Factor Asset Pricing and Statistics

A quick look

November 25, 2019

Intro

- One of the objectives of Statistics is to estimate "unknown components" (e.g. parameters) in "laws" that describe the joint probability distribution of observables
- For Statistics to be useful such laws must exist and be reasonably invariant to observation (in time, space etc.) such that you can "learn" them and then "use them to make forecasts"
- If such laws are also "stable under action" it may be possible to "make forecast conditional to actions"
- (To assess properties of distributions of observables given optional acts of "policy")

Intro

- By "law" we may intend some theory based statement or, more frequently, some empirically observed regularity (and some mix of both)
- Such "laws" are necessary for Statistics to be useful: data alone are not sufficient for Statistics to "work", the more so if data are "big"
- Statistics with no laws is just a confuse piling of data on data

Intro

- Consider, for instance, the case in which you have data on $n+1$ random variables and test at a level $\alpha = .05$ for the absence of correlation between one of these, say Y and the n others
- Even if all "true" correlations are 0, on average you shall reject the null $n * .05$ times
- With Big Data, say, 10000 variables (not so many) this amounts to identifying 500 variables which "explain" Y
- By regressing Y jointly on these 500 variables (same dataset) you'll get a quite big R^2

Intro

- This is not a good way to choose your variables
- It used to be called "data mining" in a fully negative sense
- From Wikipedia (Data mining):

> *In the 1960s, statisticians and economists used terms like data fishing or data dredging to refer to what they considered the bad practice of analyzing data without an a-priori hypothesis. [...]*
>
> *The term data mining appeared around 1990 in the database community, generally with positive connotations [...] Other terms used include data archaeology, information harvesting, information discovery, knowledge extraction, etc. [...] However, the term data mining became more popular in the business and press communities.*

Intro

- The field of Economics and Finance produces a terrible amount of stored data, this not withstanding, the amount of (known) "useful" laws in this field is quite scant

Intro

- This is not the place to discuss why but some sketch of hypothesis shall be useful
- A probable reason is that, in Economics, events happen because people act
- Now, the result of people acting is difficult to forecast, at least when people "reason" before acting
- In cases when people do not reason and decide on the basis of idiosyncratic "ticks" or maybe are constrained by necessity, fear or greed, things may become more manageable

Intro

- Let us focus on the “constrained by necessity” point
- In most cases there are many reasonable solutions to economic problems
- No solution is so clearly “the best” as to be the obvious choice
- However, there exist cases where some “solution” is so bad it generates an immediate and relevant damage if chosen
- We are free to choose such solutions, but this property is going to make such choice unlikely

Intro

- Consider an analogy
- People decide and act when, say, they cross a trafficked road or drive a car or climb a mountain or choose what to eat
- In all these examples there are many reasonable ways to solve the "problem"
- There also exist some clearly and immediately dangerous solutions which, arguably, tend to be avoided
- This constrains behaviours (exceptions are always possible) and make them follow specific and stable "laws"

Intro

- When "error", at least in the agent's understanding, has no immediate bad consequence laws tend to blur out

Intro

- This is not the sole origin of laws in fields the like of Economics and Finance
- Another possible one: humans have habits, ticks, attitudes, uses.
- Many of these are quite well widespread
- These may not be optimal solutions, may even be bad, but when not confronted by immediate bad consequences, are repeated and may imply regularities in behaviour, that is: laws
- (read, e.g. "Predictably Irrational" by Dar Ariely for many nice examples)

Intro

- We cannot go in more detail than this
- In a nutshell, if we want to find laws in Economics, it is probably reasonable to look for them where either necessity is strong or where repetitive behaviour abounds
- Be happy: Finance is a good field where to reap both

Intro

- As in the rest of Economics, in Finance too it is often difficult to state, a priori, whether a behaviour is "right" or "wrong"
- Random asset allocation (monkeys and typewriters) and fund managers, as a collective, cannot be distinguished except by their costs (fund managers, surprisingly, earn more than monkeys)
- But there is a limit which, sometimes, makes Finance similar to a trafficked road
- Most people do not like to be sure losers, at least not too often
- In Finance this becomes the "principle of no arbitrage"

Intro

- Joint to other characteristic of the marketplace, the like of frequent trading, low transaction costs and liquid markets, no arbitrage imposes strong constraints to relative prices
- These constraints are never too strong to be violated
- However, most interesting, sometimes it is possible to exploit such violations
- Most of asset pricing is to model the constraints no arbitrage imposes and to check if data satisfies the models

Intro

- In what follows I'll show you a hint of a relevant empirical, no arbitrage based, model in asset pricing: the "linear factor model"
- The presentation shall be completely informal and intuitive (the Maths very approximate)
- All the same, it should be useful in order to understand how interesting can be (and is, believe me) the interface between Finance and Statistics.
- (Or, if you prefer, "data science": a trick name statisticians use to try and get higher salaries.
- The like of "hairdresser" becoming "coiffure" or "hair stylist", or "garbage collection and disposal" mutating in "ecological processing", etc.)

