## 4. ENTRY, FORECLOSURE

**Exercise 4.1**

Consider a Hotelling circular market: heterogeneous consumers (with total mass equalized to 1) are uniformly distributed along a circle of length 1, based on their preferences. The utility of a generic consumer, whose ideal variety is $t$, when he/she buys variety $x_i$ at price $p_i$ is given by $u^*-p_i-(x_i-t)^2$. This market is populated by $n$ firms symmetrically distributed along the circle (i.e. firm $i$, with $i$=1..$n$, offers variety $x_i = i/n$).

1) Derive the demand for variety $i$ as a function of the prices.

2) Assume null production costs: compute the equilibrium prices, profits and quantities, as a function of $n$.

3) If firms have to pay an entry costs equal to 1/150 to operate in the above market, how many competitors there would be under free entry? Compute firm $i$'s profit in this case and discuss.

**Exercise 4.2***

Consider a Hotelling linear market in which consumers with total mass equal to 1 are uniformly distributed along a segment of length 1. Consumers' preferences for variety $i$ are described by the family of utility functions $4-p_i-(x_i-t)^2$ where $p_i$ is the price, $x_i$ the bought variety and $t\in$ [0,1] the ideal variety.

Assume that firms need to locate at the extremis of the segment, namely the only feasible varieties are $x_0$=0 and $x_1$=1; production costs are null.

1) Compute the equilibrium prices and profit of a multi-product monopolist ($M$) producing both varieties.

2) Assume that a new firm enters $E$ and locates in 1. Compute the Nash equilibrium in prices.

3) If firm $M$ stops producing $x_1^M$, while it keeps producing $x_0^M$, what is the Nash equilibrium in prices?

4) Consider now the extended form of the game: at time $t$=1, firm $E$ decides whether to enter ($E$), paying a fixed entry cost $F$ and locating in 1 or not to enter ($NE$), given that $M$ is offering both varieties. At time $t$=2, $M$ decides whether to stop producing variety 1 ($S$), paying a fixed exit cost $K$, or not ($NS$). At time $t$=3, firms play the market game. Assuming that entry and exit costs are defined over the intervals $K\in$[0,1] and $F\in$[0,1], find the regions in the space ($K$, $F$) in which $E$ decides to enter and not to enter.

**Exercise 4.3**

Consider a two stage game in which $m$ firms, at time $t$=1, simultaneously decide whether to enter a market (paying a fixed entry costs of 1/36) or not in; at time $t$=2, the $n\leq m$ active firms compete à la Cournot, facing an inverse demand function of the type $p=1-q$, where $p$ is the price and $q$ denotes total quantity.

Assuming zero production costs, solve the game by backward induction and derive the number $n^*$ of firms that are active in this market in a long run equilibrium. Compute also individual profits.

**Exercise 4.4**

Two firms operate in a market in which demand is given by $p=4-K_1-K_2$, where $p$ is the price and $K_i$,($i$ =1, 2) denotes firms' individual quantities. Assume zero production costs for simplicity.

At time $t$=1, firm 1 chooses its quantity $K_1$; at time $t$=2, firm 2 chooses its quantity $K_2$, given $K_1$; if $K_2$>0, firm 2 has to pay a fixed entry cost $F$=1/16.

Derive the market equilibrium.

**Exercise 4.5***

Consider a market in which homogeneous goods are produced to satisfy an inverse demand function of the type $p=1-q$, where $p$ is the price and $q$ denotes total quantity. Marginal costs are constant and null. There is already one firm ($I$) in the market, and another one ($E$) is deciding whether to enter or not. If $q_E>0$ (i.e. firm $E$ enters the market), firm $E$ has to pay a fixed cost $F^2$ with $F$ belonging to the interval [0,1/4); no cost is incurred if $q_E=0$.

Assume the following *timing*: at time 1 firm $I$ chooses the quantity $q_I$; at time 2 firm $E$ chooses $q_E \geq 0$ given $q_I$ (if $q_E=0$ we say that firm $E$ decides not to enter).

1) Find $q_I$* and $q_E$* in case $I$ accommodate entry by $E$.

2) Derive the quantity $q_I*(F)$ as a function of the fixed cost $F$ in case $I$ deters entry by $E$.

3) Find the values of $F$ for which firm $I$ prefers to accommodate entry by $E$, and the values for which it refers to deter.

4) Prove the following: in a two stage entry game (i.e. entry at stage 1, Cournot competition at stage 2) if firms have to pay a fixed cost equal to $F^2<1/16$, they both enter the market.

**Exercise 4.6**

Consider a Hotelling circular market, with three firms symmetrically distributed along a circle of length 1, such that $x_i = i/3$, with $i$=1, 2, 3.

For simplicity assume zero production costs. Consumers are uniformly distributed along the circle and utility of consumer $t$, if he/she buys $x_i$ at price $p_i$ is given by $u^*-p_i-(i/3-t)^2$.

1) Derive the demand functions faced by the three firms.

2) Compute the prices and profits in the Nash equilibrium in prices.

3) Suppose that firm 1 and 2 merge, and they keep producing the varieties offered before. Compute the prices and profits in the Nash equilibrium in prices.

4) Are there incentive sto merge? Is the merger profitable for the non merging firm (i.e. firm 3 in this example)?

**Exercise 4.7***

Consider a Hotelling linear market in which consumers with ideal variety $t$ – with $t$ uniformly distributed along a unitary segment – derive utility $u^*-p_i-(x_i-t)^2$ from buying variety $x_i$ at price $p_i$.

1) Derive the demand functions when varieties $x_1=0$ and $x_2=1$ are produced.

