Optimal Market Access Pricing*

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Abstract

We determine optimal market access pricing for an exchange or Social Planner. Exchanges optimally use rebate-based pricing (vs. strictly positive fees) when ex ante gains-from-trade and trading activity are low (high). Exchange rebate-based pricing increases (decreases) welfare when investor valuation dispersion and trading activity are low (high). A Social Planner increases welfare using rebate-based pricing. High-frequency traders strengthen exchange incentives for rebate-based pricing; a new explanation for widespread Maker-Taker and Taker-Maker pricing. With HFTs, rebate-based pricing improves total welfare, but Pareto transfers are needed to improve investor welfare. Sequential bargaining games between competing exchanges setting fees have pure-strategy equilibria.

JEL classification: G10, G20, G24, D40

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Access fees and rebates in securities markets are at the top of the agenda of financial regulators and market operators around the world. Following the introduction of Regulation National Market System (Reg NMS, 2007) in the US (and related regulation in Europe), market access pricing has become a strategic tool for trading platforms and exchanges to attract trading volume especially for liquid stocks (Cardella, Hao and Kalcheva, 2015 and O'Hara, 2015). In particular, rebates incentivize investors to submit certain types of orders, while investors using other order types are charged fees. For example, Maker-Taker pricing pays investors rebates when their limit orders (making liquidity) are executed and charges fees on market orders (taking liquidity), while under Taker-Maker (also called inverted) pricing the fees and rebates are reversed. The economic magnitude of access pricing revenue for exchanges is material. For example, for the London Stock Exchange group, it totals £407m, which represents 19 percent of the Total Group income in 2019. However, rebate–based access pricing has been criticized by some practitioners as well as by Angel, Harris, and Spatt (2013), Harris (2015), and Spatt (2019) on agency, price transparency and regulatory grounds.

Our paper models optimal access pricing in a dynamic limit order market in which traders arrive sequentially with heterogeneous random private asset valuations and choose endogenously when to submit market or limit orders and, when submitting limit orders, at which limit prices. Our analysis follows seminal theoretical research by Colliard and Foucault (2012); Foucault, Kadan, and Kandel (2013); and Chao, Yao, and Ye (2018) showing how fees and rebates for taking and making liquidity via market and limit orders can alleviate trading frictions from price discreteness. In particular, price discreteness limits the prices at which investors can transact, but access fees and rebates can be used to adjust net transaction prices, and, thereby, the rewards and costs for liquidity supply and demand.

Building from first principles, our analysis extends previous literature in two main directions. First, we show that optimal access pricing crucially depends on how heterogeneous investor gains from trade are relative to the tick size. When investors in a market have larger gains from trade, the exchange's incentive to subsidize trade by offering rebate-based pricing decreases. Second, we show that fees and rebates are still critical in resolving the discrete trading friction even when investors have an expanded choice of orders. In particular, investors can choose their limit price and choose between market orders vs limit order, and high frequency traders can decide whether or not to trade. In addition, fees and rebates continue to affect trading when, in contrast to the previous literature, liquidity supply is no longer monopolistic.

We make five main contributions towards understanding market access pricing:

- First, regulatory constraints have important effects on access pricing. With regulatory caps on access fees, an exchange's optimal access pricing and the relation between access pricing and welfare both depend on the mix of investors in a market and on the tick size. Access pricing rebates are optimal in markets populated by short-term speculative traders with small gains-from-trade, whereas strictly positive fees with no rebates are optimal in markets populated by long-term investors with large gains-from-trade. Equilibrium access pricing in a small tick market is isomorphic to the large tick market in that optimal access pricing scales linearly in the tick size.

- Second, from a policy perspective, we show that rebate-based access pricing can always be structured by a welfare-maximizing Social Planner in a way that increases total welfare. However, the welfare effects of rebate-based pricing by a profit-maximizing exchange are more nuanced: Optimal access pricing by an exchange leads to Pareto improvement when ex ante gains-from-trade are small but to deadweight welfare losses when investor gains-from-trade are larger.

- Third, we identify a seeming inconsistency between price-friction models of access pricing and how rebate-based pricing is used in practice. In a basic price-friction model, an exchange's incentives for rebate-based access pricing are weak for liquid stocks with high trading activity and large ex ante investor gains-from-trade, which appears inconsistent with current practice.