Intro

- In this short introduction we shall not consider generic financial securities (we could but ... would be too complicated)
- We shall concentrate on the simple case of stocks
- However, most of what we shall say could be extended to general financial securities

Returns

- The first thing we must understand is that, while stocks are quoted in terms of prices, most financial models do not deal with prices but with returns

Returns

- There exist two kinds of returns:
- Let P_{it} be the price of the $i - th$ stock at time t.
- The linear or simple return between times t_j and t_{j-1} is defined as:

$$r_{it_j} = P_{it_j} / P_{it_{j-1}} - 1$$

- The log return is defined as:

$$r^*_{it_j} = \ln(P_{it_j} / P_{it_{j-1}})$$

- In both these definitions of return we do not consider possible dividends.
- There exist corresponding definitions of total return where, in the case a dividend D_j is accrued between times t_{j-1} and t_j, the numerator of both ratios becomes $P_{t_j} + D_j$.

Returns

- In particular, we are going to consider here linear returns. Why?
- From now on, for simplicity, let us only consider times t and $t-1$.
- Let the value of a buy and hold portfolio, composed of k stocks each for a quantity n_i, at time t be:

$$\sum_{i=1..k} n_i P_{it}$$

- It is easy to see that the linear return of the portfolio shall be a linear function of the returns of each stock.

Returns

- in fact:

$$r_t = \frac{\sum_{i=1..k} n_i P_{it}}{\sum_{j=1..k} n_j P_{jt-1}} - 1 = \sum_{i=1..k} w_{it} r_{it}$$

- Where $w_{it} = \frac{n_i P_{it-1}}{\sum_{j=1..k} n_j P_{jt-1}}$ are (non negative, if you are only "long") "weights" summing to 1 (we suppose $\sum_{j=1..k} n_j P_{jt-1} \neq 0$)
- These represent the percentage of the portfolio value invested in the i-th stock at time $t-1$
- In words: the linear return of a portfolio is a weighted sum of the linear returns of its components.

Returns

- This simple result is very useful.
- Suppose, for instance, that you know at time $t-1$ the expected values for the returns between time $t-1$ and t.
- Since the weights w_{it} are known, hence non stochastic, at time $t-1$ we can easily compute the return for the portfolio as:

$$E(r_t) = \sum_{i=1..k} w_{it} E(r_{it})$$

- Moreover if we know all the covariances between r_{it} and r_{jt} (if $i=j$ we simply have a variance) we can find the variance of the portfolio return as:

$$V(r_t) = \sum_{i=1..k} \sum_{j=1..k} w_i w_j Cov(r_{it}; r_{jt})$$

Why returns?

- This is a nice property.
- There are at least two other relevant reasons for using returns.
- The first is that they can, roughly, be compared thru different times, invested sums and securities.
- A 20 percent return over one year means “exactly the same” be the sum invested equal to 100 Euros or 1 billion Euros, be the year 2019 or 1919, and the stock I.B.M or Microsoft
- We could discuss at length the exact meaning of this “exactly the same”: it may seem trivial at first sight but, actually, this idea implies quite strong hypotheses, for example, on the utility function of agents.

Why returns?

- The second reason has to do with statistical invariance and independence thru time.
- Whatever the idea you have on stock prices models, it is arguably untenable that the price series be stationary (in any sensible sense) and that consecutive prices be independent
- On the other hand, the idea that returns may have a more stable distribution and be, roughly, independent thru time could be a fairly good first approximation to reality

Why returns?

- Appealing to intuition: we can "learn" about statistical properties of returns.
- The (log) random walk hypothesis is a formal statement of this idea.
- This is the hypothesis that log returns are uncorrelated in time with constant expected value and variance

Important note

- IMPORTANT NOTE: to simplify the notation in what follows we shall suppose that returns are expressed in the form of excess returns from a properly chosen risk free rate.

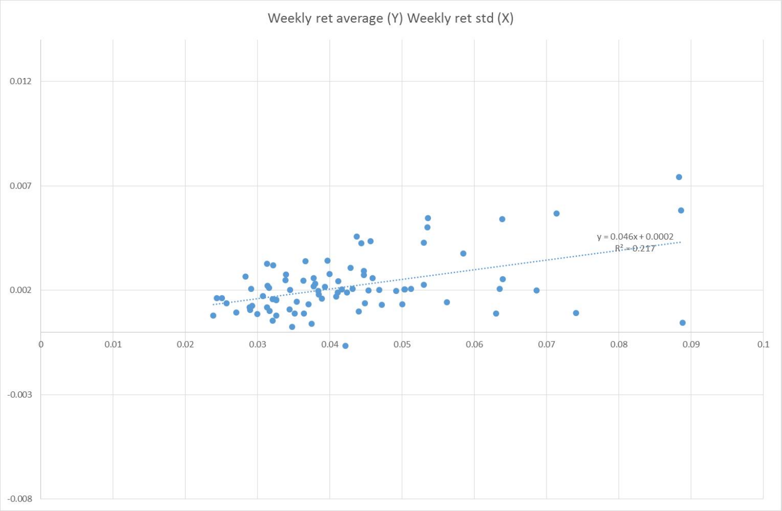
Portfolios and diversification

- Whatever the reason for using returns, the first and foremost "empirical puzzle" to solve is the fact that average returns of different stocks are wildly different, even when computed over long time intervals
- Notice the word "average": not "expected"
- Moreover, return (empirical) standard deviations are wildly different, too and, even more puzzling, it is NOT the case that average returns are proportional to standard deviations
- Why should this be puzzling?

