Life-Cycle Asset Allocation with Ambiguity Aversion and Learning

Kim Peijnenburg*

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Abstract

I show that ambiguity and learning about the equity premium can simultaneously explain the low fraction of financial wealth allocated to stocks over the life cycle and the stock market participation puzzle. Individuals are ambiguous about the size of the equity premium and are averse with respect to this ambiguity, resulting in lower stock allocations over the life cycle consistent with the data. As agents get older, they learn about the equity premium and increase their allocation to stocks. Furthermore, I find that ambiguity leads to higher saving rates. Similar results cannot be obtained by assuming higher risk aversion.

Keywords: Life-cycle portfolio choice, behavioral finance, ambiguity aversion, learning, savings

JEL classification: D14, D8, D91, G11

*Bocconi University, IGIER, and Netspar, Milan, 20136, Italy. Phone: 39 025836 2723. Email: kim.peijnenburg@unibocconi.it.

Key inputs of a life-cycle model, such as the equity risk premium, variance of stock returns, and labor income risk, are generally assumed to be known by the agent. Optimal portfolio allocations, consumption, and savings are calculated as if the agent takes these parameters as given and the resulting optimal policies are subsequently compared to the empirically observed life-cycle patterns. However, the predictions of most life-cycle models do not match well with some of the empirical findings. For instance, the overall low stock market participation rates are ill understood. Furthermore, the fraction of financial wealth allocated to stocks, conditional on participation in the stock market, appears difficult to align with the predictions from life-cycle models. I propose a standard life-cycle model, taking into account that agents are ambiguous about the equity risk premium and averse to this ambiguity (in contrast to the ambiguity-neutral approach). During their lifetime, individuals learn about the equity premium. With this parsimonious adjustment to the standard framework I can explain both the life-cycle pattern of participation in the stock market and the conditional fraction of financial wealth allocated to equity. Furthermore, I show that saving rates are substantially higher for ambiguity-averse people.

I assume that agents not only face risk, but also are uncertain about the true parameters describing this risk (Knight (1921)).[1] The ambiguity-neutral approach is a common way to deal with parameter uncertainty, where the decision maker treats the unknown parameters as random variables and combines his prior belief about the parameter with observed signals, which forms the predictive distribution. The decision maker then evaluates the expected utility with respect to this predictive distribution. In this case, the agent is *ambiguous* but is *not ambiguity averse*. However, there is substantial evidence that agents are not neutral with respect to this parameter uncertainty

[1]The difference between risk and uncertainty is that when agents face risk, they are able to attach probabilities to random events, while they do not know the probabilities when facing uncertainty. In the context of this paper, agents face risk because the returns on stocks are stochastic, but the agents are also uncertain because they do not know the expected stock returns.

(see, e.g., the classical paper by Ellsberg (1961), who argues that people are ambiguity averse by using an urn experiment). Therefore, I assume that agents are not ambiguity neutral, but ambiguity averse. Ambiguity about the equity risk premium is included, but no assumptions are made regarding the origin of this ambiguity. It can arise from, for instance, a lack of statistical evidence, a lack of theoretical evidence, a lack of investor sophistication, and so on. With a focus on, for example, statistical ambiguity, even when every agent possesses all the historical stock return data over the past 100 years and uses these to estimate the equity premium, the confidence interval will still be sizable: $[4\% - 2 * 20\%/\sqrt{100} : 4\% + 2 * 20\%/\sqrt{100}] = [+0\% : +8\%]$.

A short note on terminology is in order. As Guidolin and Rinaldi (2013) point out, in the literature ambiguity and uncertainty are not always clearly defined. Throughout the paper I use the terms *uncertainty* and *ambiguity* interchangeably and I define *ambiguity/uncertainty* as a random event where the probabilities are not known (as opposed to a coin toss), but agents have a distribution of priors over the uncertain parameter.

I use maxmin preferences to model ambiguity aversion and individuals learn about the equity premium. Gilboa and Schmeidler (1989) propose that agents have maxmin preferences in a multiple-priors framework, which entails that agents evaluate policies by maximizing utility according to the worst case belief. This atemporal framework is generalized by Epstein and Schneider (2003) to a dynamic setup. I do not assume that agents learn about the equity risk premium in a rational manner; agents weigh realized stock returns during life with a prior belief about the equity risk premium, putting no weight on returns before the year of birth. Malmendier and Nagel (2011) find that agents' "experienced return" has a larger influence on beliefs about the equity risk premium than stock return realizations before the year of birth. I assume agents learn independently of stock market participation and I employ Bayes' rule as the updating rule for beliefs about the equity risk premium. Furthermore, agents have constant relative risk aversion preferences and their labor income is risky.

The contributions of this paper are threefold. First, I find that ambiguity with respect to the equity risk premium can have a substantial effect on the optimal stock allocations. Stock market participation is substantially lower, as is the conditional allocation to equity. Both effects decline with age due to learning about the equity premium, since learning results in older agents being less ambiguous about the equity premium compared to younger agents.

Second, ambiguity about the equity premium influences the wealth levels of individuals, as well as their savings rates. The optimal wealth profile is lower for ambiguity-averse individuals, which reflects the lower investment returns due to lower optimal stock allocations. However, savings rates are higher when ambiguity about the equity premium is taken into account. Keeping up wealth levels by saving to compensate for lower investment returns is quantitatively more important than lower savings rates induced by the relatively less attractive investment opportunities perceived by ambiguity-averse investors. This is the first paper, to my knowledge, to explore the impact of ambiguity on life-cycle savings choices and wealth levels.

Third, when comparing the optimal fraction allocated to stocks to the empirical levels, I find a close match at all ages. On average, over the life cycle the model predicts 50% allocated to stocks, while the empirical average is about 45%. A comparable good fit is found when examining participation in the stock market. In addition, the model with ambiguity generates a large degree of heterogeneity in fractions allocated to stocks, in line with the data. Hence, by extending the frequently used life-cycle model with ambiguity aversion and learning, I can simultaneously explain low stock market participation and the low conditional fraction of financial wealth allocated to stocks over the life cycle. Similar results cannot be obtained by assuming no ambiguity aversion and high risk aversion instead, because higher risk aversion actually increases participation levels due to higher precautionary savings.

In their seminal works, Merton (1969) and Samuelson (1969) find that agents should hold a constant fraction in risky assets over the life cycle in the absence of labor income and complete markets. More recent work by Benzoni et al. (2007), Cocco et al. (2005), Heaton and Lucas (2000), Polkovnichenko (2007), and Viceira (2001) examines the effect of (risky) labor income on the optimal portfolio choice. If human capital is riskless, young agents have a substantial investment in this “bond-like” asset and, as a result, invest a large fraction of their liquid wealth in risky assets. This is in contrast to the empirically observed low allocation to stocks, especially early in the life cycle. In contrast to other papers, I do not need to include several additional features in the model to explain low stock participation, such as participation costs (Vissing-Jorgenson (2002)), Epstein-Zin preferences, bequests, housing, cointegration between labor income and dividends, and minimum investment requirements. The intuitive modification with ambiguity aversion alone can explain the empirical evidence very closely. Similar to this paper, Gomes and Michaelides (2005) try to match the empirically observed allocation to stocks by assuming a bequest motive, fixed entry costs of 2.5% of income, preference heterogeneity, and Epstein-Zin preferences. The participation levels match closely, except after retirement; however, the predictions about the conditional allocation to equity differ about 40% from the empirically observed levels at younger ages. In contrast to abovementioned papers, I can match both low participation levels and the allocation to equity conditional on participation in the stock market very well, especially at young ages. Benzoni et al. (2007) assume cointegration between stock and labor markets and find a hump-shaped allocation to equity; however, the absolute differences from empirical levels are substantially larger than those in this paper.

One other paper includes ambiguity and learning about the parameters in a life-cycle framework and address similar questions as in this paper.[2] Campanale (2011) assumes agents have maxmin preferences and are uncertain about the probability of a high stock return. The return on stocks can take on two values, high or low. Learning always occurs when agents invest in the stock market, but if they do not participate learning occurs with a probability below 100%. In contrast to my paper, this simplified stock return process prevents bringing this model to the data. Furthermore, I examine a broader set of choices, namely, savings and wealth levels, and use a more parsimonious model.[3]

1 The model

I extend the standard life-cycle framework by including ambiguity aversion and learning. I use the most common model for ambiguity-averse preferences, namely, maxmin preferences.[4] Agents update their beliefs about the equity premium using Bayes' rule.

1.1 Ambiguity about the equity premium

I assume that agents are uncertain about the equity premium, but certain about the volatility of stock returns. Agents update beliefs according to realized stock market returns, which can be either actively or passively observed. The updating of beliefs follows from Bayes' rule, which is described in Section 1.6. The initial prior belief about the equity premium is assumed to be

[2] Another related strand of literature explores the implications of parameter uncertainty on portfolio choice, from a non-life cycle perspective, for instance Garlappi et al. (2007), Kan and Zhou (2007), and Tu and Zhou (2010).

[3] Campanale (2011) includes a fixed annual stock market participation cost, a bequest motive, a minimum stock investment of 4% of average annual earnings in the economy (about $1,400), and a more complicated learning process.

[4] The robustness section uses smooth recursive preferences, an alternative way to model ambiguity aversion.

normally distributed with mean λ^B and standard deviation σ^B. I further restrict the domain of equity premiums that the agents think are possible at time t to belong to the set $\Lambda_t = [\lambda_t^B - 2\sigma_t^B, \lambda_t^B + 2\sigma_t^B]$, where λ_t^B and σ_t^B are the mean and the standard deviation of the agents' belief at time t, respectively.[5] The latter assumption implies that every agent expects the true equity premium to lie in a 95% confidence interval around the mean λ_t^B. Garlappi et al. (2007) make a similar assumption when incorporating ambiguity by stating that the expected return of an asset lies within a specified confidence interval of its estimated value.

In what follows, I do not take a stand on the specific source of ambiguity about the equity risk premium. Uncertainty, for example, could stem from lack of statistical evidence, since stock market returns are very volatile and it is, thus, difficult to estimate the expected return.[6] Furthermore, ambiguity about the equity premium could also result from inconsistent theoretical evidence or lack of sophistication of investors.