- Fourth, we provide a resolution to this puzzle that involves high-frequency trading and liquidity provision. When HFT market makers are added to the price-friction model, an exchange's incentives to use rebate-based pricing increase. Thus, our analysis identifies the growth of HFT market making, which is prevalent in liquid stocks, as a cause for widespread rebate-based access pricing by exchanges in US and European markets. We also show that, with HFTs, optimal rebate-based pricing by an exchange leads to increased total welfare, but Pareto transfers are needed to improve investor welfare.

- Fifth, we present a solution to a non-existence problem for pure-strategy equilibrium for exchange competition in access pricing in Chao et al. (2018). If access fee setting is modeled as sequential moves by exchanges, rather than as simultaneous moves (as in Chao et al. (2018)), then there are many pure-strategy equilibria corresponding to different amounts of exchange precommitment.

Our analysis constructs three models of optimal access pricing by a profit-maximizing exchange or, alternatively, by a welfare-maximizing Social Planner: Our first model has two periods and is solved in closed-form for a price grid with a range of different possible tick sizes. Technically, solving such a model is an open question, which Chao et al. (2018) describe as requiring "complex mathematical construction to circumvent the noncontinuity" in prices (Chao et al. (2018), page 1089). The source of the difficulty is that endogenous order choice becomes important when there are multiple possible limit prices inside the investor valuation support. However, we show that optimal access pricing for an exchange takes a simple functional form. This lets us give the first full analysis of how ex ante investor valuation heterogeneity and the tick size have different effects on access pricing and welfare given endogenous limit price choice by investors.

Regulatory restrictions on access pricing are an important element in our analysis. In the absence of a regulatory cap, widening the valuation support holding the tick size fixed proportionally re-scales optimal access pricing. In contrast, decreasing the tick size holding the valuation support fixed does not affect optimal access pricing. With regulatory restrictions, however, optimal access pricing depends on both the magnitude of investor valuation dispersion and also on the tick size. Changing the investor valuation support changes the ex ante demand for trading relative to the tick-size friction thus affecting optimal access pricing. When the dispersion of investor private valuations is low (and potential gains-from-trade are small), a combination of rebates and fees (Maker-Taker or Taker-Maker) is optimal for an exchange in order to overcome the discrete tick-size friction. In contrast, when potential gains-from-trade are large, trading is generated endogenously, and there is less incentive for the exchange to subsidize trading. As a result, a unique set of strictly positive fees on both market and limit orders (i.e., no rebates) is optimal. When fees are capped relative to the tick size, changing the tick size changes the constraint on fees allowed by regulation and therefore the optimal access pricing. We show, as a result, with a regulatory constraint, a small tick market is isomorphic to the large tick market.

Our welfare analysis of access pricing compares a market with optimal access pricing by a profit-maximizing exchange with two other related markets: One market with no access fees or rebates and another market with optimal access pricing by a Social Planner who maximizes the total welfare of all market participants. Optimal rebate-based access pricing by an exchange increases total welfare when the investor valuation support is small (i.e, investors are ex ante similar, and gains-from-trade are small). However, when investor valuations are sufficiently ex ante heterogeneous, then rebate-based pricing by an exchange reduces total welfare given the reduced need to cross-subsidize trading. In contrast, we show a Social Planer always uses rebate-based access pricing to improve total welfare. Taken together, our welfare analysis indicates a possible positive role for rebate-based access pricing and for regulation of access fees.

Our second model extends the two-period model to include a third period of trading. This change is not just quantitative (i.e., more rounds to trade). It also has a qualitative impact on endogenous order choice. In particular, with more than two trading periods, investors can choose between limit and market orders. This is new relative to Foucault et al. (2013) and Chao et al. (2018) in which the market/limit order choice is exogenous. Our three-period model offers two insights both about the market power of liquidity providers, and about the effect of trading activity on access pricing:

- Taker-Maker and Maker-Taker pricing is no longer symmetric.
- The incentive for the exchange to offer rebate-based access pricing decreases.