Portfolios and diversification

- True: average return is a very approximate measure of "possible gain" the same for standard deviation
- And, remember, these are all "ex post" measures non "ex ante"
- It is however puzzling that we can, say, choose and buy a stock that has (better: had) a bigger return mean than another stock AND a smaller return standard deviation.
- See the following plot (weekly returns, 83 components S&P 100, Dec/31/2000-Jun/5/2019)

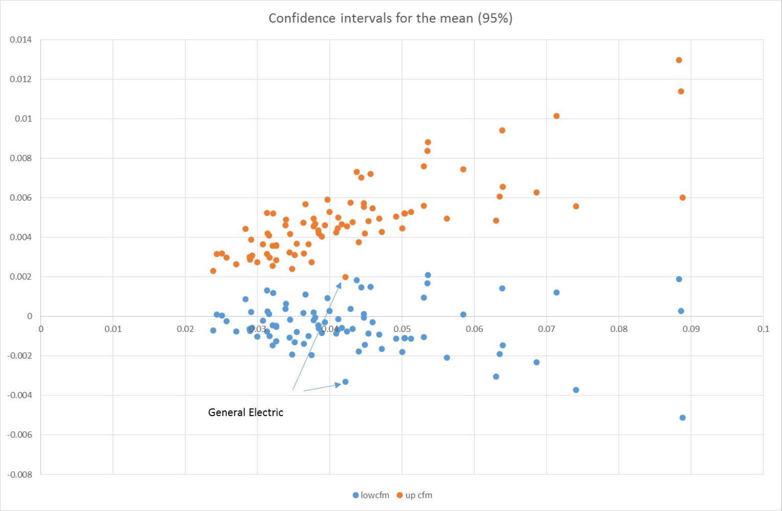
Portfolios and diversification



Portfolios and diversification

- Again: be careful, think in statistical terms and consider the following plot
- This contains $\pm 2\sigma$ confidence bounds for the expected returns on the basis of their estimates

Portfolios and diversification



Portfolios and diversification

- As you can see, while we have wildly different averages for the same std, we also have BIG confidence intervals for the expected returns
- We never see (with the almost exception of General Electric as compared with United Health Group), for a given STD, a LB of some estimate higher than the UB of some other (95% CI)
- So it could be all just sampling variability, but... forget this for the moment

Portfolios and diversification

- Suppose these ex post estimates are good for future forecasts
- Take any given standard deviation to which correspond several different means
- Why should we invest in the securities with lower means?
- Such securities exist and are priced, so...
- How is it possible that this be an equilibrium and for long periods of time?

Portfolios and diversification

- The answer comes from the first and most important intuition of Finance:
- You should never evaluate a security by itself, a correct evaluation can only be made within a portfolio
- The idea is simple, if you are free to buy/sell any quantity of any security due to the above results, your true "risk" becomes the portfolio risk
- This is in NOT "the sum" of the risk of its components
- Recall: $V(\sum_{i=1}^{n} w_i r_i) = \sum_{i=1}^{n} w_i^2 V(r_i) + \sum_{i \neq j=1}^{n} w_i w_j Cov(r_i, r_j)$

Portfolios and diversification

- You should not be interested in the expected return of the single security but in that of the portfolio
- The specific standard deviation of each single security return is not your risk, your risk is the standard deviation of the portfolio return
- Now, as seen above, while the expected return of a portfolio is the "average" of the expected returns, this is by no means true for the standard deviation.
- What are the implications of all this?

The beta model

- Consider the case where we have and already existing portfolio whose return is r_π
- We consider the addition to this portfolio of a small fraction ω of investment in a stock with return r_s
- Using a regression we can always write $r_s = \alpha_s + \beta_s r_\pi + e_s$
- Where e_s is orthogonal to (uncorrelated with) r_π and $E(e_s) = 0$

The beta model

- We have $E(\omega r_s + (1-\omega) r_\pi) = \omega E(r_s) + (1-\omega) E(r_\pi)$ so that the change in expected return due to the introduction of r_s is $\omega(E(r_s) - E(r_\pi)) = \omega(\alpha_s + (\beta_s - 1)E(r_\pi))$
- On the other hand we have $V(\omega r_s + (1-\omega) r_\pi) = V(\omega(\alpha_s + \beta_s r_\pi + e_s) + (1-\omega) r_\pi)$ that is $\omega^2 V(e_s) + (\omega\beta_s + 1 - \omega)^2 V(r_\pi)$
- If ω is small, ω^2 is very small and we can drop the first term and get $Sd((\omega r_s + (1-\omega) r_\pi) \simeq (\omega\beta_s + 1 - \omega) Sd(r_\pi)$ so that the change in the portforlio standard deviation due to the addition of the new stock is, approximately, $\omega(\beta_s - 1) Sd(r_\pi)$

