a constant coupon bond with a standard relationship between price and yield to maturity. In the case of a 10-year bond we have

$$P_{t,T}^{BTP} = \frac{C}{\left(1 + Y_{t,t+10}^{BTP}\right)} + \frac{C}{\left(1 + Y_{t,t+10}^{BTP}\right)^2} + ... + \frac{1 + C}{\left(1 + Y_{t,t+10}^{BTP}\right)^{10}}.$$

BTP Italia is an indexed bond in which the coupon paid is made of two components: the coupon, constant in each period, and the inflation adjustment for the coupon and the value of the principal. So the stream of payments for a BTP Italia goes as follows:

$$C_{t+1} = C(1 + \pi_{t+1}) + (1 + \pi_{t+1}) - 1,$$

$$C_{t+2} = C(1 + \pi_{t+2}) + (1 + \pi_{t+1})(1 + \pi_{t+2}) - (1 + \pi_{t+1}),$$

$$... = ...$$

$$C_{t+10} = 1 + C(1 + \pi_{t+10}) + \prod_{i=1}^{10}(1 + \pi_{t+i}) - \prod_{i=1}^{9}(1 + \pi_{t+i}).$$

Price and Yields can then be computed as the value of a constant stream of payments discounted with a yield in real terms $Y_{t,t+10}^{BTPi}$:

$$P_{t,T}^{BTPi} = \frac{C}{\left(1 + Y_{t,t+10}^{BTPi}\right)} + \frac{C}{\left(1 + Y_{t,t+10}^{BTPi}\right)^2} + ... + \frac{1 + C}{\left(1 + Y_{t,t+10}^{BTPi}\right)^{10}}.$$

So given $Y_{t,t+10}^{BTPi}$, the breakeven inflation rate can be computed as $Y_{t,t+10}^{BTP} - Y_{t,t+10}^{BTPi}$.

4   Appendix: YTM and Returns of Coupon Bonds

The relationship between price and yield to maturity of a constant coupon (C) bond is given by:

$$P_{t,T}^c = \frac{C}{\left(1 + Y_{t,T}^c\right)} + \frac{C}{\left(1 + Y_{t,T}^c\right)^2} + ... + \frac{1 + C}{\left(1 + Y_{t,T}\right)^{T-t}}.$$

When the bond is selling at par, the yield to maturity is equal to the coupon rate. To measure the length of time that a bondholder has invested money for we need to introduce the concept of duration:

$$D_{t,T}^c = \frac{\frac{C}{\left(1+Y_{t,T}^c\right)} + 2\frac{C}{\left(1+Y_{t,T}^c\right)^2} + \cdots + (T-t)\frac{1+C}{\left(1+Y_{t,T}\right)^{T-t}}}{P_{t,T}^c}$$

$$= \frac{C\sum_{i=1}^{T-t}\frac{i}{\left(1+Y_{t,T}^c\right)^i} + \frac{(T-t)}{\left(1+Y_{t,T}\right)^{T-t}}}{P_{t,T}^c}.$$

Note that when a bond is floating at par we have

$$D_{t,T}^c = Y_{t,T}^c \sum_{i=1}^{T-t} \frac{i}{\left(1 + Y_{t,T}^c\right)^i} + \frac{(T-t)}{\left(1 + Y_{t,T}\right)^{T-t}}$$

$$= Y_{t,T}^c \frac{\left((T-t)\frac{1}{1+Y_{t,T}^c} - (T-t) - 1\right)\frac{1}{\left(1+Y_{t,T}^c\right)^{T-t+1}} + \frac{1}{1+Y_{t,T}^c}}{\left(1 - \frac{1}{1+Y_{t,T}^c}\right)^2} + \frac{(T-t)}{\left(1 + Y_{t,T}\right)^{T-t}}$$

$$= \frac{1 - \left(1 + Y_{t,T}^c\right)^{-(T-t)}}{1 - \left(1 + Y_{t,T}^c\right)^{-1}}.$$

because when $|x| < 1$,

$$\sum_{k=0}^{n} kx^k = \frac{(nx - n - 1)x^{n+1} + x}{(1 - x)^2}.$$

Duration can be used to find approximate linear relationships between log-coupon yields and holding period returns. Extending the formula of zero-coupon bonds (where duration is equal to maturity) to coupon bonds, we have

$$r_{t+1}^c = D_{t,T}^c y_{t,T}^c - \left(D_{t,T}^c - 1\right) y_{t+1,T}^c,$$

Shiller(1979) proposes a *linearization* which takes duration as constant and considers the

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following approximation in the neighbourhood $y_{t,T} = y_{t+1,T} = \bar{y} = C$:

$$H_{t,T} \simeq D_T y_{t,T} - (D_T - 1)\, y_{t+1,T}$$

$$D_T = \frac{1 - \left(1 + \bar{Y}^c_{t,T}\right)^{-(T-t)}}{1 - \left(1 + \bar{Y}^c_{t,T}\right)^{-1}}$$

$$D_T = \frac{1 - \gamma^{T-t-1}}{1 - \gamma} = \frac{1}{1 - \gamma_T}$$

$$\gamma_T = \left\{ 1 + \bar{Y}^c_{t,T} \left[1 - 1/(1 + \bar{Y}^c_{t,T})^{T-t-1} \right]^{-1} \right\}^{-1}$$

$$\lim_{T \longrightarrow \infty} \gamma_T = \gamma = 1/(1 + \bar{y})$$

Solving this expression forward, we generate the equivalent of the DDG model in the bond market:

$$y_{t,T} = \sum_{j=0}^{T-t-1} \gamma^j (1 - \gamma)\, H_{t+j,T} + \gamma^{T-t} y_{T-1,T}.$$

In this case, by equating one-period risk-adjusted returns, we have

$$E\left[\frac{y_{t,T} - \gamma y_{t+1,T}}{1 - \gamma} \mid I_t \right] = r_t + \phi_{t,T} \tag{5}$$

From the above expression, by recursive substitution, under the terminal condition that at maturity the price equals the principal, we obtain:

$$y_{t,T} = y^*_{t,T} + E[\Phi_T \mid I_t] = \frac{1 - \gamma}{1 - \gamma^{T-t}} \sum_{j=0}^{T-t-1} \gamma^j E[r_{t+j} \mid I_t] + E[\Phi_T \mid I_t] \tag{6}$$

where the constant $\Phi_{t,T}$ is the term premium over the whole life of the bond:

$$\Phi_{t,T} = \frac{1 - \gamma}{1 - \gamma^{T-t}} \sum_{j=0}^{T-t-1} \gamma^j \phi_{t+j,T}$$

For long bonds, when $T - t$ is very large, we have:

$$y_{t,T} = y^*_{t,T} + E[\Phi_T \mid I_t] = (1 - \gamma) \sum_{j=0}^{T-t-1} \gamma^j E[r_{t+j} \mid I_t] + E[\Phi_T \mid I_t]$$

Subtracting the risk-free rate from both sides of this equation, we have

$$S_{t,T} = y_{t,T} - r_t = \sum_{j=1}^{T-1} \gamma^j E[\Delta r_{t+j} \mid I_t] + E[\Phi_T \mid I_t]$$

$$= S_{t,T}^* + E[\Phi_T \mid I_t]$$