In a two-period model, the first investor to arrive is a monopolist in providing liquidity, but in a more active three-period market, the first investor is not a monopolist. The next arriving trader may decide to compete and undercut a standing limit order or may supply liquidity on the other side of the market via a limit order rather than taking liquidity via a market order.[1] This increase in possible order choices leads to the following result: When trader valuations are not too dispersed, both Maker-Taker and Taker-Maker pricing are optimal, but they are no longer symmetric, with the take rebate in Taker-Maker pricing being larger than the make rebate in Maker-Taker pricing. The intuition is that in Taker-Maker pricing the investor posting a limit order in the first period has no rebate and, hence, tends to post less aggressive limit orders. The investor in the second period then has a choice whether to post market order or a limit order to try for price improvement. To promote executions, the exchange has an incentive to induce the investor in the second period to take, rather than make, liquidity and, therefore, offers a larger rebate compared to Maker-Taker pricing. In addition, if the maximum fee is capped by regulation, as in real markets, then we show that, moving from a two-period to a three-period market, the take rebate needs to be larger than the make rebate.

Greater trading activity — as proxied for here by the addition of a third period — increases the probability of transactions, giving exchanges less incentive to subsidize trading via rebates. Therefore, the set

[1]In a 2-period market, the investor arriving in the second period has no choice other than trading with whatever limit order is posted in the first period or not trading.

of parameterizations with rebate-based pricing shrinks, and the model departs from the empirical fact of widespread rebate-based pricing for actively traded large-cap stocks where long-term traders with large gains-from-trade are generally very active.

Our third model includes HFTs in the three-period framework. Heterogeneous investor speed and HFTs are important features of financial markets (see Brogaard, Hagströmer, Nordén, and Riordan (2015)), Our analysis shows that HFT liquidity provision has a significant impact on access pricing and welfare via its impact on investor order choice. HFTs in our model are opportunistic liquidity suppliers, who, as in Li, Wang, and Ye (2019), have zero gains-from-trade and react immediately to orders posted by regular traders. In equilibrium, HFTs increase transaction execution by augmenting the set of potential counter-parties for investors, which increases welfare and exchange profits. As a result, the presence of HFTs expands the set of market parameterizations (i.e., the set of possible investor valuation supports) in which the exchange optimally offers rebate-based access pricing.

The increased use of rebate-based pricing with HFTs, and the fact that rebates may be paid to HFTS, may appear surprising at first. In our model, fast HFTs and slower regular investors both endogenously choose limit prices. Therefore, rebates induce changes in endogenous limit prices that adjust the optimal cum-fee net prices paid or received by HFTs. Our result is new relative to Foucault et al. (2013), in which rebates are paid to slower traders but limit prices are fixed. The offsetting changes in endogenously chosen limit prices adjust the incidence of who receives rebates and pays fee. In particular, what matters for trading is not fees and rebates per se, but rather the cum-fee net prices paid or received by HFTs. Thus, either directly (via a rebate) or indirectly (via limit order prices), rebate-based access pricing incentivizes HFTs to provide liquidity thus cross-subsidizing investors posting aggressive limit prices. In addition, the fact that HFTs expands the set of markets with rebate-based access pricing is a new potential explanation for the use of rebates even in active markets and matches the empirical evidence of Cardella et al. (2015) of widespread rebate-based pricing after Regulation NMS and the concurrent increase in HFT activity.

1 Background information and prior research

Reg NMS established the regulatory foundation for the current architecture of US equity markets. This regulation includes an explicit limit on the cost of accessing (i.e., posting and trading on) quotes displayed by U.S. equity trading platforms. Rule 610 caps access fees to no more than \$0.003 per share for stocks priced over \$1, and to no more that 0.3% of the quoted price for stocks priced below \$1. In addition, the Sub-Penny Rule 612 of Reg NMS prohibits exchanges, market makers, and electronic platforms from displaying, ranking or accepting quotes on NMS securities in sub-penny increments unless a stock is priced less than \$1 per share. Thus, under Reg NMS, access fees cannot exceed one third of the tick size.[2]

In Europe, MiFID II (Directive 2014/65/EU) and MiFIR (Regulation 600/2014/EU) mandates a reduction in the tick size for European stocks and thereby implicitly reduced the maximum access fees given that the standard practice on European exchanges is to cap fees relative to the tick size.[3] MiFID II also sharpened the regulation of access fees by requiring new incentives on market making agreements under Stress Market Conditions (RTS 8), a maximum Order-To-Trade ratio for each instrument (RTS 9), and a periodic disclosure by exchanges of the percentage of fees and rebates on total turnover (RTS 27). It also bans “cliff-edge“ pricing structures in which customer-specific fees are reduced retroactively for market participants who reach a trading volume threshold (RTS 10).