The beta model

- Now consider what would happen if $\omega(E(r_s) - E(r_\pi)) \neq c\omega(\beta_s - 1)Sd(r_\pi)$, that is: $(E(r_s) - E(r_\pi)) \neq c(\beta_s - 1)Sd(r_\pi)$ where c is a constant **independent** from s
- To fix the ideas consider two securities: s and s' and suppose $(E(r_s) - E(r_\pi))/(\beta_s - 1)Sd(r_\pi) >$ $(E(r_{s'}) - E(r_\pi))/(\beta_{s'} - 1)Sd(r_\pi)$ (that is $c_s > c_{s'}$) with both numerators and both denominators positive
- This would imply that, by adding s instead of s' to the portfolio, for the same fraction, the "increase in expected value per unit of increase of the risk" would always be bigger
- Why should I ever consider adding s' to my portfolio? Why should there be a market for s'?

The beta model

- Notice some implications of this hypothesis:
- If r_s is the excess return of an investment at risk free rate, that is 0, the above condition becomes $E(r_\pi) = cSd(r_\pi)$ so that the requirement $(E(r_s) - E(r_\pi)) = c(\beta_s - 1)Sd(r_\pi)$ is equivalent to $E(r_s) = c\beta_s Sd(r_\pi) = \beta_s E(r_\pi)$
- (By the way, this implies $\alpha_s = 0$)
- In other words: since $E(r_\pi)$ is the "risk premium for one unit of exposure to the "market" (index) and since exposure to one unit of "security s risk" contributes β_s units of market risks, we have that the risk premium for one unit of exposure to security s risk must imply a risk premium equal to $E(r_s) = \beta_s E(r_\pi)$
- Different quantities of the same "good" must have proportional prices (barred premiums and discounts)
- How could this be not true?

The beta model

- Let us find possible "non rational reasons" ☺ for $E(r_s) = c\beta_s Sd(r_\pi) = \beta_s E(r_\pi)$ NOT to be valid
- It may be that c is not the same for all s
- Or it may be that we have a non zero "alpha": $E(r_s) = \alpha_s + \beta_s E(r_\pi)$
- The reason why this should not happen is that in both cases we could exploit this to increase expected return and not risk

The beta model

- In the first case simply invest in the higher c_s stocks for a given level of β_s
- The second case is the most frequently discussed: suppose $\alpha_s > 0$, this means that, if you go "long" one unit of s and "short" β_s units of "index" the return of your investment shall be $\alpha_s + e_s$
- And its β shall be 0
- However, e_s is "idiosyncratic" that is: is not correlated across stocks, so, if you put some (not too much!) of this long/short investment in any existing big portfolio you are going to add to that portfolio some "alpha" and no new risk (idiosyncratic risks cancel out due to diversification"

The beta model

- This strategy did receive many names in time: “portable alpha”, “smart beta”, “factor investing”
- More on this follows

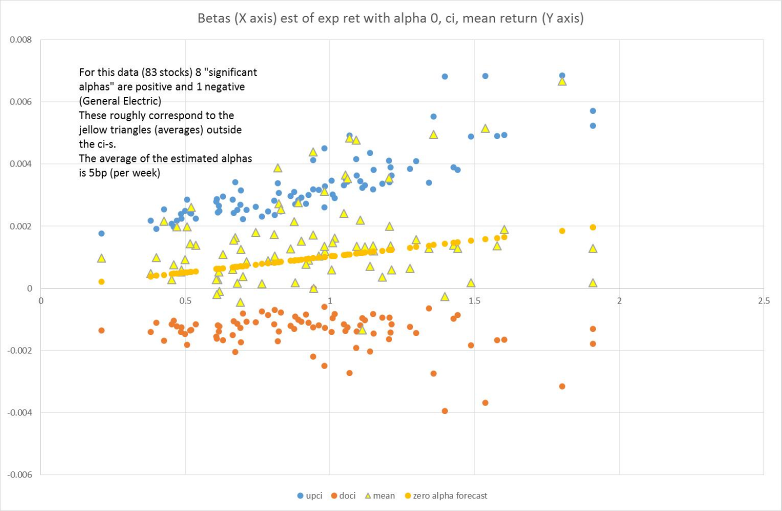
The beta model

- Just a quick peek to our data
- If we run the "beta regressions" for our 83 components of the S&P100 of the Fama & French (see below) general market index (both in excess return form) we get 9 ".05 significant" (8 positive 1 negative) "alphas"
- This means the model is not fully successful (we would expect only 4 .05 significant alphas, why?) but, for such a simple thing like this, is quite good