2) Assume that the market is populated by a monopolist producing both varieties with zero costs. Find the optimal $p_1$ and $p_2$. Being able to choose, do you think that the monopolist would prefer a different "location" for $x_1$ and $x_2$?

3) Assume a new firm enters offering variety $x_3=1/2$, i.e. locating in the middle of the segment, with zero costs. Derive the new demand functions and the equilibrium prices.

4) Consider the case in which the entrant bears a fixed cost $F=1/6$. Is entry still profitable?

5) Assume that the monopolist does not bear any exit costs if it stops producing one variety. Prove that, in this case, the entrant may find it profitable to enter and locate in one of the two extremes.

**Exercise 4.8**

Consider Hotelling linear model: heterogeneous consumers (with total mass equal to 1) are uniformly distributed along a circle of length 1, based on their preferences. The utility of a consumer with ideal variety $t$, when he/she buys variety $x_i$ at price $p_i$ is given by $10-p_i-(x_i-t)^2$.

This market is populated by $n$ firms symmetrically distributed along the circle (i.e. firm $i$, with $i$=1..$n$, offers variety $x_i=1/n$).

1) Compute the demand for variety $i$, as a function of the prices.

2) Assume null production costs. Compute the equilibrium prices, profit and quantities, as a function of $n$.

3) If firms have to pay a fixed entry cost of 1/200, how many competitors do you expect to see on the market, under free entry? Compute firm $i$'s profit in this case and discuss.

**Exercise 4.9**

There is a market in which three firms have to decide how many plants they want to build (0, 1, 2…). Production costs are null, building costs equal 3.5 per plant. The market price is given by 6-$K$ where $K$ denotes the total number of plants (i.e. the total quantity, assuming that each plant produces one unit of good).

1) Prove that a monopolist would build only one plant.

2) Prove that Cournot duopolists would build only one plant each.

3) Assume now that firms do not act simultaneously. In particular, firm 1 builds before firm 2. Prove that the first one builds two plants, while the second stays out of the market.

**Exercise 4.10***

Consider a market in which the inverse demand function is linear and given by $p=1-q$, where $p$ is the price and $q$ denotes total quantity. For simplicity assume zero production costs.

Firm $I$ (the Incumbent) is already on the market, while firm $E$ (the Entrant) has to decide whether to enter or not.

1) Assume the following *timing:* at time $t$=1, $I$ decides whether to accommodate ($A$) or to deter ($NA$, i.e. choose a quantity such that firm $E$ makes zero profit) entry by firm $E$; at time $t$=2, the Entrant decides whether to enter ($E$, in this case it pays a fixed entry cost $F$=1/64) or not ($NE$).

a. Represent the extended form of the game.

b. Through backward induction, find the Entrant's profit at the second stage, in case it enters and it does not enter.

c. Compute the quantities produced by the two firms in case the Incumbent accommodates entry by firm $E$.

d. Compute the quantities produced by the two firms in case the Incumbent deters entry by firm $E$.

e. Compute the two firms' profits in c. and d.?

f. Find the subgame perfect equilibrium.

2) Assume now a different *timing*: at time $t$=1 the Entrant states irrevocably its decision on whether to enter the market ($E$) or not ($NE$); at time $t$=2 the Incumbent accommodates ($A$) or deter ($NA$, i.e. start a price war) entry by firm $E$.

a. Represent the extended form of the game.

b. Find the two firms' profits in case firm $E$ enters and does not enter.

c. From the Incumbent's point of view, is it better if the Entrant enters or not?

d. Do you think that the Incumbent's threat to deter actually discourages entry by the Entrant? Why?

**Exercise 4.11***

There is a market populated by two types of consumers: $l$ (corresponding to a share λ of the total population) and $h$ (corresponding to a share 1- λ of the population).

Consumers of any kind buy at the most one unit of good $A$, produced by monopolist Gamma, and $q_i$ $i=\{l,h\}$ units of good $B$, for which firm Gamma competes à la Bertrand with a few rivals.

For simplicity, assume zero production costs for both goods.

Assume also that the demand good $B$ is $q_l=v_l(1-p_B)$, for the $l$-type consumers, and $q_h=v_h(1-p_B)$ for the $h$-type consumers, with $v_l<v_h$.

1) Find the condition according to which consumers buy both goods.

2) Suppose that firm Gamma sells $A$ and $B$ independently to both types of consumers.

a. What is the optimal price for the two goods?

b. Compute firm Gamma's profit and consumers' surplus.

c. Compute total surplus.

3) Suppose now that firm Gamma sells $A$ and $B$ independently, but only to the high demand consumers.

a. What is the optimal price for the two goods?
b. Compute firm Gamma's profit and consumers' surplus.
c. Compute total surplus.

4) Suppose that firm Gamma forces consumers acquiring good *A*, to buy also good *B*.
a. What kind of foreclosure is this?
b. Write down the expression for firm Gamma's profit.
c. What is the optimal price for the two goods?

**Exercise 4.12**

Consider a two stage simultaneous entry game. In the first stage, $m$ potential entrants decide whether to enter or not a market with demand $p=a-q$ where $p$ is the price and $q$ denotes total quantity. In case of entrance, every firm has to pay a fixed sunk cost $F$ but it entails zero production costs. In the second stage, the $n$ firms that have entered the market compete à la Cournot.

1) Derive the individual and total quantities, the price and profits at the second stage, as a function of $n$.
2) Compute the optimal number of firms in the long run equilibrium with free entry. How does the number of firms change with market size and fixed cost $F$?

**Exercise 4.13**

Consider $n$ firms competing à la Bertrand, with marginal (and average) costs constant and equal to $c$. The demand function is linear and given by $p=a-q$, where $p$ is the price and $q$ denotes total quantity.