Access pricing has been investigated in a small number of theoretical papers. The starting point for work

[2]According to the more recent S.E.C. (2018) Release No.34-82873 on Transaction Fee Pilot for NMS Stocks “For maker-taker exchanges, the amount of the taker fee is bounded by the cap imposed by Rule 610(c) on the fees the exchange can charge to access its best bid/offer for NMS stocks. This cap applies to the fees assessed on an incoming order that executes against a resting order or quote, but does not directly limit rebates paid. The Rule 610(c) cap on fees also typically indirectly limits the amount of the rebates that an exchange offers to less than \$0.003 per share in order to maintain net positive transaction revenues. For taker-maker exchanges, the amount of the maker fee charged to the provider of liquidity is not bounded by the Rule 610(c) cap, but such fees typically are no more than \$0.003, and the taker of liquidity earns a rebate." If the price of a protected quotation is less than \$1.00, the access fee is no more than 0.3% of the quotation price per share SEC (2009).

[3]See Article 49 of MiFID II and the following Regulatory Technical Standard 11 (RTS 11, ESMA 2017). ESMA (2015)

on price discreteness frictions is Colliard and Foucault (2012), which shows that the breakdown between make and take fees has no effects on the cum-fee-spread (net of fees spread) in a competitive market with continuous prices. The reason is that traders can neutralize changes in fees by making offsetting changes in the pricing aggressiveness of limit orders. Subsequent research has identified and studied two channels through which a price-discreteness friction affects access pricing and trading: Market monitoring and limit order price choice. Foucault et al. (2013) show how price discreteness and access pricing affect investor monitoring incentives and the order-arrival process in a coordination game matching buyers and sellers. However, investor gains-from trade are non-random and known, and there is no decision about posted limit prices (which are exogenously fixed in their analysis) or choice between limit and market orders. Foucault et al. (2013) show that, in single market with a discrete tick size, the make-take breakdown affects market quality. In contrast, we study a trading game in which potential buyers and sellers and an exchange decide how to split random investor gains-from-trade. In particular, investors endogenously choose the limit prices at which limit orders are posted so as to maximize their expected share of the gains-from-trade, and the exchange's access pricing affects both the probability of transactions and the exchange's profit per trade. Panayides, Rindi, and Werner (2017) show how a change in trading fees affects market quality when two trading platforms compete for the provision of liquidity.

Our analysis is closely related to Chao et al. (2018), which models optimal access pricing both in a single monopolistic market and also with competition between multiple markets. In terms of modeling structure, we extend their model by expanding the scope of endogenous order choice (i.e., by wider parametric changes in both the support of investor valuations and the price grid), adding additional rounds of trading, and including HFTs. These changes lead to three sets of new insights: First, optimal access pricing is strictly rebate-based in Chao et al. (2018), which provides an explanation for widespread rebate-based pricing post Reg NMS. We show that, with a more realistic regulatory constraint just on the fee size but not on rebates, optimal access pricing depends on the investor population in the market. As a result, strictly positive access pricing by exchanges is possible when the ex ante gains-from-trade are large. In addition, we provide an alternative explanation for widespread rebate-based access pricing due to the growing importance of HFTs after Reg-NMS. Second, Chao et al. (2018) show the absence of pure-strategy equilibria with competing exchanges and simultaneous moves. In contrast, we show that pure-strategy equilibria exist if competition between exchanges occurs through sequential-move bargaining. We specifically construct pure-strategy equilibria with price discrimination and then show that Bertrand competition, as in Chao et al. (2018), still need not drive exchange profits to zero. Third, our analysis identifies an asymmetry between make and take fees. In Chao et al. (2018), there are pairs of symmetric Taker-Maker and Maker-Taker pricing schedules that are both optimal for exchanges. In contrast, we show that optimal Taker-Maker and Maker-Taker can be asymmetric when there are more than two periods of trading (i.e., in our three-period model). Our analysis below shows that the reason for these asymmetries is that make fees affect the choice of *which* limit orders investors submit, whereas take fees only affect the decision of *whether or not* to use a market order. Thus, allowing for endogenous choice of posted limit prices in our model leads to new insights.