1.2 Preferences

I consider a life-cycle investor of age $t = 1, ..., T$, where t is the adult age, T is the maximum age possible, and I denote by $K < T$ the retirement age. Individuals maximize utility over consumption and preferences are represented by a time-separable utility function over consumption. The agent's decision variables at time t are consumption, C_t, and the fraction of wealth invested in stocks, w_t. I assume investors' preferences are described by maxmin expected utility, which implies that agents maximize expected utility according to the belief that generates the lowest utility. Gilboa and Schmeidler (1989) axiomatize this behavior in a static setting and Epstein and

[5]The set of priors satisfies the rectangularity condition (Epstein and Schneider (2003)).

[6]Even when using all the stock market returns from the past 100 years to estimate the equity premium, we would still end up with a large 95% confidence interval: $[4\% - 2 * 20\%/\sqrt{100}, 4\% + 2 * 20\%/\sqrt{100}] = [+0\%, +8\%]$.

Schneider (2003) in a dynamic framework.

As described, the agent is uncertain about the equity premium and at every time t has a set of beliefs Λ_t. Moreover, at the beginning of period t the agent also observes the realization of wealth W_t (which depends on the realized past return) and income Y_t. The agent then selects optimal consumption and fraction of wealth invested in stocks so as to solve the following problem:

$$V_t(W_t, Y_t, \lambda_t^B) = \max_{w_t, C_t} \min_{\lambda \in \Lambda_t} \left\{ u(C_t) + \beta p_{t+1} \mathbb{E}_t^{\lambda} \left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B)) \right] \right\}, \text{ with} \quad (1)$$

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \quad (2)$$

subject to all the constraints described in section 1.3, where β is the time preference discount factor. In addition, the probability of surviving to age $t+1$, conditional on having lived to period t, is indicated by p_{t+1}. The term $\mathbb{E}_t^{\lambda}$ denotes the conditional expectation computed using λ as the true equity premium. I assume a CRRA utility function, u, where γ is the risk aversion coefficient. It turns out that the minimum in the problem above is achieved when the agent uses $\lambda_t^B - 2\sigma_t^B$ as the equity premium. Note that the assumption that the agent's beliefs are limited to a bounded interval of possible equity premiums is not only intuitive, but also necessary since beliefs are normally distributed and, hence, the worst case belief would be unboundedly negative.

1.3 Constraints

The individual faces a number of constraints on consumption and investment decisions. First, I assume that the agent faces borrowing and short-sales constraints

$$w_t \geq 0 \text{ and } w_t \leq 1. \quad (3)$$

Second, I impose the following liquidity constraints on the investor:

$$C_t \leq W_t + Y_t, \tag{4}$$

which implies that the individual cannot borrow against future income to increase consumption today. The intertemporal budget constraint equals

$$W_{t+1} = (W_t - C_t + Y_t)R^P_{t+1}, \tag{5}$$

where R^P_{t+1} denotes the portfolio return

$$R^P_{t+1} = 1 + R^f + (R_{t+1} - R^f)w_t, \tag{6}$$

and R_{t+1} and R^f denote the stock return between time t and $t+1$ and the risk-free rate, respectively.

1.4 Financial market

I consider a financial market with a constant interest rate R^f and a stock with independent and identically distributed returns R_{t+1}. The stock returns, R_{t+1}, are normally distributed, with an annual mean equity return $R_f + \lambda^R$ and a standard deviation σ_R, where λ^R is the "true" equity risk premium. As described above, the agent does not observe λ^R and, at time t, believes that the equity premium lies in the set Λ_t, which changes over time due to learning. All the parameters employed are chosen in Section 1.7.

1.5 Labor income process

I assume that labor income is uncertain and given by

$$Y_t = \exp(f_t + v_t + \epsilon_t) \text{ for } t < K, \tag{7}$$

where

$$v_t = v_{t-1} + u_t. \tag{8}$$

After the retirement age K, income is riskless and equals a fraction of the labor income at age 65 (the replacement rate). Labor income exhibits a hump-shaped profile over the life cycle that is accommodated by f_t, where f_t is a deterministic function of age. The error term consists of a transitory component and a permanent component. The term ϵ_t is a transitory shock and is distributed as $N(0, \sigma_\epsilon^2)$; u_t presents a permanent shock, where $u_t \sim N(0, \sigma_u^2)$. This representation follows Cocco et al. (2005) and I calibrate the labor income process according to their estimates. The function f_t is modeled by a third-order polynomial in age,

$$f_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2/10 + \alpha_3 t^3/100. \tag{9}$$

1.6 Learning and updating beliefs

Agents learn about the equity risk premium from birth throughout their lifetime and become less uncertain with age because they have received more information. The updating process for the set of priors follows Bayes' rule.[7] Before observing any signals, the set of priors is normally

[7] Other updating rules for beliefs are explored by Epstein et al. (2010), Epstein and Schneider (2007), and Hanany and Klibanoff (2009).

distributed, with mean λ^B and variance $(\sigma^B)^2$. I consider $t = 1$ to correspond to age 20, thus, an individual of age t has received $t + 19$ independent signals about λ^R, given by past returns $R_t = R^f + \lambda^R + \epsilon_t$, where ϵ_t is normally distributed with mean zero and a known variance σ_R^2. Signals are observed annually. The updated set of beliefs about λ^R are normally distributed, with mean λ_t^B and variance $(\sigma_t^B)^2$, where

$$\lambda_t^B = \lambda^B \underbrace{\frac{\frac{1}{(\sigma^B)^2}}{\frac{1}{(\sigma^B)^2} + \frac{t+19}{\sigma_R^2}}}_{\text{weight mean prior}} + \frac{1}{t+19} \sum_{\tau=-19}^{t-1} (R_\tau - R^f) \underbrace{\frac{\frac{t+19}{\sigma_R^2}}{\frac{1}{(\sigma^B)^2} + \frac{t+19}{\sigma_R^2}}}_{\text{weight returns}} \tag{10}$$

$$(\sigma_t^B)^2 = \frac{1}{\frac{1}{(\sigma^B)^2} + \frac{t+19}{\sigma_R^2}}. \tag{11}$$

The posterior mean λ_t^B is a precision-weighted average of the prior mean at birth and the average signal. Also, unlike λ_t^B, the posterior variance $(\sigma_t^B)^2$ does not depend on the specific realizations of the signals, only the number of signals. This variance, which measures the uncertainty/ambiguity about λ^R, decreases as the number of signals t increases (learning reduces uncertainty), that is, $(\sigma_{t+1}^B)^2 < (\sigma_t^B)^2$.

I assume agents update their beliefs irrespective of whether or not they participate in the stock market, e.g., since everyone receives similar information via the newspapers, television, and other media. People start with prior beliefs about the equity risk premium when born and update those beliefs according to realized returns from their birth year onward. The updating rule places no explicit weight on stock returns before the year of birth and thus only takes into account realizations during one's lifetime. The priors at birth could be thought of as containing to some extent the realized stock returns before birth, but I do not assume that prior beliefs at birth are equal to the confidence interval from the stock return data available. I choose this specific starting age for updating beliefs instead of, for instance, adult age or long before birth, because Malmendier and Nagel (2011) find that stock returns experienced receive a much larger weight when beliefs are formed about expected stock market returns compared to stock returns before birth. In their baseline model, the authors use experienced returns dating back to the year of birth. In contrast to Malmendier and Nagel (2011), I apply equal weights to all experienced returns and update beliefs according to Bayes' rule.

Two additional underlying assumption are that (1) the level of ambiguity, σ^B, is the same for every person at birth, independent of birth year, and (2) the mean of the beliefs about the equity risk premium, λ^B, at birth is independent of birth year and hence independent of stock return realizations before birth. In regard to assumption (1), the reason I assume that the level of ambiguity (standard deviation of belief) about the equity risk premium is the same in 1970 and 2000 is that data going back more than, for instance, 70 years may, according to the agent, not be that relevant for estimating the equity premium today, due to, for example, structural changes (Pastor and Veronesi (2009)). Structural changes, induced by, say, technological innovations might permanently change the equity risk premium. Hence the amount of uncertainty does not disappear with time and is thus irrespective of the year in which the agent is born.

Assumption (2) is that the mean of the belief is the same for every person at birth and does not depend on birth year. Different priors at birth could generate additional cohort effects; however, I assume that the prior is independent of birth year, because agents incorporate realized stock returns during their life more heavily into beliefs than returns before birth (Malmendier and Nagel (2011)). In Section 4, I explore the impact of starting updating from age 20 onward and different levels of initial ambiguity on the main findings.

1.7 Benchmark parameters for the life-cycle model with ambiguity

I set the risk aversion coefficient (γ) equal to five, which is the same as used by Benzoni et al. (2007) and Gomes and Michaelides (2005). Time ranges from $t = 1$ to time T, which corresponds to ages 20 and 100, respectively. Agents retire at time $K = 45$, corresponding to age 65. The survival probabilities are the current male survival probabilities in the United States, which are obtained from the Human Mortality Database.[8] I assume certain death at age 100.

The true equity premium λ^R is assumed to be normally distributed, with an annual mean of 4% and an annual standard deviation σ_R of 16%, which is in accordance with historical stock returns. The risk-free rate is 2%; hence the expected stock return is 6%. The mean of the priors about the equity premium at birth is equal to the true equity premium, $\lambda^B = 4\%$. The standard deviation of the beliefs at birth, σ^B, is 2%.

I use the parameters for the labor income process estimated by Cocco et al. (2005). The deterministic hump-shaped profile of income is generated by the parameters $\alpha_1 = 0.1682$, $\alpha_2 = -0.0323$, and $\alpha_3 = 0.002$. I choose the constant α_0 to accommodate different income levels at time $t = 1$. The benchmark income level at age 20 is \$15,000, as in Cocco et al. (2005). The variance of the transitory shock to labor income, σ_u^2, is 7.38% and the variance of the permanent shock, σ_ϵ^2, is 1.06%. The replacement rate of labor income at age 65 is 68% of the wage at age 65. Income during retirement is riskless. These numbers are for a high school graduate as estimated by Cocco et al. (2005) and used as the benchmark parameters in their analysis.

[8] I refer to further information at the website at www.mortality.org.

1.8 The individual's optimization problem

The timing during one year is as follows: First, an individual receives labor or retirement income, after which the individual consumes. Subsequently the remaining wealth is invested. The optimization problem is solved via dynamic programming and I proceed backward to find the optimal investment and consumption strategy. In the last period the individual consumes all the remaining wealth; hence the individual's utility from terminal wealth is known.