The beta model

- The following plot summarizes the results
- The X axis is the value of the estimated betas
- For each estimated beta, on the Y axis I plot the observed mean return (yellow triangles), the estimate (yellow circles) and the upper and lower bound (blue and orange circles) of confidence intervals for the expected return under the hypothesis that alpha be 0
- Each time you see a yellow triangle out of its confidence interval, you reject the hypothesis that the estimate of the return mean is compatible with an alpha set to 0

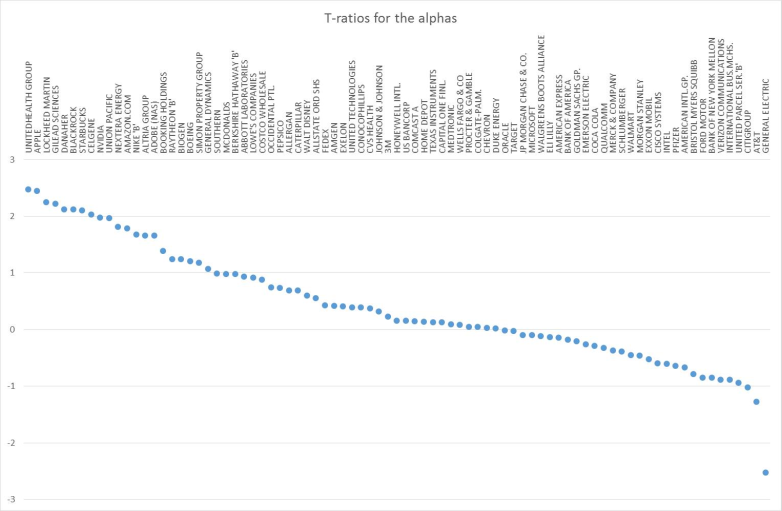
The beta model



The beta model

- And here the T-ratios for the alphas (ordered by size) with the names of the corresponding companies
- You can see that most T-ratios are positive, corresponding to positive alpha estimates
- In fact the average of the estimated alphas is 5 bp
- This is 5 bp per week, there is a huge standard error of the estimate, still the result is not small

The beta model



The beta model

- One possible reason for this is selection bias
- The 83 stocks in the sample are the stocks that TODAY are in the S&P100 index and for which we have 19 years of data
- The S&P100 is not exactly the index containing the biggest companies in the S&P500
- http://us.spindices.com/idsenhancedfactsheet/file.pdf?calcFrequency=M&force_download=true&hostIdentifier=48190c8c-42c4-46af-8d1a-0cd5db894797&indexId=2431
- However Its constituents tend to be the biggest and most "important" companies in the S&P500 and we may expect it to perform better than a general market index

The beta model

- Moreover, since our 83 companies are in the S&P100 NOW, most likely they did not have a bad performance in the past
- Compare this with, say, the 100 companies who WERE in the index at the beginning of year 2000 of to a random sample of 100 companies from the NYSE (in 2000)
- This two possible choices do not require companies to perform in a particular way after the beginning of year 2000 while our sample requires them to perform not badly at least at the end of the period
- Since the overall index is computed with the return of all stocks in the index at the time of computation, it shall tend to perform worse than 83 companies which are in the index at the end of the period
- This "selection bias" may explain, at least in part, why our estimates of the alphas tend to be positive

The beta model

- Using proper Statistics we could correct this
- We would then see less “significant” positive alphas
- (Still, we would get on average a positive alpha, see below for a possible explanation)
- The important point, now, is to realize we may have a selection problem
- Let us now forget this important point and go back to the model

The beta model

- Can we make the model work better?
- Can we "explain" the alphas (beyond selection bias)?
- Any improvement would have relevant implications for asset management, performance evaluation and risk management
- In fact we can and, while there is not much time left, we can try and see how this could work out
- ...And how much good Statistics has to do with this

The beta model

- Obviously, here we have some hidden hypothesis, and such "anomalies" as non zero alphas could be the result of some of these hypotheses not being true
- We did not characterize the properties of the "index portfolio"
- In order to make our arguments rigorous we should specify in which sense this index portfolio should "represent the market"
- Then: "the market" may not judge investments in terms of a ratio between expected return and expected (non diversifiable) risk

The beta model

- We do not have the time for this, but we still want to hint at one more explanation on which to work for improving the model
- Implied in the previous analysis is the fact that a return risk (correlation with a given market index) is evaluated in the same way, independently on the variables it can be correlated with beyond the reference portfolio.