Angel et al. (2013) and Spatt (2019) take a different approach from the price-friction literature. They emphasize that access fees and rebates have important potential effects via the transparency of economic prices (price + access pricing) vs quoted prices, the efficacy of regulatory protections based on quoted prices, agency issues when brokers do not pass through fees and rebates to their clients, and impeding intermarket competition. Harris (2015) points out further that negative fees allow for intra-tick trading, thus by-passing the Reg NMS trade-through rule. Li, Ye, and Zheng (2020) show how fees affect order routing decision in fragmented markets and create demand for complex order types. In contrast, our analysis is based on the idea that constraining trade to a discrete price grid creates trading frictions and that access pricing potentially reduces those frictions. Both sets of considerations are likely to be important. Moreover, a complete understanding of access pricing is likely to involve interactions between these various effects and price frictions.

A sizable empirical literature investigates different aspects of access pricing.[4] Malinova and Park (2015) find evidence following changes in access fees and rebates on the Toronto Stock Exchange (TSX) that appears to support the Colliard and Foucault (2012) irrelevance prediction provided that the TSX price tick-size is interpreted as being economically small. However, using Rule 605 data, O'Donoghue (2015) finds that changes in the split of trading fees between liquidity suppliers and demanders affect order choice and execution quality as predicted by Foucault et al. (2013). Battalio, Corwin, Jennings (2016) find that access fees and rebates appear to affect broker order-routing decisions. Panayides et al. (2020) find that quoted and cum-fee spreads are affected by change in total fees on the BATS European platforms, CXE and BXE.

Empirical research finds that rebate-based access pricing is related to HFTs. Menkveld (2013) shows that access rebates are a significant part of HFT profits. We show that HFTs activity makes rebate–based pricing more likely in equilibrium. This is also consistent with evidence in Cardella, Hao, and Kalcheva (2015) that Reg NMS was followed by the adoption of rebate–based access pricing by most trading platforms in U.S. markets and by a sharp increase in HFT firm trading. O'Hara (2015) also links HFT trading activity and the increased use of rebate–based access pricing around the world.

2 Two-Period Model

We begin our analysis with a parsimonious model with two dates, t_1 and t_2, on which investors arrive sequentially over a trading day and potentially submit orders. With only two periods, the model dynamics can be solved analytically. We use this model to develop basic intuition about access pricing in a setting in

[4]In addition to the research discussed here, see also Bourke, DeSantis, and Porter (2019), Baldacci, Possamaï, and Rosenbaum (2019), Brauneis, Mestel, Riordan, and Theissen (2019), Skjeltorp, Sojli, and Tham (2012), He, Jarnecic, and Liu (2015), Clapham, Gomber, Lausen, and Panz (2017), Anand, Hua, and McCormick (2016), Comerton-Forde, Grégoire, and Zhong (2019), Lin, Swan, and Harris (2019) and Brolley and Malinova (2013).

which investors endogenously choose their posted limit prices. Section 3 extends this basic model to more than two periods, and Section 4 introduces trading speed heterogeneity with high-frequency traders (HFTs).

At each period $t_z \in \{t_1, t_2\}$ a risk-neutral trader arrives characterized by a private valuation equal to β_{t_z} which is an i.i.d drawn from a uniform distribution, $U[\underline{\beta}, \overline{\beta}]$, where $\underline{\beta}$ and $\overline{\beta}$ are the limits of the trader valuation supports. The mean of the valuation support v is constant over time and denotes the ex ante *asset value*. Traders with more extreme β_{t_z} realizations have stronger demands to trade, whereas traders with β_{t_z} realizations close to v are more willing to supply liquidity. Therefore, the *support width* $\Delta = \overline{\beta} - \underline{\beta}$ measures the ex ante gains-from-trade and, thus, the associated demand for trade. The wider the support, $[\underline{\beta}, \overline{\beta}]$, the higher is the probability that arriving traders will have strong heterogeneous directional demands to trade, such as, e.g., long–term asset managers. The smaller the support $[\underline{\beta}, \overline{\beta}]$, the higher is the probability that arriving traders will prefer to profit as passive liquidity providers.