The problem cannot be solved analytically, so I employ numerical techniques following Brandt et al. (2005) and Carroll (2006) with several extensions by Koijen et al. (2010). Brandt et al. (2005) adopt a simulation-based method that can deal with many exogenous state variables. In this model, the mean of the beliefs about the equity premium, λ_t^B, and income, Y_t, are the relevant exogenous state variables. Wealth acts as an endogenous state variable. For this reason, following Carroll (2006), I specify a grid for wealth *after* income and consumption. As a result, I do not need numerical root finding to obtain the optimal consumption decision. The details of the numerical method I use to solve the life-cycle problem with maxmin preferences are described in Appendix A.

1.9 Data

When comparing the predicted stock allocation to the data, I use the 2010 Survey of Consumer Finances, which is the most comprehensive dataset on U.S. household assets and liabilities. High-income households are over-sampled to obtain a sufficient number of wealthy households in the study. I employ a measure for financial wealth and stock investment according to the method suggested by the Survey of Consumer Finances. The same measures are used by Gomes and Michaelides (2005). Financial wealth consists of both retirement and non-retirement wealth and stock investment is calculated as the sum of direct investment in stock and stock mutual funds, as well as stock investments of pension wealth. More details on the data from the Survey of Consumer Finances can be found in Appendix B.

2 Effect of ambiguity aversion on optimal allocations

We determine the optimal life-cycle choices using simulations and analyze the importance of ambiguity about the equity premium on optimal stock allocations and savings decisions.

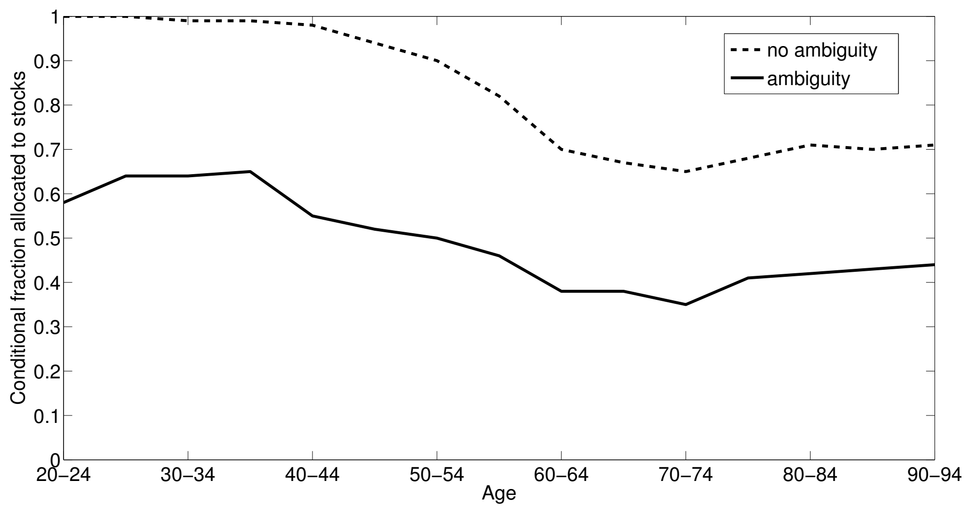
2.1 Effect of ambiguity aversion on optimal portfolio choice

The optimal fraction allocated to stocks, conditional on participation in the stock market, is plotted in Figure 1a. Comparing optimal allocations including ambiguity (solid line) with those with no ambiguity (dashed line) shows that the allocation to stocks when agents are ambiguity averse is much lower. At all ages the fraction allocated to stocks is around 50%. The impact of ambiguity aversion is substantial at young ages, but this effect declines slightly with age as the level of ambiguity about the equity risk premium decreases over time as agents learn.

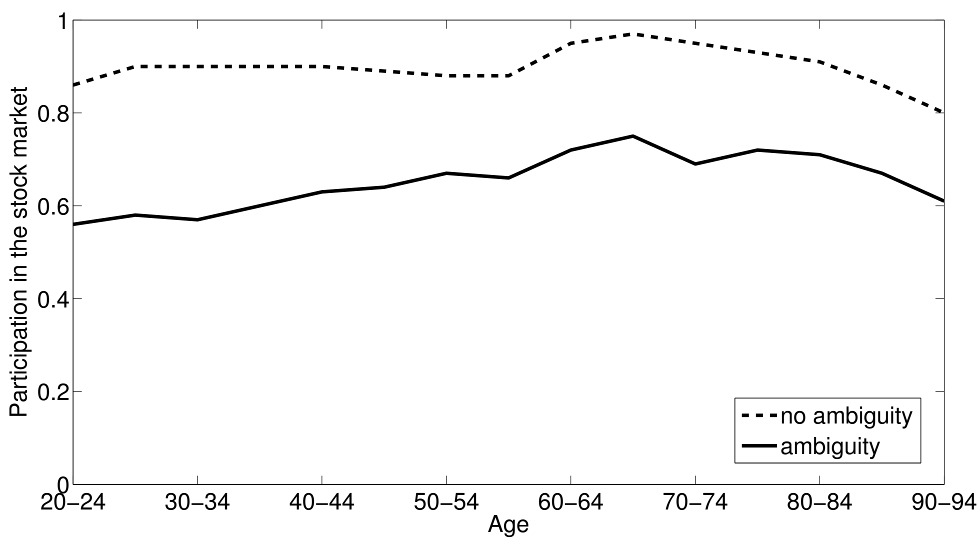
Focusing on the analysis without ambiguity, I find that if agents are fully certain about the values of all the parameters in the model, they allocate 100% of financial wealth to stocks before age 40. Similar results are found by Cocco et al. (2005). The reason for this high fraction is that young agents have only a small amount of financial wealth compared to a high level of human capital. Since human capital is like an implicit investment in a riskless asset, agents allocate their entire financial wealth to equity. Between ages 40 and 65, the conditional allocation to risky assets decreases. At these ages, (retirement) savings are high while at the same time the net present value

Figure 1: Optimal fraction allocated to stocks and optimal participation in the stock market
These figures show the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for (1) agents ambiguous about the equity risk premium and averse to this ambiguity and who learn about this parameter and (2) agents not ambiguous about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near-zero financial wealth level (below $100), optimal participation is assumed to be zero.

(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation



of labor income decreases; hence the fraction of financial wealth to human capital increases. This results in a decline of the relative allocation to the riskless asset human capital and, consequently, the optimal fraction of financial wealth invested in stocks decreases to maintain a similar risk profile. After retirement, the allocation to stocks increases slightly, as shown by Cocco et al. (2005). At that time the agent depletes financial wealth more rapidly due to the additional implicit discount factor, survival probabilities, and hence the fraction of financial wealth to human capital decreases, which induces a higher fraction of financial wealth invested in stocks.

Figure 1b displays optimal participation levels in the stock market. The effect of ambiguity is substantial and participation levels drop by 25%, on average, over the life cycle. When agents are not ambiguous about the equity premium, the participation levels in the stock market are high. Since labor income is not correlated with returns on the stock market, it is optimal for all agents, even those with low financial wealth, to allocate at least a small fraction of financial wealth to stocks. The reason for less than 100% participation is that I assume that agents with financial wealth less than $100 do not invest in stocks. Taking these agents with near-zero wealth into account would distort the subsequent comparison of the model predictions to the data, since in reality people with less than $100 of wealth would not invest, due to participation costs and minimum balance requirements.[9] As before, the impact of ambiguity aversion decreases with age, since the ambiguity about the equity risk premium declines as agents learn by observing the realized stock returns. In a non–life-cycle framework, Cao et al. (2005) and Easley and O'Hara (2009) confirm that ambiguity aversion can limit participation levels.

Vissing-Jorgenson (2002) examines the implications of fixed participation costs on optimal

[9] In addition, the simulation inaccuracy of optimal stock allocations is higher for these low wealth levels, since the difference in the utility of the agent investing 100% or 0% in stocks is negligible.

participation levels and finds that it can explain why less wealthy households do not participate, but not the low participation levels of the wealthy. I find that ambiguity about the equity risk premium can provide an explanation for the low participation levels of wealthy individuals as well. Furthermore, ambiguity about the equity premium can simultaneously explain the low participation levels, as well as the low conditional fraction allocated to stocks, while fixed participation costs only impact participation levels.

The previous paragraphs explore the optimal allocations for agents who are ambiguous *and* averse to this ambiguity. In contrast, in the more standard ambiguity-neutral framework, agents are only uncertain about the parameters, but not averse with respect to this uncertainty. When this is the case, the optimal allocations hardly change. In the benchmark model, the agents' beliefs about the equity risk premium are normally distributed, with a mean of 4% and a standard deviation equal to 2% at birth. If agents are ambiguity neutral, their behavior is induced by the so-called predictive distribution. The standard deviation for the compound distribution of the volatility of the return on equity, σ_R and the volatility of the belief, σ_t^B, can be reduced to the predictive volatility $\sqrt{\sigma_R^2 + (\sigma_t^B)^2}$. For the benchmark parameters, this results in a standard deviation of 16.1% (note that σ_R is 16%). Hence uncertainty about the equity risk premium will have (almost) no effect on optimal portfolio choices when uncertainty neutrality is assumed.

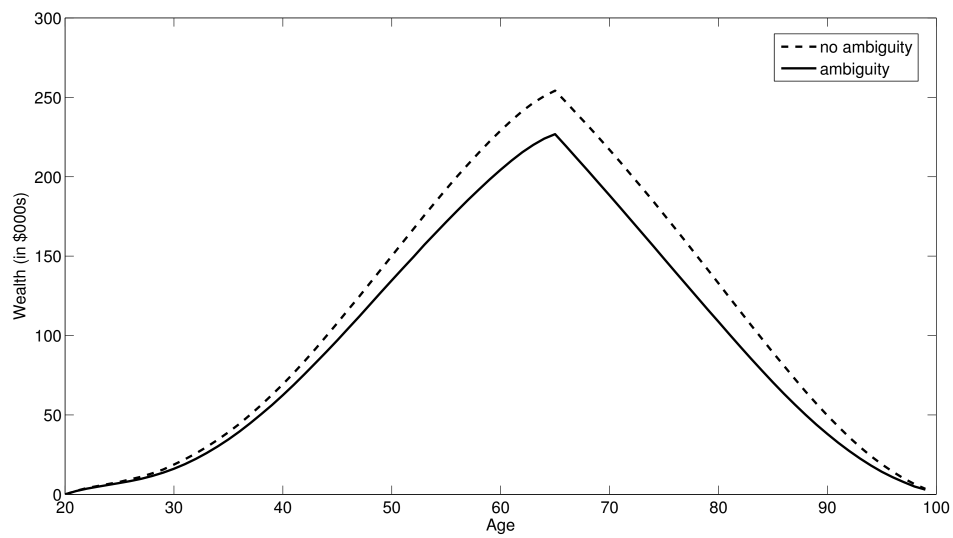
2.2 Effect of ambiguity aversion on savings

The optimal mean wealth levels are plotted in Figure 2. Agents who face ambiguity about the equity risk premium have lower amounts of wealth compared to individuals not facing ambiguity. For instance, at age 65 the mean wealth level when facing ambiguity is $225,000, compared to

$250,000 when exposed to only risk. Note that the sizes of these wealth levels are comparable to the findings of Cocco et al. (2005).