1) Derive the individual and total quantities, the prices and the profits in equilibrium.

2) Assume a previous stage in which $m \geq n$ potential entrants decide whether to enter or not the above market. If entry involves a fixed cost $F>0$, how are many firms are there in the market? Why?

**4. ENTRY, FORECLOSURE – solutions**

**Exercise 4.1**

1) To derive $D_i$ we first identify the consumer indifferent between buying variety *i* and buying variety *i-1* (called $t_{i,i-1}$) and the consumer indifferent between buying variety *i* and buying variety *i+1* (called $t_{i,i+1}$).

$$t_{i,i-1}: u^* - p_i - (x_i - t)^2 = u^* - p_{i-1} - (x_{i-1} - t)^2$$

$$u^* - p_i - \left(\frac{i}{n} - t\right)^2 = u^* - p_{i-1} - \left(\frac{i-1}{n} - t\right)^2$$

This yields:

$$t_{i,i-1} = \frac{p_i - p_{i-1} + \frac{2i-1}{n^2}}{\frac{2}{n}}$$

$$t_{i,i+1}: u^* - p_i - (x_i - t)^2 = u^* - p_{i+1} - (x_{i+1} - t)^2$$

$$u^* - p_i - \left(\frac{i}{n} - t\right)^2 = u^* - p_{i+1} - \left(\frac{i+1}{n} - t\right)^2$$

This yields:

$$t_{i,i+1} = \frac{p_{i+1} - p_i + \frac{2i+1}{n^2}}{\frac{2}{n}} \text{i}$$

As a result:

$$D_i = t_{i,i+1} - t_{i,i-1} = \frac{p_{i+1} + p_{i-1} - 2p_i + \frac{2}{n^2}}{\frac{2}{n}}$$

2) Firm *i* solves the following maximization problem:

$$\max_{p_i} \pi_i = p_i D_i = p_i \frac{p_{i+1} + p_{i-1} - 2p_i + \frac{2}{n^2}}{\frac{2}{n}}$$

$$FOC_{p_i}: \frac{p_{i+1} + p_{i-1} - 2p_i + \frac{2}{n^2}}{\frac{2}{n}} - \frac{2p_i}{\frac{2}{n}} = 0$$

Given that the $n$ firms are symmetric on the cost side: $p_{i+1} = p_i = p_{i-1}$. Imposing this condition in the $FOC_{p_i}$ we get:

$$p_i^* = \frac{1}{n^2}$$

Substituting into $D_i$:

$$D_i^* = \frac{1}{n}$$

Firm $i$'s profit is given by:

$$\pi_i^* = \frac{1}{n^3}$$

3) New firms keep entering the market as long as they make a positive profit. Therefore, we impose the zero profit condition:

$$\frac{1}{n^3} - F = 0$$

Which yields $n^*=5$.

Since $n^*$ has to be an integer number, each firm makes a positive (although very small) profit: $\frac{1}{125} - \frac{1}{150}$.

**Exercise 4.2***

1) The monopolist charges the highest feasible price in line with the participation constraint of the most distant consumer, located at ½. Doing this way, it maximizes its profit and serves the entire market. So we impose:

$$4 - p - \frac{1}{4} = 0$$

As a result:

$$p^M = \frac{15}{4}$$

$$\pi^M = \frac{15}{4}$$

2) Firm $M$ and firm $E$ compete à la Bertrand in 1:

$$p_1^M = p_1^E = 0$$

$$\pi_1^M = \pi_1^E = 0$$

Now we need to compute firm $M$'s profit from selling variety 0. We first identify the consumer indifferent between variety 0 and 1:

$$t_{0,1} : 4 - p_0^M - (0-t)^2 = 4 - 0 - (1-t)^2$$

This yields:

$$t_{0,1} = \frac{1 - p_0^M}{2}$$

$$D_0^M = \frac{1 - p_0^M}{2}$$

Firm $M$ solves the following maximization problem:

$$\max_{p_0^M} \pi_0^M = p_0^M D_0^M = p_0^M \frac{1 - p_0^M}{2}$$

$$FOC_{p_0^M} : \frac{1 - p_0^M}{2} - \frac{p_0^M}{2} = 0$$

As a result:

$$p_0^{M*} = \frac{1}{2}$$

$$D_0^{M*} = \frac{1}{4}$$

$$\pi_0^{M*} = \pi^{M*} = \frac{1}{8}$$

3) Here we find again the consumer indifferent between variety 0, offered by $M$, and variety 1, offered only by $E$:

$$t_{0,1}^{'} : 4 - p_0^M - (0-t)^2 = 4 - p_1^E - (1-t)^2$$

This yields:

$$t'_{0,1} = \frac{p_1^E - p_0^M + 1}{2}$$

$$D_0^M = \frac{p_1^E - p_0^M + 1}{2}$$

$$D_1^E = \frac{1 - p_1^E + p_0^M}{2}$$

Firm *M* solves the following maximization problem:

$$\max_{p_0^M} \pi_0^M = p_0^M D_0^M = p_0^M \frac{p_1^E - p_0^M + 1}{2}$$

$$FOC_{p_0^M} : \frac{p_1^E - p_0^M + 1}{2} - \frac{p_0^M}{2} = 0$$

Firm *M*'s best reply function is:

$$p_0^M = \frac{p_1^E + 1}{2}$$

Firm *E* solves the following maximization problem:

$$\max_{p_1^E} \pi_1^E = p_1^E D_1^E = p_1^E \frac{1 - p_1^E + p_0^M}{2}$$

$$FOC_{p_1^E} : \frac{p_0^M - p_1^E + 1}{2} - \frac{p_1^E}{2} = 0$$

Firm *E*'s best reply function is:

$$p_1^E = \frac{p_0^M + 1}{2}$$

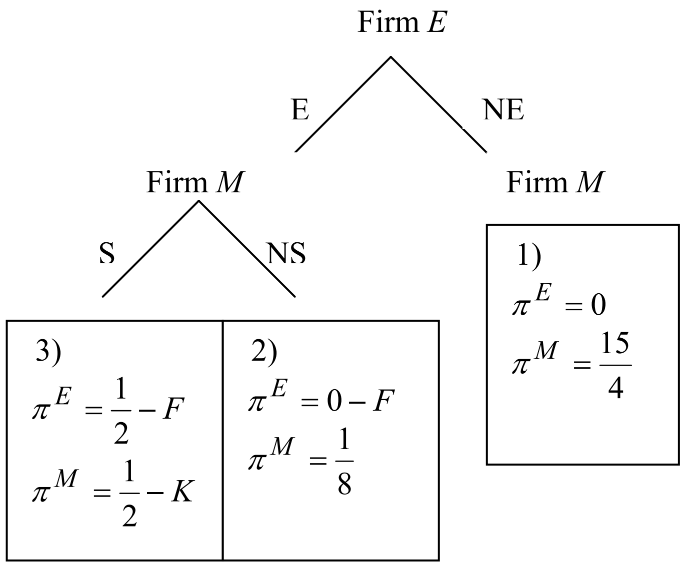
Solving the system of best replies, we get:

$$p_0^{M**} = p_1^{E**} = 1$$

$$D_0^{M**} = D_1^{M**} = \frac{1}{2}$$

$$\pi_0^{M**} = \pi^{M**} = \pi_1^{E**} = \pi^{E**} = \frac{1}{2}$$

4) The extended form of the game is as follows:




Firm $E$ finds it profitable to enter the market if and only if:

$$K \leq \frac{3}{8} \text{ and } F \leq \frac{1}{2}$$

Firm $E$ does not find it profitable to enter if and only if:

$$K > \frac{3}{8} \text{ or } K \leq \frac{3}{8} \text{ and } F \geq \frac{1}{2}$$

**Exercise 4.3**

Through backward induction, at stage 2, firm $i$ competes à la Cournot with $n$-1 competitors, and it solves the following maximization problem:

$$\max_{q_i} \pi_i = pq_i = \left(1 - \sum_{j=1}^{n} q_j\right) q_i$$

$$FOC_{q_i}: -q_i + 1 - Q = 0$$

By symmetry, we impose $Q=nq_i$, therefore:

$$q_i^* = \frac{1}{1+n}$$

$$p^* = \frac{1}{1+n}$$

$$\pi_i^* = \frac{1}{(1+n)^2}$$

At stage1, based on the zero profit condition:

$$\pi_i^* - F = 0$$

$$\frac{1}{(1+n)^2} - \frac{1}{36} = 0$$

As a results, there are $n$*=5 firms in the long run equilibrium with free entry. Each of them makes zero profit.

**Exercise 4.4**
Moving backward, at stage 2, firm 2 has to decide whether to:

Enter (i.e. $K_2$>0 and $F$>0)
or
Non to enter (i.e. $K_2$=0 and $F$=0)

We first compute firm 2's profits in the two cases.

If firm 2 enters, it solves the following maximization problem:

$$\max_{K_2} \pi_2 = (4 - K_1 - K_2)K_2$$

$$FOC_{K_2} : -K_2 + 4K_1 - K_2 = 0$$

$$K_2^{E*}(K_1) : K_2^E = \frac{4 - K_1}{2}$$

$$\pi_2^E = \frac{(4 - K_1)^2}{4} - \frac{1}{16}$$

If firm 2 does not enter:

$$K_2^{NE} = 0$$

$$\pi_2^{NE} = 0$$

At stage 1, firm 1 has to decide whether to:

Accommodate (this results in a Stackelberg game)
or
Deter (i.e. $K_1$ is such that $\pi_2^E = 0$)

We first compute firm 1's profits in the two cases.

If firm 1 accommodates, it solves the following maximization problem:

$$\max_{K1} \pi_2 = \left(4 - K_1 - \frac{4 - K_1}{2}\right) K_1$$

As a result:

$$K_1^A = 2$$

$$K_2^A = 1$$

$$\pi_1^A = 2$$

If firm 1 deters, we impose $\pi_2^E = 0$ to derive the followings:

$$K_1^{NA} = \frac{7}{2}$$

$$K_2^{NA} = 0$$

$$\pi_1^{NA} = \frac{7}{4}$$

Given $\pi_1^{NA} < \pi_1^A$, firm 1 accommodates entry by firms 2, pushing the equilibrium towards its most preferable option. This is possible due to firm 1's first mover advantage.

**Exercise 4.5***

1) Moving by backward induction:

At stage 2, firm *E* decides whether to enter or not:

If it does not enter, $\pi_E^{NE} = 0$

If it enters, firm *E* solves the following maximization problem:

$$\max_{q_E} \pi_E^E = (1 - q_I - q_E) q_E - F^2$$

From $FOC_{q_E}$ we get:

$$q_E^*(q_I) : q_E = \frac{1 - q_I}{2}$$

$$\pi_E^E = \left(\frac{1 - q_I}{2}\right)^2 - F^2$$

At stage 1, firm *I* decides whether to accommodate or to deter:

If it accommodates, firm *I* compete à la Stackelberg with firm 2, thus solving the following maximization problem:

$$\max_{q_I} \pi_I^A = \left(1 - q_I - \frac{1 - q_I}{2}\right) q_I$$

So:

$$q_I^A = \frac{1}{2}$$

$$q_E = \frac{1}{4}$$

Firm *I*'s profit, in this case, equals:

$$\pi_I^A = \frac{1}{8}$$

2) If firm *I* deters, it produces a quantity $q_I^{NA}$ such that the rival makes zero profit. Therefore:

$$\left(\frac{1 - q_I}{2}\right)^2 - F^2 = 0$$

This yields:

$$q_I^{NA} = 1 - 2F$$

$q_E$=0

Firm *I*'s profit, in this case, equals:

$$\pi_I^{NA} = 2F(1 - 2F)$$

3) Firm *I* accommodates if and only if $\pi_I^A \geq \pi_I^{NA}$, i.e.:

$$4F^2 - 2F + \frac{1}{8} \leq 0$$

This is true for:

$$F \leq \frac{2 - \sqrt{2}}{4} \vee F \geq \frac{2 + \sqrt{2}}{4}$$

Given the interval for $F$, the above inequality is verified for:

$$F \leq \frac{2-\sqrt{2}}{8}$$

4) Moving backward, at the second stage, firms compete à la Cournot, having a best reply function analogous to that of 1):

$$q_E = \frac{1-q_I}{2}$$

By symmetry:

$$q_E^* = \frac{1}{3} = q_I^*$$

$$\pi_E^* = \frac{1}{9} - F^2 = \pi_I^*$$

At the first stage, through the zero profit condition, we get:

$$\frac{1}{9} - F^2 \geq 0$$

If $F^2<1/16$, $F^2<1/9$, (indeed 1/16<1/9) therefore both firms enter the market.

**Exercise 4.6**

1) First we identify the consumer indifferent between varieties 1 and 2:

$$t_{1,2} : u^* - p_1 - \left(\frac{1}{3} - t\right)^2 = u^* - p_2 - \left(\frac{2}{3} - t\right)^2$$

$$t_{1,2} = \frac{3}{2}(p_2 - p_1) + \frac{1}{2}$$

Second we identify the consumer indifferent between varieties 2 and 3:

$$t_{2,3} : u^* - p_3 - (1-t)^2 = u^* - p_2 - \left(\frac{2}{3} - t\right)^2$$

$$t_{2,3} = \frac{3}{2}(p_3 - p_2) + \frac{5}{6}$$

Third we identify the consumer indifferent between varieties 1 and 3:

$$t_{1,3} : u^* - p_1 - \left(\frac{1}{3} - t\right)^2 = u^* - p_3 - (0-t)^2$$

$$t_{1,3} = \frac{3}{2}(p_1 - p_3) + \frac{1}{6}$$

Demands for the three varieties, as a function of the prices, are as follows:

$$D_2 = \beta(t_{2,3} - t_{1,2}) = \beta\left[\frac{3}{2}\left(p_3 + p_1 - 2p_2) + \frac{1}{3}\right)\right]$$

$$D_3 = \beta(1 - t_{2,3} + t_{1,3}) = \beta\left[\frac{3}{2}\left(p_2 + p_1 - 2p_3) + \frac{1}{3}\right)\right]$$

$$D_1 = \beta(t_{1,2} - t_{1,3}) = \beta\left[\frac{3}{2}\left(p_2 + p_3 - 2p_1) + \frac{1}{3}\right)\right]$$

2) Since firms are symmetric on the cost side, we can focus only on one of them (say firm 2) and then reason by symmetry.
Firm 2 solves the following maximization problem:

$$\max_{p_2} \pi_2 = p_2 D_2 = p_2 \beta\left[\frac{3}{2}(p_3 + p_1 - 2p_2) + \frac{1}{3}\right]$$

From $FOC_{p_2}$ we get:

$$p_1^* = p_2^* = p_3^* = \frac{1}{9}$$

$$D_1^* = D_2^* = D_3^* = \frac{1}{3}\beta$$

$$\pi_1^* = \pi_2^* = \pi_3^* = \frac{1}{27}\beta$$

3) Call *I* the firm resulting from the merger between 1 and 2. Firm *I* competes in prices with firm 3.

Firm *I* solves the following maximization problem:

$$\max_{p_1, p_2} \pi_I = D_1 p_1 + D_2 p_2 = \beta\left[\frac{3}{2}(p_2 + p_3 - 2p_1) + \frac{1}{3}\right]p_1 + \beta\left[\frac{3}{2}(p_3 + p_1 - 2p_2) + \frac{1}{3}\right]p_2$$

From first order conditions, we obtain:

$$p_1 = \frac{3p_2 + \frac{3}{2}p_3 + \frac{1}{3}}{6}$$

$$p_2 = \frac{3p_1 + \frac{3}{2}p_3 + \frac{1}{3}}{6}$$

By symmetry: $p=p_1=p_2$, therefore:

$$p^*(p_3): p = \frac{1}{2}p_3 + \frac{1}{9}$$

Firm 3 solves the following maximization problem:

$$\max_{p_3} \pi_3 = D_3 p_3 = \beta\left[\frac{3}{2}(p_2 + p_1 - 2p_3) + \frac{1}{3}\right]p_3$$

Which yields:

$$p_3^*(p): p_3 = \frac{1}{2}p + \frac{1}{18}$$

Solving the system of best replies:

$$p^{**} = \frac{5}{27}$$
$$p_3^{**} = \frac{4}{27}$$

The more concentrated the market, the higher the prices charged by the two firms.

$$D^{**} = D_1^{**} + D_2^{**} = \beta\frac{5}{9}$$

$$D_3^{**} = \beta\frac{4}{9}$$

$$\pi^{**} = \beta\frac{5}{9}\frac{5}{27}$$

$$\pi_3^{**} = \beta\frac{4}{9}\frac{4}{27}$$

4) There are incentives to merge because firms 1 and 2 make higher profits: $\pi_1^{**} = \pi_2^{**} = \frac{\pi^{**}}{2} = \beta\frac{5}{9}\frac{5}{27}\frac{1}{2} = \beta 0{,}051 > \beta 0{,}037 = \beta\frac{1}{27} = \pi_1^* = \pi_2^*$.