Multiple betas

- This may be problematic
- It may be that two securities have the same beta with respect to the given index/portfolio
- However, one of this is more correlated with, say, the returns of the "big" stocks in the index and the other with the return of the "value" stocks of the index.
- If, for any reason, "the market" gives a different "value" to the fact of being more exposed to the "size" or the "price to book value" risk in the portfolio a "single index" model would be insufficient.
- Let then $r_\pi = wr_{1\pi} + (1-w)r_{2\pi}$ and suppose, for simplicity $Cov(r_{1\pi}; r_{2\pi}) = 0$

Multiple betas

- In this case, a replica of the above arguments would imply something the like of: $E(r_s) = \beta_{1s} w E(r_{1\pi}) + \beta_{2s}(1 - w)E(r_{2\pi})$
- This boils down to the above result in two possible ways:
- If $\beta_{1s} = \beta_{2s} = \beta_s$we have:
  $E(r_s) = \beta_s(wE(r_{1\pi}) + (1 - w)E(r_{2\pi})) = \beta_s E(r_s)$
- If $E(r_{1s}) = E(r_{2s}) = E(r_s)$ we have:
  $E(r_s) = (\beta_{1s} w + \beta_{2s}(1 - w))E(r_s) = \beta_s E(r_s)$

Multiple betas

- The first possibility is just a chance happening which may be true for some stocks and is uninteresting
- The second is much more interesting: it tells us that, if the risk premia of the two sub indexes are the same, we are back to the single index model
- How can it happen that the premia are not identical? (after all both are premia for undiversifiable risk expressed in the variance of a return of "the same money"!)
- The standard and interesting interpretation is that r_{1s} and r_{2s} could be correlated with different economic variables so that their risk "mimics" the risk of such variables
- In our view (utility?) such different risks could have different weights

Multiple betas

- For instance: r_{1s} could have a high positive correlation with and index expressing the change in value of a bundle of consumption goods, while r_{2s} could have correlation with other economic factors
- If I want to maintain my purchasing power at the level of the cost of consumption goods and am less worried by other economic variables I shall prefer to invest in r_{1s} than in r_{2s}
- If my opinion is shared by a majority of agents, this shall imply a lower risk premium for r_{1s} (I like its variance because it gives me value when I need it to buy more costly consumption goods) and higher for r_{2s}

Multiple betas

- Here notice an interesting point: suppose you linearly regress r_s on just $r_{1\pi}$.
- The intercept of this regression, under the above hypotheses, is $E(r_s) - \beta_{1s} w E(r_{1\pi}) = \beta_{1s} w E(r_{1\pi}) + \beta_{2s}(1-w)E(r_{2\pi}) - \beta_{1s} w E(r_{1\pi}) = \beta_{2s}(1-w)E(r_{2\pi})$
- This is not 0 if $\beta_{2s} \neq 0$ and $E(r_{2\pi}) \neq 0$ so that you have an "alpha" different than 0.

Multiple betas

- If you "believe" that only $r_{1\pi}$"implies a risk premium" you may be induced to "believe" that the above hypotheses are not working and you could gain a alpha "for free" by investing in s
- A "portable alpha/smart beta/factor investing" strategy of the kind mentioned above, would only imply that you are taking a (non diversifiable) risk you are not measuring (very bad idea!)
- Your "alpha" would simply be the average amount of the risk premium of this non measured risk

Multiple betas

- Notice that the setup could be as follows: you are an asset manager, you have a multifactor model in which you believe and there is no alpha.
- However you know that the "market", or any potential client, evaluates you with, say, a single beta model
- If this is the case you can simply take a position which is perfectly priced by your model, with no alpha, buy yields a considerable alpha for the evaluation model

Multiple betas

- Or consider a second setup
- You know this trick is common in the market, so you go to a big fund management client (e.g. a pension fund) and show them how they "misvalued" their managers
- (Today you would do this with a lot of mumbo-jumbo about deep learning, data mining, knowledge engineering by auto encoding time dynamic networks, etc.)
- Then, you offer them your deep data understanding as a consulting service

Multiple betas

- Let us go back to the starting point.
- What is relevant is that, now, we have an argument to potentially explain the initial empirical puzzle.
- If $E(r_s) = \beta_s E(r_\pi)$ or, maybe $E(r_s) = \beta_{1s} w E(r_{1\pi}) + \beta_{2s}(1-w)E(r_{2\pi})$
- There is no need for Expected return and overall return standard deviation to be proportional
- What is relevant is NOT the total standard deviation of r_s but only the part of it which is correlated with “non diversifiable/systematic risk factors”.

Multiple betas

- We can have any expected value and any standard deviation for returns, what we need is that the expected return be proportional to the risk we cannot diversify.
- As an added bonus we have the idea that market/non diversifiable/systematic risk can have many components ("factors")
- These may have different risk premia due to their different correlation with more or less "dangerous" underlying economic variables
- Moreover we now know that an "alpha" could come from an "omitted risk factor"

Multiple betas

- The existence of "multi factors" that is, the fact that what is relevant for correct pricing is not only the undiversifiable market exposure but how this exposure covariates with other risk factors is something known from a long time
- A classic in this setting is the Value/Growth controversy
- Since the beginning of the 20th century it has been noticed that portfolios long low price to book value stocks (value) and short high price to book value (growth) stocks, tend to make average over/under performances for long stretches of time (overall in favor of value)