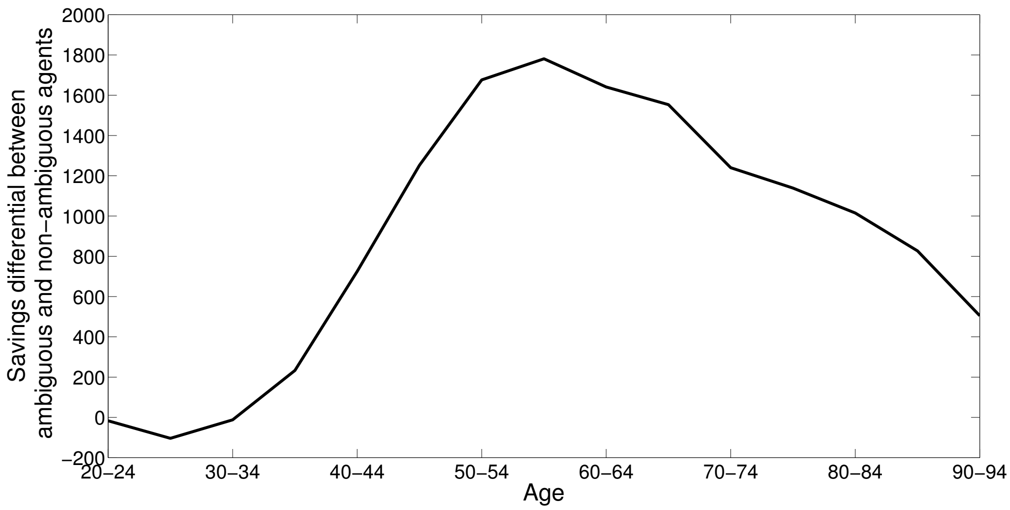
Figure 2: Optimal wealth levels

This figure shows the average optimal wealth for (1) agents ambiguous about the equity risk premium and averse to this ambiguity and who learn about this parameter and (2) agents who are not ambiguous about the equity premium.



These lower wealth levels for ambiguity-averse agents do not necessarily imply lower savings out of income and wealth. Several factors are resulting in differential wealth levels. First, these lower wealth levels are (partly) an automatic result of ambiguity-averse agents investing less in equity and thus having less wealth (savings plus investment return) accumulated. An additional rationale for lower wealth levels could be that individuals have fewer incentives to save, because they make investment decisions based on the worst case equity premium, which does not generate sufficient investment income on their savings. Going in the opposite direction are ambiguity-averse agents saving more to compensate for their lower wealth levels due to lower investment returns. Figure 3 shows the difference in savings levels between individuals facing ambiguity and those not facing ambiguity with similar wealth and income levels. Ambiguity-averse individuals save substantially more than non ambiguity-averse individuals. If the lower wealth levels of ambiguity-

Figure 3: Difference in savings between individuals facing ambiguity and not facing ambiguity
This figure shows the difference in the average savings out of income and wealth between agents who are ambiguity averse and those who are not. For a given wealth and income level (median wealth and income at each age for the non-ambiguous agent), savings are calculated for both types of agents, ambiguity averse and not ambiguity averse. The optimal savings for the ambiguity-averse agent minus the optimal savings for the non–ambiguity-averse agent are displayed.



averse agents are merely a reflection of lower investment returns, then the savings out of income and wealth should be the same for both and the difference in Figure 3 should be zero. However, the savings out of income and wealth are higher for ambiguity-averse agents. Thus, keeping up wealth levels by saving extra is quantitatively more important than lower savings rates induced by the relatively less attractive investment opportunities perceived by ambiguity-averse investors.[10]

3 Comparing optimal stock allocations to the empirical evidence

In this section I compare the predictions from the life-cycle model with ambiguity aversion and learning with the stock allocation data from the Survey of Consumer Finances in 2010.

[10]Sutter et al. (2013) find that more ambiguity-averse adolescents save less; however, these results are insignificant.

3.1 Fraction of financial wealth allocated to stocks

Figure 4: Fraction of financial wealth allocated to stocks, conditional on participation: Empirical and optimal allocations

This figure shows the empirical fraction of financial wealth allocated to stocks, conditional on participation, and the optimal fraction invested in the stock market for (1) agents ambiguous about the equity risk premium and averse to this ambiguity and who learn about this parameter and (2) agents who are not ambiguous about the equity premium. The data are from the 2010 Survey of Consumer Finances.

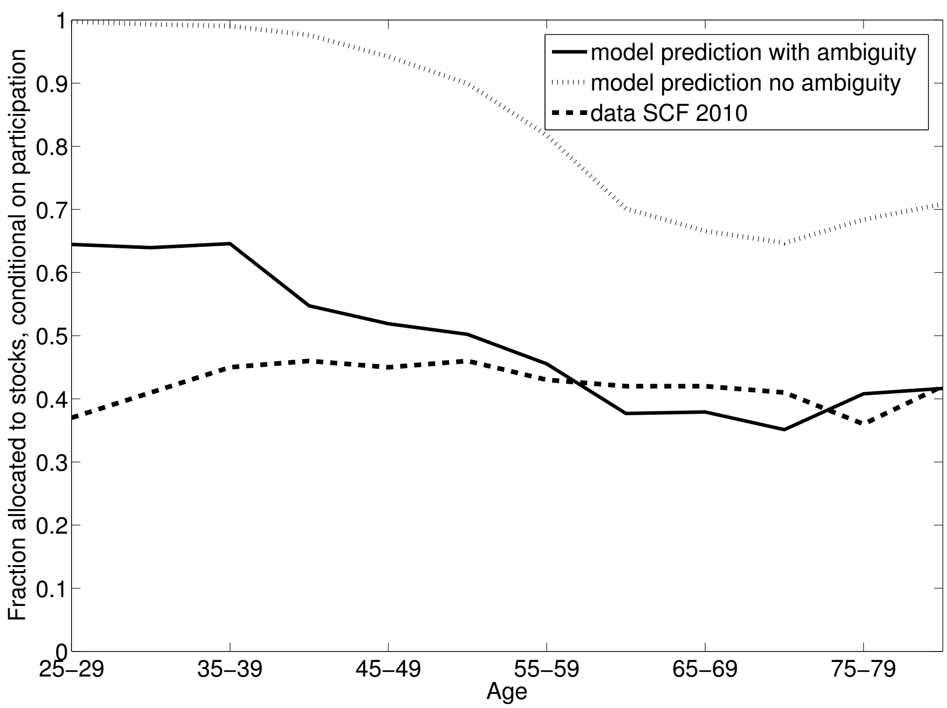


Figure 4 compares the optimal fraction allocated to stocks with the empirical levels in 2010. We can see that the fraction of financial wealth allocated to stocks for agents facing ambiguity is much closer to the empirical levels. The model without ambiguity predicts an optimal average fraction over the life cycle of about 85%, versus 50% when ambiguity is taken into account. The empirical average is slightly less than 45%.

The previous graph focuses on the means of the conditional allocation to stocks, not examining other moments. Table 1 displays the stock allocations predicted by the model and the empirical estimates for different percentiles. For all ages grouped together (ages 25–84), the median matches

Table 1: Percentiles of optimal and empirical fractions of financial wealth allocated to stocks
The conditional fractions allocated to stocks in 2010 are calculated using the Survey of Consumer Finances. The optimal fraction is calculated using the simulation results of the benchmark life-cycle model for (1) agents ambiguous about the equity risk premium and averse to this ambiguity and who learn about this parameter and (2) agents who are not ambiguous about the equity premium. Both the optimal and empirical fractions are conditional on stock market participation.

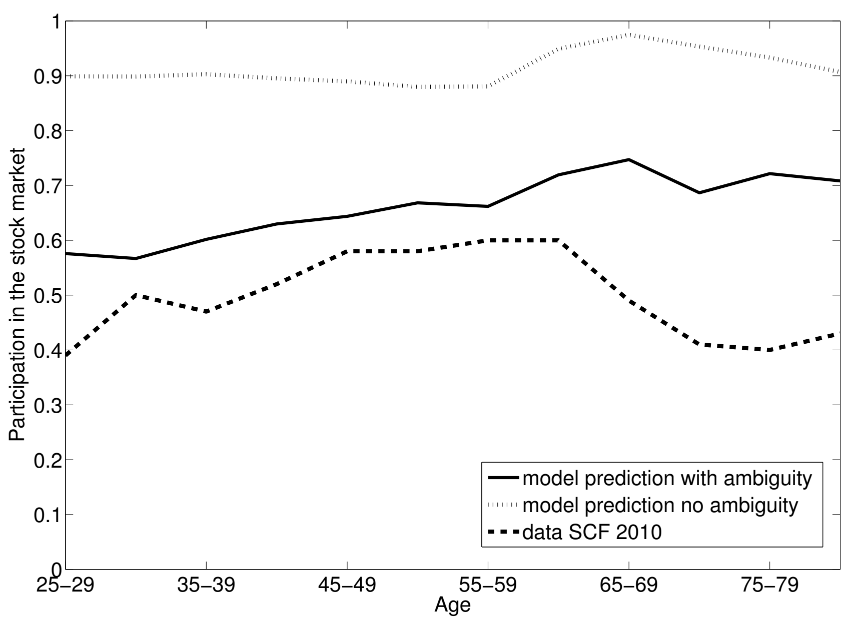
Data	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
Ages 25–84	7	19	40	65	86
Ages 25–34	6	15	35	62	84
Ages 35–44	9	22	42	69	89
Ages 45–54	9	21	43	67	89
Ages 55–64	6	19	41	62	84
Ages 65–74	7	18	37	65	81
Ages 75–84	5	12	36	58	80
Model with ambiguity	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
Ages 25–84	12	25	44	72	98
Ages 25–34	17	36	67	99	100
Ages 35–44	17	35	59	90	100
Ages 45–54	13	26	47	76	100
Ages 55–64	9	19	35	59	99
Ages 65–74	7	15	27	51	91
Ages 75–84	10	20	33	57	98
Model without ambiguity	10th percentile	25th percentile	50th percentile	75th percentile	90th percentile
Age 25–84	67	75	84	97	100
Ages 25–34	100	100	100	100	100
Ages 35–44	98	100	100	100	100
Ages 45–54	71	91	100	100	100
Ages 55–64	49	59	79	98	100
Ages 65–74	40	47	61	91	100
Ages 75–84	42	50	67	96	100

well, with 40% of financial wealth allocated to stocks in the data compared to 44% according to the model with ambiguity. Furthermore, for instance, the 10th percentile is 7% in the data, 12% according to the model that includes ambiguity, and 67% for the model without ambiguity. When splitting the fraction invested in stocks into different age groups, the fit to the data is similar. In line with the data, the model with ambiguity generates a large degree of heterogeneity in conditional fractions allocated to stocks, in contrast to the model without ambiguity.