Also firm 3, benefits from the merger: $\pi_3^{**} = \beta\frac{4}{9}\frac{4}{27} = \beta 0{,}066 > \beta 0{,}037 = \beta\frac{1}{27} = \pi_3^*$.

**Exercise 4.7***

1) The consumer that is indifferent between varieties 1 (located in 0) and 2 (located in 1) is derived as follows.

$$t_{1,2} : u^* - p_1 - (x_1 - t)^2 = u^* - p_2 - (x_2 - t)^2$$

This yields:

$$t_{1,2} = \frac{1 + p_2 - p_1}{2}$$

$$D_1 = t_{1,2} = \frac{1 + p_2 - p_1}{2}$$

$$D_2 = 1 - t_{1,2} = \frac{1 - p_2 + p_1}{2}$$

2) The monopolist chooses the highest possible price in line with the participation constraint of the most distant consumer. Therefore:

$$u^* - p - \left(0 - \frac{1}{2}\right)^2$$

$$p^* = u^* - \frac{1}{4}$$

If possible, the monopolist would choose locations 1/4 and3/4, to charge a higher price ($u^*$-1/16).

3) The consumer that is indifferent between varieties 1 and 3 is derived as follows:

$$t_{1,3} : u^* - p_1 - (x_1 - t)^2 = u^* - p_3 - (x_3 - t)^2$$

This yields:

$$t_{1,3} = p_3 - p_1 + \frac{1}{4}$$

The consumer that is indifferent between varieties 2 and 3 is derived as follows:

$$t_{2,3} : u^* - p_3 - (x_3 - t)^2 = u^* - p_2 - (x_2 - t)^2$$

This yields:

$$t_{2,3} = p_2 - p_3 + \frac{3}{4}$$

$$D_1 = t_{1,3} = p_3 - p_1 + \frac{1}{4}$$

$$D_3 = t_{2,3} - t_{1,3} = p_2 + p_1 - 2p_3 + \frac{1}{2}$$

$$D_2 = 1 - t_{2,3} = p_3 - p_2 + \frac{1}{4}$$

Firm 1 solves the following maximization problem:

$$\max_{p_1} \pi_1 = p_1 D_1 = p_1\left(p_3 - p_1 + \frac{1}{4}\right)$$

From $FOC_{p_1}$ we get:

$$p_1^*(p_3): p_1 = \frac{p_3}{2} + \frac{1}{8}$$

Through analogous reasoning:

$$p_2^*(p_3): p_2 = \frac{p_3}{2} + \frac{1}{8}$$

$$p_3^*(p_1, p_2): p_3 = \frac{p_1 + p_2}{4} + \frac{1}{8}$$

Solving the system of best replies:

$$p_3^{**} = p_2^{**} = p_1^{**} = \frac{1}{4}$$

4) The entrant's profit is:

$$\pi_3^{**} = \frac{1}{8} - F$$

with $F = \frac{1}{6}$ the entrant's profit is negative, so it does not find it profitable to enter the market.

5) If the entrant locates in 0, the two firms compete à la Bertrand there. As a result:

$$p_1^{***} = p_3^{***} = 0$$
$$\pi_1^{***} = \pi_3^{***} = 0$$

The consumer that is indifferent between varieties 0 and 1 is computed as follows:

$$t_{1,2} : u^* - 0 - (0 - t)^2 = u^* - p_2 - (1 - t)^2$$

$$t_{1,2} = \frac{p_2 + 1}{2}$$

$$D_2 = 1 - t_{1,2} = \frac{1 - p_2}{2}$$

The incumbent solves the following maximization problem:

$$\max_{p_2} \pi_2 = p_2 \frac{1 - p_2}{2}$$

So:

$$p_2^{****} = \frac{1}{2}$$

$$\pi_2^{****} = \frac{1}{8} \text{ (incumbent's profit)}$$

$$\pi_3^{****} = 0 \text{ (entrant's profit)}$$

If the monopolist withdraws the variety in 0, the demand function would be equal to those of 1). (The only difference is that varieties 2 and 3 are involved, instead of 1 and 2).

The incumbent solves the following maximization problem:

$$\max_{p_2} \pi_2 = p_2 \frac{1 - p_2 + p_3}{2}$$

From $FOC_{p_2}$ we get:

$$p_2^*(p_3) : p_2 = \frac{1 + p_3}{2}$$

By symmetry:

$$p_2^{*****} = 1 = p_3^{*****}$$

$$\pi_2^{*****} = \frac{1}{2} = \pi_3^{*****}$$

Both firms prefer this situation, compared to the previous one, because the make higher profits.

**Exercise 4.8**

1) For details and computations see Exercise 4.4. Here we simply restore the main results.
The demand for variety *i*, as a function of the prices, is:

$$D_i = \frac{\frac{1}{n} + (p_{i+1} + p_{i-1} - 2p_i)}{\frac{2}{n}}$$

2) $p_i^* = \frac{1}{n^2}$; $D_i^* = \frac{1}{n}$; $\pi_i^* = \frac{1}{n^3}$

3) Since the number of firms has to be an integer, $n^*$=5. As a result, each firm makes a positive (although very small) profit in equilibrium: 1/125-1/200=0.003.