Prices are quoted on a discrete price grid $\{\ldots, P_{-k}, \ldots, P_{-1}, P_1, \ldots, P_k, \ldots\}$ centered around the mean investor private valuation v with a fixed tick size τ. The state of the limit order book at time t_z is a vector $L_{t_z} = [D_{t_z}^{P_k}]$, where $D_{t_z}^{P_k}$ indicates the total limit order depth at price P_k at time t_z. Investors trade using limit orders, which supply depth to the book, and market orders, which hit standing limit orders and take depth from the book. The initial limit order book L_{t_0} at the start of the day is assumed to be empty, and then we model how the book evolves over time. Let x_{t_z} denote a generic action taken by an investor at a date t_z. Trading and limit order book dynamics take a particularly simple form in a two-period market: An investor arriving at time t_1 chooses between submitting a limit buy order LBP_k or limit sell order LSP_k at one of the available price levels P_k on the price grid or a no trade NT. Market orders are not possible at t_1 given the empty initial book. Let X^L denote the set of possible limit buy and sell orders at all possible limit prices. Next, the investor arriving at t_2 chooses between submitting a market buy order MBP_k or sell order MSP_k that is executed immediately given the standing book L_{t_1} at the best bid (for market sells) or offer (for market buys) price P_k or, instead, does not trade NT.[5] In the two-period model, limit orders are not used at t_2 since, after the final round of investor arrival, a limit orders posted at t_2 would not be executed.

Consistent with common practice in today's financial markets, the trading platform may set different access fees $\xi(x)$ for different order types x. An investor offering liquidity by posting a limit order pays a *make fee* MF. An investor taking liquidity via a market order (or via a marketable limit order) pays a *take fee* TF. Access pricing for an exchange is denoted as the set $\Xi = \{\xi(x)\}_{\forall x} = \{MF, TF\}$. Rebates are negative fees, which are a cost for the trading platform and a reward for the investor receiving them. Under a *Maker-Taker* structure, investors submitting market orders pay a take fee ($TF > 0$) to the trading platform, and investors posting limit orders receive a make rebate ($MF < 0$) whenever their limit order executes. In a *Taker-Maker* structure, the fees and rebates are reversed so that now limit-order submitters pay make fees ($MF > 0$), and market-order submitters receive take rebates ($TF < 0$). Also, consistent with current practice and with Foucault et al. (2013), access fees and rebates in our model are subject to regulation. For notational simplicity, we assume the maximum allowable fee (whether take or make) is one tick (i.e., rather than a fraction of a tick as in, e.g., Reg NMS). Thus, the regulatory constraint on fees is more binding for smaller tick sizes. There are no direct regulatory constraints on rebates in our model, but Lemma 1 in Appendix A shows that if fees are capped at one tick, then in equilibrium exchange rebates are never larger than one tick. The welfare impact of a regulatory cap on fees is considered below in Section 5.

Quoted prices and the access fees and rebates determine the net prices paid and received by investors when trading, which we call *cum-fee prices*. Let $P_k^{cum,MS} = P_k - TF$ denote the cum-fee price received from a market order to sell at the quoted price P_k (net of take fees paid to the exchange) , and let $P_k^{cum,MB} = P_k + TF$ be the cum-fee price paid on a market order to buy at P_k (net of take fees paid to the exchange). Similarly, $P_k^{cum,LS} = P_k - MF$ is the cum-fee price for a limit order to sell and $P_k^{cum,LB} = P_k + MF$ is the cum-fee price

[5] Marketable limit orders that cross with the best available bid/ask on the opposite side of the standing book $L_{t_{z-1}}$ are treated as market orders in terms of both order execution and exchange access pricing.

for a limit order to buy.

Our model determines optimal fees for an exchange or, alternatively, a Social Planner in a Stackelberg game in which the exchange/Social Planner is a Stackelberg leader and investors are Stackelberg followers. We solve the model in two steps: Taking market access pricing Ξ as given, we first solve for optimal investor trading strategies — i.e., the optimal responses of the Stackelberg followers — in the trading subgame. These optimal investor strategies are computed by solving the trading subgame by backward induction. Given this characterization of optimal investor trading, we then solve for the optimal access pricing Ξ given an exchange's profit-maximization problem or a Social Planner's total welfare-maximization problem.

Given a standing book $L_{t_{z-1}}$ and access pricing Ξ, the expected payoff on an order x_{t_z} for an investors arriving at time $t_z$with a private valuation β_{t_z} is:

$$\pi_{t_z}^{INV}(x_{t_z}|\beta_{t_z},\Xi,L_{t_{z-1}}) = \begin{cases} [\beta_{t_z} - P(x_{t_z}) - \xi(x_{t_z})]\,Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}}) & \text{if } x_{t_z} \text{ is a buy order} \\ [P(x_{t_z}) - \beta_{t_z} - \xi(x_{t_z})]\,Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}}) & \text{if } x_{t_z} \text{ is a sell order} \\ 0 & \text{if } x_{t_z} \text{ is } NT \end{cases} \tag{1}$$