3.2 Stock market participation

Figure 5 plots the optimal stock market participation and empirically observed participation levels. Comparing the empirical participation levels to the optimal participation levels when taking into account ambiguity, we find a close match. In 2010, averaged over the entire life cycle, about 50% of the people invest in the stock market and the model predicts about 65%. To compare, Gomes and Michaelides (2005) find optimal allocation levels of almost 100% at young ages, while the model with ambiguity predicts optimal levels of less than 60%. Note that it is not insightful to present the percentiles for the participation levels, since this is a binary variable and all the information is already contained in Figure 5.

Figure 5: Stock market participation: Empirical and optimal allocations
I display the empirical participation levels and optimal participation levels in the stock market for (1) agents ambiguous about the equity risk premium and averse to this ambiguity and who learn about this parameter and (2) agents who are not ambiguous about the equity premium. In case an agent has a near-zero financial wealth level (below $100), optimal participation is assumed to be zero.



4 Sensitivity analysis and an alternative ambiguity model

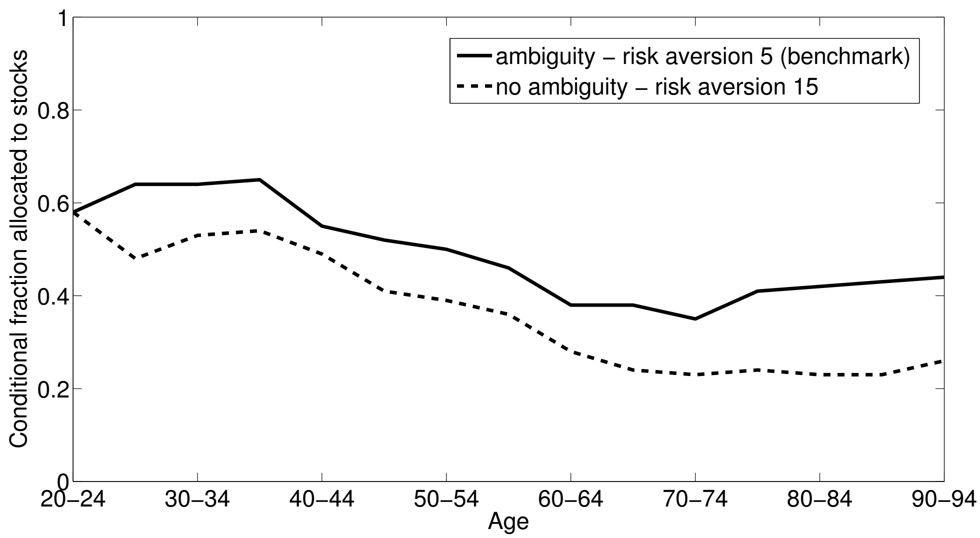
This section tests the importance of several features of the model. I explore whether higher risk aversion can substitute for ambiguity aversion and thereby change the optimal stock allocation to be in line with the data. Furthermore, the impact of the age at which the agent starts learning and the influence of the initial level of ambiguity about the equity premium are shown. Finally, an alternative ambiguity model, smooth recursive preferences, is examined.

4.1 Can higher risk aversion substitute for ambiguity aversion?

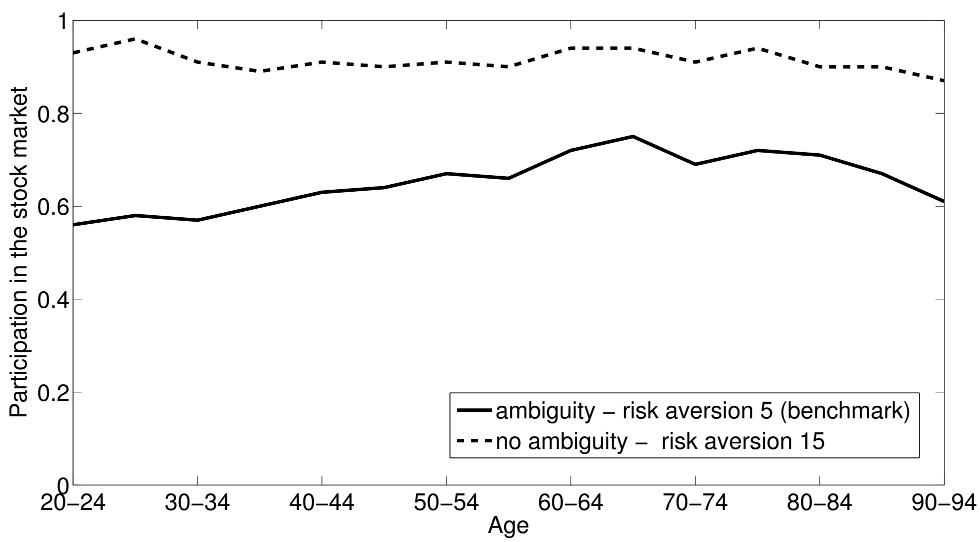
This paper shows that ambiguity about the equity risk premium can help solve the participation puzzle and explain the low fraction of financial wealth allocated to stocks over the life cycle. In this section I show that similar findings cannot be obtained by assuming higher risk aversion. The results are presented in Figure 6. Arguably, when agents have a risk aversion of 15 and are not ambiguity averse, the optimal fraction matches well the empirically observed fractions allocated to stocks. However, in Figure is shown that high risk aversion has almost no influence on optimal participation levels, almost 100% participation is predicted. Higher risk aversion actually increases participation, since it increases precautionary savings. The key difference is that while ambiguity aversion depresses participation levels, risk aversion cannot. Risk averse agents will always optimally invest a positive fraction of their financial wealth in stocks, when the equity premium is positive. Ambiguity averse agents, if the worst case belief is zero or negative, do not participate in the stock market, whereas if the worst case equity risk premium is positive, the agent participates. Hence risk aversion does not act as a substitute for ambiguity aversion and I do not obtain the same results with higher risk aversion compared to those obtained when including ambiguity aversion.

Figure 6: Stock allocations: Can risk aversion substitute for ambiguity aversion?
These figures show the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for (1) agents ambiguous about the equity risk premium and averse to this ambiguity and who learn about the parameters and (2) agents who are highly risk averse (gamma=15) and not ambiguous about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near-zero financial wealth level (below $100), optimal participation is assumed to be zero.

(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation



4.2 Impact of starting learning at a later age

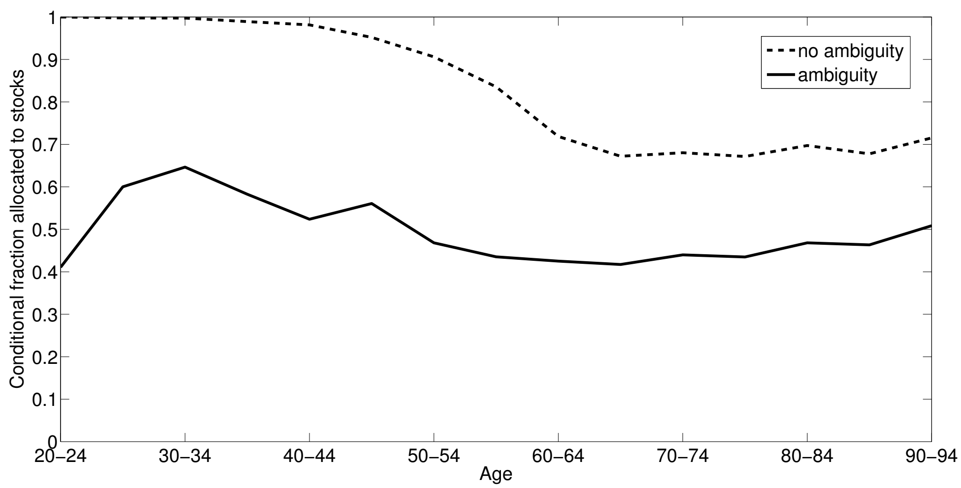
In the benchmark model, I assume that people incorporate in their beliefs returns realized from the year of birth onward, equally weighting each stock market realization. This section explores the impact of a different approach for updating beliefs about the equity premium. Specifically, I assume that agents learn from age 20 onward instead of from birth and the results are plotted in Figure 7. Both the optimal participation levels and the fraction of financial wealth allocated to stocks are lower if agents update beliefs from age 20 onward (compare to Figure 1). The reason is that the level of ambiguity, the standard deviation of beliefs about the equity premium, is higher when updating starts at a later age. When individuals are assumed to incorporate stock market realizations from age 20 onward, they will have seen only one stock market realization at age 21, compared to 21 realizations when all realizations after birth are incorporated. Hence, assuming learning starts from age 20 strengthens the main results in this paper; the impact of ambiguity about the equity premium on optimal stock allocations is larger.