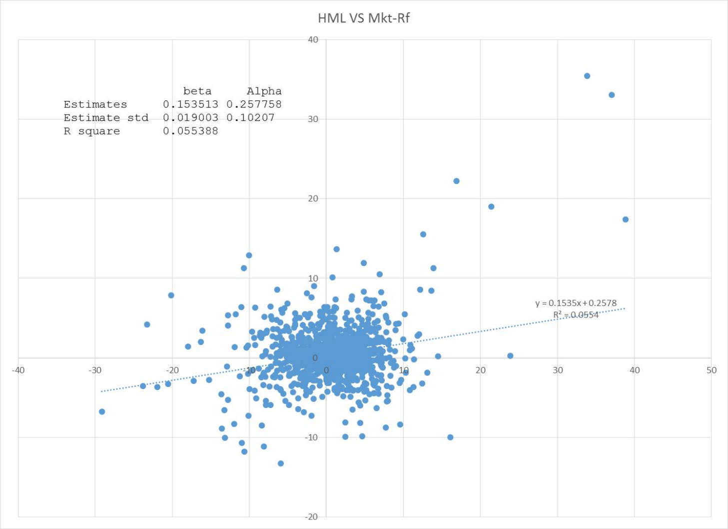
Multiple betas

- Indeed, using one of the classic Kenneth R. French dataset

(https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html)

- and regressing the return of a long value, short growth portfolio against the market excess return we get the following:
- (Monthly Returns: July 1926 -May 2019)

Multiple betas



Multiple betas

- From the statistical point of view, an estimate of alpha equal to .25 (25 bp per month) with a standard deviation of .1 (a t ratio of about 2.5) cannot be considered a 0 so this long short portfolio "violates CAPM"
- On average, for about a century, going long value short growth, while almost uncorrelated with the market (R square .05) gives you a relevant positive average return
- There would be lots of other considerations to make, for instance: this is an ex post return, we see what we already know to be true while we do not test other possible strategies because they did not reach the footlights.
- How to avoid this "sample selection bias"?

Multiple betas

- This is not the place or the time, those interested could begin with these papers:
- Fama, E. F.; French, K. R. (1993). "Common risk factors in the returns on stocks and bonds". Journal of Financial Economics. 33: 3–56.
- Fama, E. F.; French, K. R. (1992). "The Cross-Section of Expected Stock Returns". The Journal of Finance. 47 (2): 427
- (and the huge literature which followed them)

Multiple betas

- For beginners in the field of Finance with an interest in Statistics:
- Even at this very introductory level, you should see that wonderful things can be done with sound Statistics in finding and assessing risk factors
- By the way: any development in this field may have direct consequence on the asset management industry
- About 80 trillion dollars: roughly the same as the GNP/GDP of the planet (GNP and GDP of the planet should be the same)
- But there is more, much more. Let's get some hint

The conditional beta model

- Most of what we wrote is just reasonable intuition
- This is not the place where to make these reflections mathematically rigorous (follow an asset pricing theory course for this)
- However, we know enough in order to be able to introduce a new dimension in the picture
- Interestingly, this may imply a different interpretation of "alphas".

The conditional beta model

- Up to now we, implicitly, suppose knowledge of all the parameters (expected values, variances, covariances) and these to be non random
- In we do not know the parameter values (and we do not) we may estimate them
- Parameter estimates are going to be conditional to available information, not parameter themselves.

The conditional beta model

- For the moment forget estimation (and estimation risk) and simply change your parameter model into a stochastic one
- Parameters are now random variables
- Suppose we are at time t, prices of stocks are available and we want to build a reasonable portfolio for time $t+1$
- In order to do this, suppose we are able to exactly compute the conditional distribution of each parameter to the available info at time t

The conditional beta model

- Using this, we build an "optimal portfolio" at each time t for time $t+1$
- We then collect the results of such portfolio and ask the following question: what shall be, on average (i.e. unconditional to info or "marginally") the behaviour of the returns of such portfolios?
- In the previous setup we had constant parameters, hence the "conditional" version is identical to the unconditional one
- Now things are different, and more interesting

The conditional beta model

- In this new setting the market/index excess return is indexed to time: $r_{\pi t}$, the same for r_{st}
- What used to be β_s, a parameter, becomes now a random variable indexed by time β_{st}
- We call $E_t(.)$ the conditional expectation given what is known up to t.
- We can replicate the previous "approximate" reasoning and imply a conditional (single index) expected excess return of: $E_t(r_{st+1}) = E_t(r_{\pi t+1})\beta_{st+1}$
- This should be true in each time period (according to the above arguments), what shall happen "on average"?
- The question is important as the "average" is what we can compute after many time periods (again, hypothesis are required for this to be true but...)