where $P(x_{t_z})$ is the posted price at which order x_{t_z} trades if it is executed and $\xi(x_{t_z}) = TF$ for market orders and MF for limit orders. $\theta_{t_z}^{x_{t_z}}$ denotes the set of future trading states in which an order x_{t_z} submitted at time t_z is executed, and $Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}})$ is the associated probability of execution. If x_{t_z} is a market order, then $P(x_{t_z})$ is the best standing quote on the other side of the market at time t_z, and $Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}}) = 1$, since market orders are executed immediately at the standing bid or ask (if that side of the book is non-empty). If x_{t_z} is a non-marketable limit order, then the execution price $P(x_{t_z})$ is its limit price, and the execution probability $Pr(\theta_{t_z}^{x_{t_z}}|\Xi,L_{t_{z-1}})$, which is the probability of later investors choosing to hit standing limit orders with market orders, is between 0 and 1. Limit order execution probabilities depend parametrically on the valuation support S and the tick size τ. Liquidity is endogenous, and, thus, the order-execution probabilities

$Pr(\theta_{t_z}^{x_{t_z}}|\Xi, L_{t_{z-1}})$ in (1) are endogenous. Tables A1 and B1 in the Appendix detail the actions available to traders, their associated expected payoffs, and order submission and execution probabilities.

An investor arriving at time t_z chooses his order x_{t_z} to maximize his expected payoff from (1):

$$\max_{x_{t_z} \in X_{t_z}} \pi_{t_z}^{INV}(x_{t_z}|\beta_{t_z}, \Xi, L_{t_{z-1}}) \tag{2}$$

given his private value realization β_{t_z}. This is a discrete-choice problem where the optimal order-submission strategy $x_{t_z}(\beta_{t_z}|\Xi, L_{t_{z-1}})$ assigns orders that maximize (2) to each of the different possible investor valuations β_{t_z} in the support $[\underline{\beta}, \overline{\beta}]$ at time t_z conditional on the standing book $L_{t_{z-1}}$ at t_z and the access pricing Ξ. The optimization problem in (2) is tractable because the investor expected payoffs $\pi_{t_z}^{INV}(x_{t_z}|\beta_{t_z}, \Xi, L_{t_{z-1}})$ from (1) for different orders x_{t_z} are linear in the investor valuation β_{t_z}. Consequently, the maximized expected profit in (2) for β_{t_z} valuations in the support S is the upper envelope of the linear expected payoff functions, and optimal orders are associated with different intervals of β_{t_z} valuations in between points where the optimized linear expected payoff functions intersect. A key intuition is that the investor's optimal order choice depends on a trade-off between order-execution probabilities and price improvement: More aggressive limit order prices reduce the payoff conditional on execution, but can increase the probability of execution.

A novel feature of our analysis, relative to other access pricing models, is endogenous choice of posted limit prices. This means we need to consider the market order submission decision given different hypothetical limit orders at multiple hypothetical possible posted limit prices. To simplify the backward induction analysis, we first reduce the number of prices we need to consider by identifying a priori prices at which investors might actually post limit orders rather than considering hypothetical limit order at all prices on the price grid. Lemma 2 in Appendix A shows that investors only submit limit buys at posted prices for which the corresponding cum-fee market sell price is $P_k^{cum,MS} \leq \overline{\beta}$ and, by symmetry, limit sells at such that the cum-fee market buy price is $P_k^{cum,MB} \geq \underline{\beta}$.

Consider the last round of trading at t_2 in the two-period market when investors submit market orders or decide not to trade. An investor is willing to submit a market sell MSP_{k,t_2} to hit a limit buy at a posted price P_k if his payoff $P_k^{cum,MS} - \beta_{t_2} > 0$ is positive, where $P_k^{cum,MS} = P_k - TF$ is the cum-fee price for a market sell at the posted price P_k.[6] Given the investor valuation β_{t_2} is drawn from $U[\underline{\beta}, \bar{\beta}]$ with support width Δ, the order-submission probability of a market sell, MSP_{k,t_2}, at t_2 given the cum-fee price $P_k^{cum,MS}$ is:

$$Pr(x_{k,t_2}^{MS}|\Xi, L_{t_1}) = \max\left\{0, \frac{P_k - TF - \underline{\beta}}{\Delta}\right\} = Pr(\theta_{t_1}^{x_k^{LB}}|\Xi, L_{t_0}) \tag{3}$$