4.3 Impact of initial ambiguity about the equity risk premium

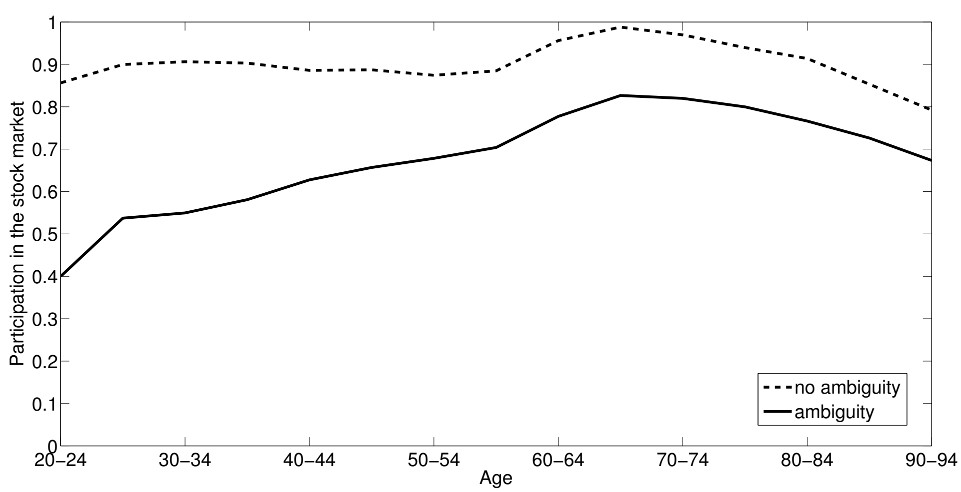
The initial level of ambiguity, that is, the standard deviation of the belief about the equity risk premium, is set at 2% in the benchmark case. Intuitively, a standard deviation of 2% seems reasonable, since this ensures that the 95% confidence interval of the equity risk premium that the agent believes is possible is between 0% and 8% at birth. Compelling evidence that this is not overstating the degree of ambiguity can be derived from the financial literacy literature. When answering questions to establish financial literacy levels, Rooij van et al. (2011) find that 22% of survey respondents answer that they do not know whether "considering a long time period, stocks,

Figure 7: Stock allocations: Learning from age 20 onward
These figures show the impact of learning from age 20 onward instead of from the year of birth. I display the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for (1) agents ambiguous about the equity risk premium and averse to this ambiguity and who learn about this parameter and (2) agents who are not ambiguous about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near-zero financial wealth level (below $100), optimal participation is assumed to be zero.

(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation

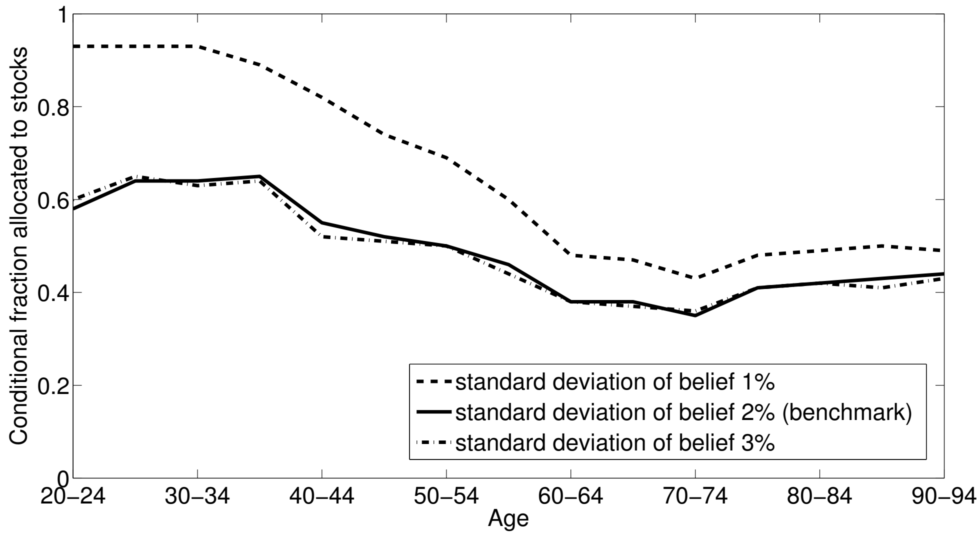


bonds, or savings accounts give the highest return". Furthermore, 30% gives the wrong answer and less than half gives the correct answer. This at least indicates that a large fraction of agents being ambiguous about the equity premium and, generally, about financial market parameters is a valid assumption. Furthermore, even assuming that agents look up all previous stock market returns, the confidence interval about the equity premium would still be large. However, since I have no means to determine the range of equity risk premium that agents deem possible, this section examines the influence of the initial ambiguity level on the results. These results can also be viewed in light of agents having different levels of ambiguity and how this influences the optimal fraction allocated to stocks and optimal participation levels.

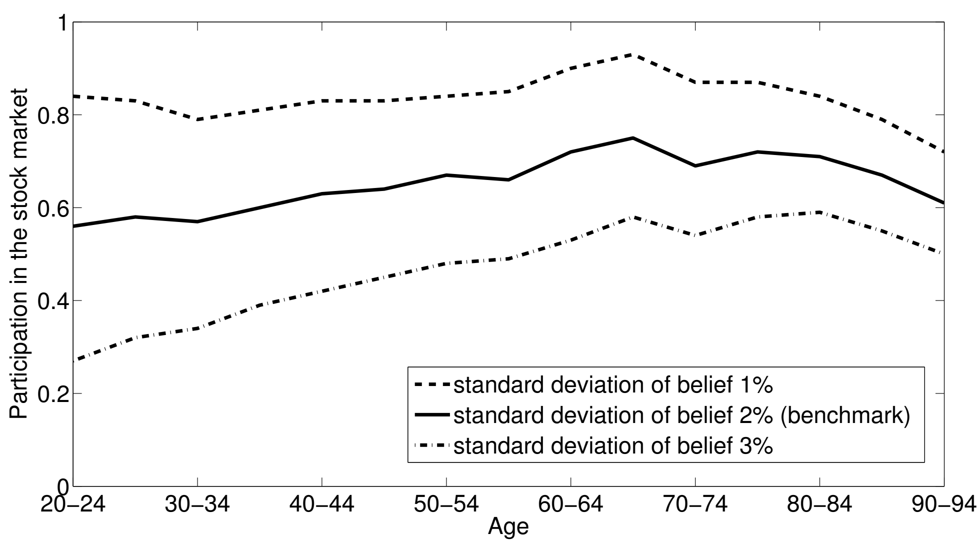
Figure 8 displays conditional allocations to equity and participation in the stock market for three different levels of initial ambiguity. The optimal conditional fraction allocated to stocks is approximately the same if the *initial* standard deviation of beliefs is 3% or 2%. The reason is that agents only participate in the stock market if, after updating, they have a positive worst case belief. Then, conditional on participation, the average belief about the equity premium is the same for agents starting out with a standard deviation of beliefs of 2% and 3%, and thus the average conditional fraction is the same. The fraction allocated to stocks if agents have a standard deviation of beliefs of 1% is much higher at young ages, because the average worst case belief, conditional on having a positive worst case belief, is higher. Participation levels differ substantially, since the agent with a standard deviation of 3% needs a much larger positive average realized return to have a worst case belief higher than 0% and participate, compared to an agent with a 2% standard deviation.

Figure 8: Stock allocations: Three different initial levels of ambiguity
These figures show the impact of the initial level of ambiguity. A standard deviation of the belief about the equity premium of 1%, 2%, or 3% is assumed. I display the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for agents ambiguous about the equity risk premium and averse to this ambiguity and who learn about this parameter. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near-zero financial wealth level (below $100), the optimal participation is assumed to be zero.

(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation



4.4 Alternative ambiguity model: *Smooth recursive preferences*

The previous sections show that ambiguity aversion has a large effect on optimal portfolio choices for agents with maxmin preferences. However, there is no consensus on whether agents exhibit maxmin preferences or smooth recursive preferences. Hence this section examines the influence of ambiguity aversion when agents behave according to smooth recursive preferences with moderate ambiguity aversion. I assume the preferences are as specified in Klibanoff et al. (2005), which includes an ambiguity function ϕ and total optimal lifetime utility equal to

$$V_t(W_t, Y_t, \lambda_t^B) = \max_{w_t, C_t} u(C_t) + \beta p_{t+1} \phi^{-1} \left(\int_{\Lambda_t} \phi \left(\mathbb{E}_t^{\lambda} \{ V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B) \} \right) \pi_t(\lambda) d\lambda \right), \quad (12)$$

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \quad (13)$$

$$\phi(x) = -\exp(-\alpha x), \quad (14)$$

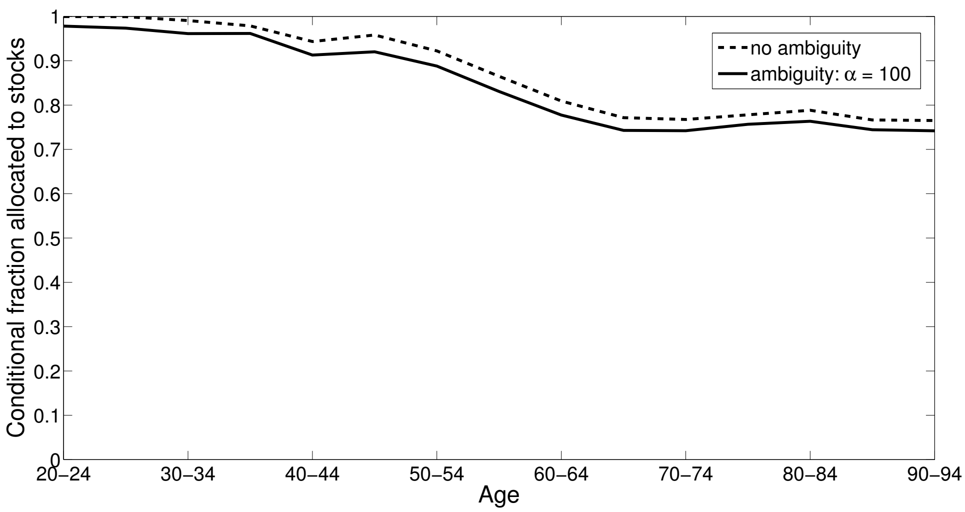
subject to the constraints described in section 1.3. Here, ϕ represents the constant absolute ambiguity aversion function and α is the ambiguity aversion coefficient, which we set equal to 100.[11] Think of each prior $\lambda \in \Lambda_t$ as describing a possible scenario (a possible equity risk premium) and $\pi_t(\lambda)$ as the probabilistic belief over the different scenarios. This utility function can be interpreted as being solved in two stages. First, the expected utility for all the priors in Λ_t is calculated to obtain a set of expected utilities. Using maxmin, one would then take the minimum of these expected utilities, while smooth preferences take an expectation over the distorted probabilities. The ambiguity aversion function ϕ distorts the probabilities, giving a higher weight to lower ex-

[11] The literature on smooth recursive preferences is relatively new and the preferences require as an input the ambiguity aversion parameter. As of yet only a few papers try to estimate this parameter and arrive at varying numbers. Chen et al. (2014) find values in the range of 50-90, while Ju and Miao (2012) use a calibrated ambiguity parameter of 8.9. I choose $\alpha = 100$.

pected utilities, reflecting ambiguity aversion. Details on the numerical solution method are given in Appendix C.