**Exercise 4.9**

1) The monopolist solves the following maximization problem:

$$\max_K \pi = (6 - K)K - 3{,}5K$$

From $FOC_K$, we get:

$K^*$=1

(Remember that the number of plants must be an integer)

2) Each duopolist (for instance firm 1) solves the following maximization problem:

$$\max_{K_1} \pi_1 = (6 - K_1 - K_2)K_1 - 3{,}5K_1$$

From first order conditions, reasoning by symmetry, we get:

$$K_1^{**} = K_2^{**} = 1$$

(Remember that the number of plants must be an integer)

3) Consider now the sequential entry game: at stage 1, firm 1 decides whether to accommodate or to deter entry by the rival; at stage 2, firm 2 decides whether to enter or not. The game is solved by backward induction:

Stage 2:

If firm 2 enters (i.e. $K_2$>0), it solves the following maximization problem:

$$\max_{K_2} (6 - K_1 - K_2)K_2 - 3{,}5K_2$$

From $FOC_{K_2}$ we derive:

$$K_2^*(K_1) : K_2 = \frac{2{,}5 - K_1}{2}$$

Therefore:

$$\pi_2^E = (6 - K_1 - K_2^*(K_1))K_2^*(K_1) - 3{,}5K_2^*(K_1)$$

If firm 2 does not enter (i.e. $K_2$=0) it makes zero profit:

$$\pi_2^{NE} = 0$$

Stage 1:
If firm 1 deters, $K_1$ is such that firm 2 makes zero profit:

$$\left(6 - K_1 - \frac{2{,}5 - K_1}{2}\right)\frac{2{,}5 - K_1}{2} - 3{,}5\frac{2{,}5 - K_1}{2} = 0$$

Therefore:

$$K_1^{NA} = 2$$

$$K_2^{NA} = 0$$

Firm 1's profit, in this case, is:

$$\pi_1^{NA} = 1$$

If firm 1 accommodates, it compete à la Stackelberg with firm 2, solving the following maximization problem:

$$\max_{K_1} \pi_1 = \left(6 - K_1 - \frac{2{,}5 - K_1}{2}\right)K_1 - 3{,}5K_1$$

This yields:

$$K_1^A = 1$$

$$K_2^A = 1$$

Firm 1's profit, in this case, is:

$$\pi_1^A = 0{,}5$$

Comparing $\pi_1^A$ and $\pi_1^{NA}$, it is possible to conclude that firm 1 deters entry by firm 2. Therefore, in equilibrium:

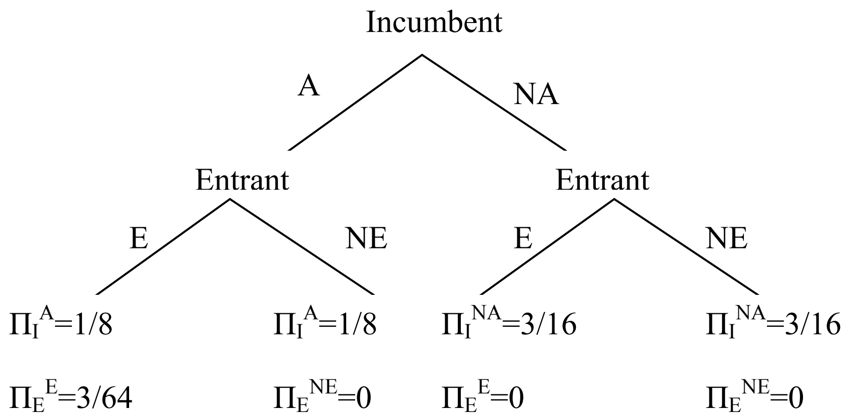
$$K_1^{***} = 2$$

$$K_2^{***} = 0$$

**Exercise 4.10***

1)

a. The extended for of the game is as follows:
(Profits correspond to the next points in the exercise)




b. At the second stage, if the Entrant does not enter (*NE*), it produces 0, pays no entry costs, and makes profit $\pi_E^{NE} = 0$
If the Entrant enters (*E*), it solves the following maximization problem:

$$\max_{q_E} \pi_E = (1 - q_I - q_E)q_E - F$$

From $FOC_{q_E}$ we get:

$$q_E^*(q_I) : q_E = \frac{1 - q_I}{2}$$

$$\pi_E^E = \frac{(1 - q_I)^2}{4} - \frac{1}{64}$$

c. At the first stage, if the Incumbent accommodates (*A*), it solves the following maximization problem (as a Stackelberg leader):

$$\max_{q_I} \pi_I = (1 - q_I - \frac{1 - q_I}{2})q_I$$

From $FOC_{q_I}$ :

$$q_I^A = \frac{1}{2}$$

$$q_E^E = \frac{1}{4}$$

$$\pi_I^A = \frac{1}{8}$$

$\pi_E^E = \frac{3}{64}$

d. At the first stage, if the Incumbent deters (*N*A), it sets a quantity such that the Entrant's profit is 0:

$$\frac{(1-q_I)^2}{4} - \frac{1}{64} = 0$$

Therefore:

$$q_I^{NA} = \frac{3}{4}$$

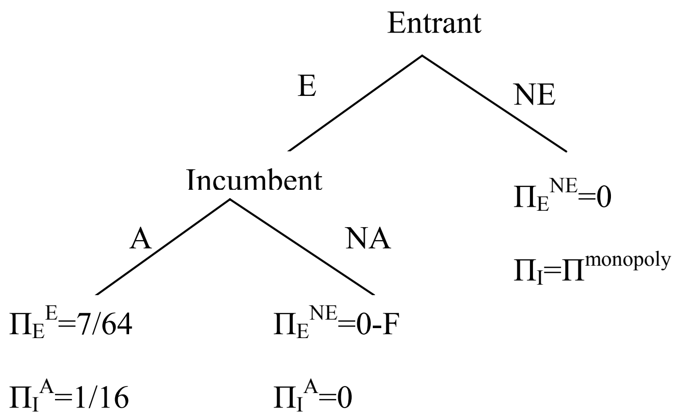
$$q_E^E = 0$$

$$\pi_E^E = 0$$

$$\pi_I^{NA} = \frac{3}{16}$$

e. The subgame perfect equilibrium is derived by backward induction: {NA; (E, NE)}