The conditional beta model

- Recall how to pass from the conditional to the marginal expectation and compute:
- $E(E_t(r_{st+1})) = E(r_{s,t+1}) = E(E_t(r_{\pi t+1})\beta_{st+1}) = Cov(\beta_{st+1}, E_t(r_{\pi t+1})) + E(\beta_{st})E(E_t(r_{\pi t+1})) = Cov(\beta_{st+1}, E_t(r_{\pi t+1})) + E(\beta_{st+1})E(r_{\pi t+1})$
- This is quite interesting: the result is NOT, in general, $E(E_t(r_{st+1})) = E(r_{st+1}) = E(\beta_{st+1})E(r_{\pi t+1})$
- We cannot simply say that, marginally, we still have the beta model with β_s changed into $E(\beta_{st+1})$ so that we should still observe a risk premium proportional to the (now average) beta

The conditional beta model

- With stochastic parameters we get a new term $Cov(\beta_{st+1}, E_t(r_{\pi t+1}))$
- This does not depend of the size of the risk exposure times the risk premium but on the covariance between the two
- In other word, this is an alpha, because is a component of the risk premium which is not proportional to the market exposure

The conditional beta model

- However, the new “alpha” is not a free lunch (as was the old one)
- This depends on the fact that, being both random variables, β_{st+1} and $E_t(r_{\pi t+1})$ may have a non zero covariance
- (In the previous model β_s was a constant so this term would have been always equal to 0)
- This covariance is in itself a measure of risk: the risk that an exposure changes in a (positively or negatively) correlated way with the conditional expected risk premium

The conditional beta model

- Let us stress a point we already mentioned above.
- Clearly, in the original model the parameters, while constant, are unknown.
- You must estimate them and these estimates shall be a function of your information.
- However, they are not random variables and as information grows, in the end you shall "know" parameters.

The conditional beta model

- While this could be argued against, this "estimation risk" is, at least at the simplest level, not considered in (basic) financial models
- Modellers act as if "the market knew" the true values of these parameters.

- However, if the parameters are considered as random variables, even with perfect knowledge of their distribution, conditioning information shall have a relevant financial implications on the model results.
- The presence of $Cov(\beta_{st+1}, E_t(r_{\pi t+1}))$ in the "marginal" models testifies this.
- Now, in the original model, an alpha different than 0 would be equivalent to "unreasonable" returns maybe due to omitted risk factors

The conditional beta model

- The new model offers a different interpretation
- A non zero alpha in the marginal expected return comes from a fully reasonable conditional model with no omitted factors
- The "alpha" is the trace of the fact that risk weights and risk premia may covariate

The conditional beta model

- Just to understand how this is possible let us be inspired by the already mentioned idea of "covariance with relevant economic variables"
- Suppose X_t is any random variable whose value is in the info set at t (i.e. if you are at t you know the value of X_t)
- Now, model the random β_{st+1} and $E_t(r_{\pi t+1})$ as:
- $\beta_{st+1} = \gamma_{1s} X_t$ and $E_t(r_{\pi t+1}) = \delta_1 X_t$ (here γ_{1s} and δ_1 for simplicity, are not stochastic and positive)
- Notice: in this simple model β_{st+1} and $E_t(r_{\pi t+1})$ are perfectly (positively) correlated

The conditional beta model

- In the end we have, Conditionally:
- $E_t(r_{s,t+1}) = E_t(r_{\pi t+1})\beta_{st+1} = \gamma_{1s}\delta_1 X_t^2$
- Unconditionally:
- $E(E_t(r_{s,t+1})) = E(r_{s,t+1}) = \gamma_{1s}\delta_1 V(X_t) + E(\beta_{st+1})E(r_{\pi t+1})$
- (This because $E(\beta_{st+1}) = \gamma_{1s}E(X_t)$ and $E(E_t(r_{\pi t+1})) = \delta_1 E(X_t)$ so that $\gamma_{1s}\delta_1 E(X_t)^2 = E(\beta_{st+1})E(r_{\pi t+1}))$

The conditional beta model

- The "alpha" here is seen as $\gamma_{1s}\delta_1 V(X_t)$, this is due to the perfect correlation between β_{st+1} and $E_t(r_{\pi t+1})$
- If the γ_{1s} and δ_1 have, say, the same sign (so that the covariance between the risk premium and the risk exposure is positive) this shall be positive
- In an unconditional model this would be interpreted as an "excess return" beyond what justified by the beta: an "alpha", an anomaly.
- In the new model this is simply to be read as the non anomalous consequence of a positive covariance between risk premium and beta

The conditional beta model

- Go back to the value/growth alpha.
- Suppose you find an alpha when regressing a stock excess return on the market return and this disappears if you add the value vs growth portfolio
- A first possible interpretation was that the alpha is just the average risk premium for the omitted "risk factor" (value/growth or other)

The conditional beta model

- We have now a second possible interpretation: the conditional risk factor is just one.
- However, both the beta and the risk premium are correlated (with the same sign) with the return of the value vs growth portfolio (our X_t)
- This makes $\gamma_{1s}\delta_1 V(X_t)$ positive even if, conditionally, the only risk factor is "the market".
- Which interpretation would you choose?
- Could you distinguish the two interpretations? (suggest a test, recall that you now know the "formula" of the alpha)