Figure 9: Stock allocations: Alternative ambiguity aversion model - smooth recursive preferences
These figures show the impact of an alternative model for ambiguity aversion, that is, smooth recursive preferences. I display the optimal conditional fraction of financial wealth allocated to stocks and optimal participation in the stock market for (1) agents ambiguous about the equity risk premium and moderately averse to this ambiguity and who learn about this parameter and (2) agents who are not ambiguous about the equity premium. The upper panel shows the fraction of financial wealth allocated to stocks, conditional on stock market participation. The lower panel shows the optimal participation level. In case an agent has a near-zero financial wealth level (below $100), optimal participation is assumed to be zero.

(a) Fraction allocated to stocks, conditional on participation



(b) Stock market participation

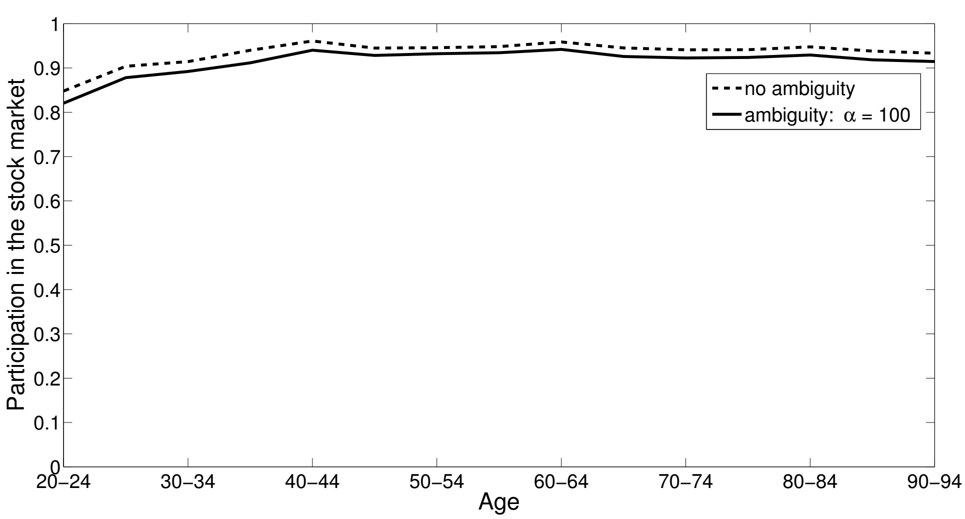


Figure 9a shows the optimal fraction of financial wealth allocated to stocks, conditional on participation in the case that (1) the parameters are not ambiguous and (2) the parameters are ambiguous and the agent is moderately averse to this ambiguity ($\alpha = 100$). There is a small decrease in allocation to stocks. When comparing the two cases in Figure 9b, similarly I find that the participation levels decrease slightly. Overall, the effect of ambiguity aversion on optimal portfolio allocation is negligible. Note that smooth ambiguity preferences with infinite ambiguity aversion is equivalent to maxmin preferences. Hence, in the limit, when ambiguity aversion goes to infinity, the effect of ambiguity about the equity risk premium is sizable and can help explain the empirically observed low allocation to stocks. Consistent with the findings in this paper, experimental evidence suggests that agents behave more according to kinked (maxmin) preferences than according to smooth ambiguity preferences (Ahn et al. (2014)).

5 Conclusion

This paper develops a realistically calibrated life-cycle model with ambiguity aversion and learning to explore the impact of ambiguity about the equity risk premium on optimal portfolio allocations. Taking into account ambiguity about the equity premium reduces optimal participation levels, on average, by 25% and the optimal conditional fraction of financial wealth allocated to stocks by 40%. Furthermore, wealth levels over the life cycle are reduced substantially while savings out of income and wealth are increased. I compare the model predictions with data from the Survey of Consumer Finances. Two important empirical facts are matched: the low participation levels in the stock market over the life cycle and the low fraction of financial wealth allocated to equity, conditional on participation.

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A Numerical method for model with maxmin preferences

Investors preferences are described by maxmin expected utility, which, in effect, means that agents maximize their utility with respect to the worst case belief. Agents are uncertain about the equity risk premium. I solve the following Bellman equation:

$$V_t(W_t, Y_t, \lambda_t^B) = \max_{w_t, C_t} \min_{\lambda \in \Lambda_t} \{u(C_t) + \beta p_{t+1} E_t^\lambda \left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B)\right]\}, \text{ with} \tag{A.1}$$

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \tag{A.2}$$

subject to (5). As described in Section 1.1, I restrict the domain of beliefs about the equity risk premium to lie in $[\lambda_t^B - 2\sigma_t^B, \lambda_t^B + 2\sigma_t^B]$. This is necessary otherwise the worst case belief would be unbounded, since the beliefs are normally distributed.

In each period the optimal asset weights are given by the first-order condition:

$$E_t^{\lambda_t^{min}}[C_{t+1}^{*-\gamma}(R_{t+1} - R^f)] = 0, \tag{A.3}$$

where λ_t^{min} is the lowest equity premium belief in Λ_t and C_{t+1}^* denotes the optimal real consumption level. Optimal consumption follows from

$$C_t^{*-\gamma} = \beta p_{t+1} E_t^{\lambda_t^{min}} \left[C_{t+1}^{*-\gamma} R_{t+1}^{P*}\right], \tag{A.4}$$

where R_{t+1}^{P*} is given by (6) evaluated at the optimum.

As described in Section 1.8, I regress the realizations of the Euler condition on a polynomial expansion in the state variables to obtain an approximation of the conditional expectation in the Euler condition

$$E_t^{\lambda_t^{min}} \left[C_{t+1}^{*-\gamma}(R_{t+1} - R^f)\right] \simeq \rho(w_t, A_t)' h(Y_t, \lambda_t^B), \tag{A.5}$$

where $A_t = W_t + Y_t - C_t$ is defined as the after consumption wealth. In addition, I employ a further approximation, introduced by Koijen et al. (2010), who show that the regression coefficients ρ are smooth functions of the asset weights and, consequently, I approximate the regression coefficients ρ by projecting them further on a polynomial expansion in the asset weights:

$$\rho(w_t, A_t) \simeq \Psi(A_t) g(w_t). \tag{A.6}$$

The Euler condition must be set to zero to find the optimal asset weights:

$$g(w_t)'\Psi(A_t)'h(Y_t, \lambda_t^B) = 0. \tag{A.7}$$

Similarly, I approximate the Euler condition for optimal consumption, equation (A.4), via regressing the realization of the Euler conditions on a polynomial expansion in the state variables. This results in optimal policies for every simulation path. I simulate 5000 paths and repeat this 10 times. Results are obtained by using the total of 50,000 simulation paths. A more detailed explanation of the numerical approach is described in Koijen et al. (2010).

B Survey of Consumer Finances and stock allocations

The Survey of Consumer Finances is a triennial survey on the financial assets of the household. It provides information on assets on the balance sheet, pensions, income, and demographics of the household. Participation in the survey in strictly voluntary and about 4500 families are interviewed. It is a repeated cross section and only the years 1983 to 1989 are part of the panel study. The median length of an interview is about 75 minutes, but an interview with a family with complex finances can take up to several hours. High-income households are over-sampled to measure asset holdings more accurately, since wealth in the United States is highly concentrated among a relatively small number of households. About two-thirds of the sample, 3000 households, is drawn from a national area probability sample that represents the entire population. The remaining one-third, 1500 households, is drawn from tax records to obtain the list of high-income households. Weights are used to account for both non-response and the difference between the initial sample design and the actual distribution of population characteristics. In the case of missing data, multiple imputation is used to solve this problem.

Financial wealth is the sum of liquid assets (checking, savings, money market, and call accounts); certificates of deposit; directly held mutual funds; stocks; bonds; quasi-liquid retirement accounts that consists of IRAs/Keoghs, thrift accounts, and future pensions; savings bonds; the cash value of whole life insurance; other managed assets (trusts, annuities, and managed investment accounts); and other financial assets (loans from the household to someone else, future proceeds, royalties, futures, non-public stock, deferred compensation, oil/gas/mineral investment). The part of financial assets invested in stocks consists of directly held stock, stock mutual funds, and retirement assets invested in stocks. I follow the Survey of Consumer Finances in calculating this. Stock investment includes all directly held stock, all stock mutual funds, half of the value of combination mutual funds, and a fraction of the value of IRAs/Keoghs that is invested in stocks. Similarly, the fraction of the value of other managed assets invested in stocks is added and the part of the value of the thrift account that is allocated to stocks.

The fraction of agents participating in the stock market is determined by calculating which weighted fraction in the total sample has a stock investment larger than zero. Furthermore, the conditional allocation to equity is the fraction allocated to stocks, conditional on participation in the stock market. Note that I use weights to calculate the participation rate and the conditional allocation to stocks to adjust for non-response and the non-equal probability design of the survey.

C Asset allocation with alternative ambiguity preferences

C.1 Summary of the life-cycle problem with smooth recursive preferences

The investor solves the following Bellman equation at time $t \neq T$:

$$V_t(W_t, Y_t, \lambda_t^B) =$$

$$\max_{w_t, C_t} u(C_t) + \beta p_{t+1} \phi^{-1} \left(\int_{\Lambda_t} \phi \left(\mathbb{E}_t^\lambda \{ V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B) \} \right) \pi_t(\lambda) d\lambda \right), \tag{C.1}$$

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \tag{C.2}$$

$$\phi(x) = -\exp(-\alpha x), \tag{C.3}$$

subject to the constraints in 1.3. Agents employ an uncertainty aversion function ϕ. The exogenous state variables are income, Y_t, and the mean of the belief about the equity premium λ_t^B. Wealth, W_t, is an endogenous state variable.