2)
a. The extended for of the game is as follows:
(Profits correspond to the next points in the exercise)




b. Moving backward, at the second stage, the Incumbent decides whether to accommodate or to deter.
If the Incumbent accommodates, it solves the following maximization problem:

$$\max_{q_I} \pi_I = (1 - q_I - q_E) q_I$$

From $FOC_{q_I}$:

$$q_I^*(q_E): q_I = \frac{1-q_E}{2}$$

$$\pi_I^A = \frac{(1-q_E)^2}{4}$$

If the Incumbent deters, it starts a price war, choosing $p$=$MC$=0. As a result: $\pi_I^{NA} = 0$.

At the first stage, the Entrant decides whether to enter or not.
If the Entrant enters, it solves the following maximization problem:

$$\max_{q_E} \pi_E = (1-q_E-\frac{1-q_E}{2})q_E - F$$

From $FOC_{q_E}$ we get:

$$q_E^E = \frac{1}{2}$$

$$q_I^A = \frac{1}{4}$$

$$\pi_E^E = \frac{7}{64}$$

$$\pi_I^A = \frac{1}{16}$$

c. From the Incumbent's point of view it is better if the Entrant does not enter. In this case the Incumbent makes monopoly profit.

d. The Incumbent's threaten to deter is not credible: once the Entrant has entered, the Incumbent finds it profitable to accommodate, rather than deter, to earn more. Anticipating this, the Entrant enters and the subgame perfect equilibrium is $\{E; A\}$.

**Exercise 4.11***
1) Consider first $l$-type consumers:

$$CS_l = \frac{(1-p_B)^2 v_l}{2} - p_A \geq 0$$

Consider then $h$-type consumers:

$$CS_h = \frac{(1-p_B)^2 v_h}{2} - p_A \geq 0$$

2)

a. Production of good $B$ results from Bertrand competition, so: $p_B^{NT} = MC_B = 0$.
As far as good $A$ is concerned, making $l$-type consumers' surplus equal to zero, we obtain:

$$p_A^{NT} = \frac{v_l}{2}$$

b. Consumers' surplus is:

$$CS^{NT} = \lambda CS_l^{NT} + (1-\lambda)CS_h^{NT} = \frac{(1-\lambda)(v_h - v_l)}{2}$$

The overall profit of firm Gamma, from selling good $A$ and good $B$ is:

$$\pi^{NT} = \frac{v_l}{2}$$

c. Total surplus is:

$$W^{NT} = \frac{(1-\lambda)v_h + \lambda v_l}{2}$$

3)
a. Production of good $B$ results from Bertrand competition, so: $p_B^{NTh} = MC_B = 0$.
As far as good $A$ is concerned, making $h$-type consumers' surplus equal to zero, we obtain:

$$p_A^{NTh} = \frac{v_h}{2}$$

b. Consumers' surplus is:

$$CS^{NTh} = 0$$

The overall profit of firm Gamma, from selling good $A$ and good $B$ is:

$$\pi^{NTh} = \frac{v_h(1-\lambda)}{2}$$

c. Total surplus is:

$$W^{NTh} = \frac{(1-\lambda)v_h}{2} < W^{NT}$$

4)
a. This i san example of tying.

b. Firm Gamma's profit equals:

$$\pi = p_B\left[\lambda v_l(1-p_B)+(1-\lambda)v_h(1-p_B)\right]+p_A$$

c. The equilibrium prices result from the following maximization problem:

$$\max_{p_B} \pi = p_B\left[\lambda v_l(1-p_B)+(1-\lambda)v_h(1-p_B)\right]+\frac{v_l(1-p_B)^2}{2}$$

From the first order conditions, we find:

$$p_B^T = \frac{(1-\lambda)(v_h-v_l)}{2v_h-v_l-2\lambda(v_h-v_l)}$$

This can be substituted into the expression for $p_A$, to derive the following:

$$p_A^T = \frac{v_l(1-\lambda)^2(v_h-v_l)^2}{2\left[2v_h-v_l-2\lambda(v_h-v_l)\right]^2}$$

**Exercise 4.12**

1) At the second stage, each firm solves the following maximization problem:

$$\max_{q_i} \pi_i = pq_i = (a-\sum_{j=1}^{n} q_j)q_i$$

By symmetry, from $FOC_{q_i}$ we get:

$$q_i^* = \frac{a}{n+1}$$

$$q^* = \frac{an}{n+1}$$

$$p^* = \frac{a}{n+1}$$

$$\pi_i^* = \frac{a^2}{(n+1)^2}$$

2) Imposing the zero profit condition:

$$\pi_i^* - F = 0$$

$$\frac{a^2}{(n+1)^2} - F = 0$$

As a result:

$$n^* = \frac{a}{\sqrt{F}} - 1$$

The higher the market size $a$ (it is at the numerator) the higher $n^*$; the higher the fixed cost $F$ (it is at the denominator) the lower $n^*$.

**Exercise 4.13**

1) In the Bertrand equilibrium, all firms charge the same price, equal to the marginal cost:

$p_i^* = c$ for $i$=1..$n$

Therefore:

$$q^* = a - c$$

$$q_i^* = \frac{a-c}{n}$$

$$\pi_i^* = 0$$

2) Based on previous results, with a positive entry cost, no firm enters the market.