C.2 Beliefs about the equity risk premium

As described before, each agent has beliefs about the equity risk premium with mean λ_t^B and standard deviation σ_t^B. I limit the set of beliefs that the agent thinks are viable to be bounded by a 95% confidence interval. Hence the beliefs about the equity premium lie in the range of $[\lambda_t^B - 2\sigma_t^B, \lambda_t^B + 2\sigma_t^B]$. I make a grid for the possible equity premiums by dividing this confidence interval into K intervals, each of them having equal probability according to a normal distribution with mean λ_t^B and standard deviation σ_t^B.

C.3 First-order conditions: smooth recursive preferences

In period T, the optimal policies are easily determined: Namely, the agent consumes the entire wealth level and no optimal investment strategy needs to be made. In all the other time periods, optimal decisions on consumption and investment are calculated by deriving the first-order conditions of the problem. The optimization problem is solved via dynamic programming and I proceed backward.

Consider the agent at time t, who has to choose C_t and w_t. The first-order conditions for this problem are:

$$\begin{aligned}\frac{\partial V_t}{\partial w_t} &= \beta p_{t+1}(\phi^{-1})' \left(\int_{\Lambda_t} \{\phi\left(\mathbb{E}_t^{\lambda}\left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B)\right]\right)\}\pi_t(\lambda)d\lambda \right) \\ &\quad \int_{\Lambda_t} \left\{\phi'\left(\mathbb{E}_t^{\lambda}\left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B)\right]\right)\mathbb{E}_t^{\lambda}\left[C_{t+1}^{*-\gamma}(R_{t+1} - R^f)\right]\right\}\pi_t(\lambda)d\lambda = 0.\end{aligned} \tag{C.4}$$

and

$$\begin{aligned}C_t^{*-\gamma} &= \beta p_{t+1}(\phi^{-1})' \left(\int_{\Lambda_t} \{\phi\left(\mathbb{E}_t^{\lambda}\left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B)\right]\right)\}\pi_t(\lambda)d\lambda \right) \\ &\quad \int_{\Lambda_t} \left\{\phi'\left(\mathbb{E}_t^{\lambda}\left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B)\right]\right)\mathbb{E}_t^{\lambda}\left[C_{t+1}^{*-\gamma}(1 + R^f + w_t^*(R_{t+1} - R^f))\right]\right\}\pi_t(\lambda)d\lambda.\end{aligned} \tag{C.5}$$

C.4 Optimization procedure asset weights: Smooth recursive preferences

I use numerical optimization techniques to solve the problem, which I will describe in detail since this is a more complicated problem with smooth recursive preferences compared to maxmin preferences. In this section I explain this procedure, which combines the methods of Brandt et al. (2005) and Carroll (2006) with several extensions added by Koijen et al. (2010). Brandt et al.

(2005) propose approximating the conditional expectations by regressing the realizations of the Euler conditions on a polynomial expansion of the state variables. All state variables except for wealth can be simulated, since only wealth is endogenous. To deal with this endogenous state variable, I follow Carroll (2006), who proposes a grid for wealth *after* consumption, A_t, instead of a grid for wealth, W_t. This choice allows us to solve the Euler conditions analytically instead of numerically and I form an M-dimensional grid for wealth after consumption. Additionally, I use the extensions of Koijen et al. (2010) to increase the optimization speed. I construct H test portfolios and let the weight invested in the risky asset range from 0% to +100%, in steps of 5%; hence H is 21. The return on the test portfolios is defined as R_{t+1}^{test}. Furthermore, we simulate N trajectories of T periods for every state variable.

The problem is solved via backward recursion. To solve the optimal policies at time t, I have available the endogenous wealth grid at time $t+1$ and optimal consumption at time $t+1$.

First, I need to determine the two conditional expectations in equation (C.4):

$$\mathbb{E}_t^{\lambda}\left[C_{t+1}^{*-\gamma}(R_{t+1}-R^f)\right], \tag{C.6}$$

$$\mathbb{E}_t^{\lambda}\left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^B)\right]. \tag{C.7}$$

The conditional expectation in equation (C.6) is straightforward to calculate. Optimal consumption in period $t+1$ is determined via backward recursion. I make sure that C_{t+1}^* belongs to the grid point for after-consumption wealth at time t, A_t, by interpolating it linearly. I then approximate the conditional expectation with a polynomial expansion in the state variables:

$$\mathbb{E}_t^{\lambda}\left[C_{t+1}^{*-\gamma}(R_{t+1}-R^f)\right] \simeq \rho^{\lambda}(w_t, A_t)' h(Y_t, \lambda_t^B). \tag{C.8}$$

This is done for each simulation path, N, and $M \times K$ grid points.

The second conditional expectation (C.7) requires some more steps. The goal is to determine the realizations of V_{t+1} and regress these on the state variables at time t to obtain the conditional expectation. The value function at time $t+1$ is

$$V_{t+1}(W_{t+1}, Y_{t+1}, \lambda^B_{t+1}) =$$
$$u(C^*_{t+1}) + \beta p_{t+2} \phi^{-1} \left(\int_{\Lambda_{t+1}} \{\phi \left(\mathbb{E}^{\lambda}_{t+1} \left[V_{t+2}(W_{t+2}, Y_{t+2}, \lambda^B_{t+2}) \right] \right) \} \pi_{t+1}(\lambda) d\lambda \right). \quad \text{(C.9)}$$

As before, I use interpolation to make sure that optimal consumption belongs to the grid points for after-consumption wealth at time t. Furthermore, I use interpolation to guarantee that V_{t+2} belongs to the grid points for after-consumption wealth at time t. At time t, the value of the Bellman equation for $t+2$, V_{t+2}, is given through backward recursion. As before, to obtain $\mathbb{E}^{\lambda}_{t+1}[V_{t+2}]$ I regress V_{t+2} on the state variables at time $t+1$. Finally, at time $T-1$, I use $V_{T+1} = 0$ and $V_T = u(W_T)$.

Since agents are uncertain about the equity risk premium and are ambiguity averse, the optimization problem requires applying function ϕ to the expectation $\mathbb{E}_{t+1}[V_{t+2}]$. Remember that by construction the grid for the beliefs of the agent are such that there is an equal $1/K$ probability that the equity risk premium falls on each grid point. Hence, I can take the simple average of $\phi\left(E^{\lambda}_{t+1}[V_{t+2}]\right)$ over the grid for the beliefs to calculate $\left(\int_{\Lambda_{t+1}} \{\phi\left(\mathbb{E}^{\lambda}_{t+1}\left[V_{t+2}\right]\right)\}\pi_{t+1}(\lambda)d\lambda\right)$. Next, I obtain V_{t+1} through equation (C.9). As before, I regress the realizations of V_{t+1} on the state variables and obtain the conditional expectation, $\mathbb{E}^{\lambda}_{t}[V_{t+1}]$ (equation (C.7)).

I now have all the objects inside the integrals in (C.4). Thus, I can compute the left-hand side of (C.4) by simply using (C.3) and taking averages over the grid for the beliefs.

Following Koijen et al. (2010), the optimal asset weights are determined in two steps. First, I approximate the conditional expectation with polynomial state variables:

$$
\begin{aligned}
&\beta p_{t+1}(\phi^{-1})' \left(\int_{\Lambda_t} \{\phi\left(\mathbb{E}_t^{\lambda}\left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^{B})\right]\right)\}\pi_t(\lambda)d\lambda \right) \\
&\left(\int_{\Lambda_t} \left\{\phi'\left(\mathbb{E}_t^{\lambda}\left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^{B})\right]\right)\mathbb{E}_t^{\lambda}\left[C_{t+1}^{*-\gamma}(R_{t+1}-R^f)\right]\right\}\pi_t(\lambda)d\lambda \right) \\
&= \tilde{\rho}(w_t, A_t)' h(Y_t, \lambda_t^B).
\end{aligned} \tag{C.10}
$$

Subsequently, the projection coefficients, $\tilde{\rho}$, depend on the stock fraction w_t. Also, since $\tilde{\rho}$ is a smooth function of w_t, I can further approximate $\tilde{\rho}$ by projecting it on a polynomial expansion in the stock fraction $\tilde{g}$:

$$
\tilde{\rho}(w_t, A_t) \simeq \tilde{\Psi}(A_t)\tilde{g}(w_t), \tag{C.11}
$$

for some coefficients $\tilde{\Psi}(A_t)$. In turn, this implies that the conditional expectation of the Euler condition is approximated via

$$
\begin{aligned}
&\beta p_{t+1}(\phi^{-1})' \left(\int_{\Lambda_t} \{\phi\left(\mathbb{E}_t^{\lambda}\left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^{B})\right]\right)\}\pi_t(\lambda)d\lambda \right) \\
&\int_{\Lambda_t} \left\{\phi'\left(\mathbb{E}_t^{\lambda}\left[V_{t+1}(W_{t+1}, Y_{t+1}, \lambda_{t+1}^{B})\right]\right)\mathbb{E}_t^{\lambda}\left[C_{t+1}^{*-\gamma}(R_{t+1}-R^f)\right]\right\}\pi_t(\lambda)d\lambda \\
&= \tilde{g}(w_t)'\tilde{\Psi}(A_t)' h(Y_t, \lambda_t^B).
\end{aligned} \tag{C.12}
$$

It turns out that a polynomial expansion of order one is sufficient for this life-cycle problem; hence, for every simulation path I solve

$$
0 = \begin{pmatrix} 1 \\ w_t^* \end{pmatrix}' \tilde{\Psi}(A_t)' h(Y_t, \lambda_t^B), \tag{C.13}
$$

which can be solved analytically, taking into account the portfolio constraints.

C.5 Optimization procedure consumption - smooth recursive preferences

The first-order condition with respect to C_t is given by (C.5). Since the timing is such that the agent consumes before making the investment decision and since I solve the problem via backward recursion, when finding optimal consumption, the optimal fraction invested in stocks, w_t^*, is already computed as explained above. For the first-order condition (C.5), I proceed as before. First, I calculate the conditional expectations inside the integrals, then I apply function ϕ, and finally I compute the integrals by taking simple averages. The optimal consumption strategy then follows analytically from (C.5). Note, however, that I want to impose that the conditional expectation $\mathbb{E}_t^\lambda \left[C_{t+1}^{*-\gamma} (1 + R^f + w_t^* (R_{t+1} - R^f)) \right]$ is always positive. Hence, following Koijen et al. (2010), I approximate the logarithm of this conditional expectation.

Once I approximate the policy function for consumption and investment decisions for each time t as a function of the state variables, I start from the initial states and simulate forward. I simulate 5000 paths and repeat this 4 times. Results are obtained by using those 20,000 simulation paths.