which is, therefore, the execution probability $Pr(\theta_{t_1}^{x_k^{LB}}|\Xi, L_{t_0})$ of a limit buy LBP_{k,t_1} posted at P_k at t_1.[7] In particular, this probability is well-defined (i.e., ≤ 1) for all priori possible limit prices from Lemma 2. By symmetry, the order-submission probability of a market buy MBP_{k,t_2} at a posted price P_k at t_2 given a cum-fee market-buy price $P_k^{cum,MB} = P_k + TF$ is:

$$Pr(x_{k,t_2}^{MB}|\Xi, L_{t_1}) = \max\left\{0, \frac{\bar{\beta} - P_k - TF}{\Delta}\right\} = Pr(\theta_{t_1}^{x_k^{LS}}|\Xi, L_{t_0}), \tag{4}$$

which is the execution probability $Pr(\theta_{t_1}^{x_k^{LS}}|\Xi, L_{t_0})$ of a limit sell, LSP_{k,t_1} posted at t_1.

Next, consider the initial time t_1 in the two-period market. The limit order book opens empty, and so the investor arriving at t_1 chooses between submitting limit orders and submitting no order (NT). From Lemma 3 in Appendix A, an investor with $\beta_{t_1} > v$ is a potential buyer who only submits a limit buy or a NT. This investor optimally posts a limit buy LBP_{k,t_1} at a price P_k if two conditions hold: First, the expected payoff

[6] We extended our previous notation so that, for example, x_{k,t_2}^{MS} and MSP_{k,t_2} are used interchangeably for a market sell order at P_k at t_2. When possible, we simplify the notation to make it consistent with the notation used in the figures.

[7] The book opens empty at t_1 and therefore the only possible limit buy a seller at t_2 can hit is a limit buy posted at t_1

from LBP_{k,t_1} given a private valuation β_{t_1} is positive so that it dominates NT:

$$(\beta_{t_1} - P_k^{cum.LB}) \times Pr(\theta_{t_1}^{x_k^{LB}} | \Xi, L_{t_0}) > 0 \tag{5}$$

and, second, it is greater than the expected payoff from all other alternative limit buys $LBP_{\sim k,t_1}$:

$$(\beta_{t_1} - P_k^{cum,LB}) \times Pr(\theta_{t_1}^{x_k^{LB}} | \Xi, L_{t_0}) > (\beta_{t_1} - P_{\sim k}^{cum,LB}) \times Pr(\theta_{t_1}^{x_{\sim k}^{LB}} | \Xi, L_{t_0}) \tag{6}$$

where $\sim k$ indexes any other possible limit price $P_{\sim k}$, and where $P_k^{cum,LB} = P_k + MF$ and $P_{\sim k}^{cum,LB} = P_{\sim k} + MF$ are the associated cum-fee limit-buy prices. Hence, the order-submission probability of LBP_{k,t_1} at t_1 is the probability that conditions (5) and (6) are both satisfied:

$$\begin{aligned} & Pr(x_{k,t_1}^{LB} | \Xi, L_{t_0}) \\ & = Pr\Big[(\beta_{t_1} - P_k^{cum,LB}) \times Pr(\theta_{t_1}^{x_k^{LB}} | \Xi, L_{t_0}) > 0, \\ & \qquad (\beta_{t_1} - P_k^{cum,LB}) \times Pr(\theta_{t_1}^{x_k^{LB}} | \Xi, L_{t_0}) > (\beta_{t_1} - P_{\sim k}^{cum,LB}) \times Pr(\theta_{t_1}^{x_{\sim k}^{LB}} | \Xi, L_{t_0}), \forall \sim k\Big] \end{aligned} \tag{7}$$

A potential seller at t_1 with $\beta_{t_1} < v$ submits a limit sell LSP_{-k,t_1} if symmetric conditions hold:

$$(P_{-k}^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-k}^{LS}} | \Xi, L_{t_0}) > 0 \tag{8}$$

$$(P_{-k}^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-k}^{LS}} | \Xi, L_{t_0}) > (P_{\sim -k}^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{\sim -k}^{LS}} | \Xi, L_{t_0}) \tag{9}$$

where $P_{-k}^{cum,LS} = P_{-k} + MF$ and $P_{\sim -k}^{cum,LS} = P_{\sim -k} + MF$ are the cum-fee limit-sell prices. Thus, the order-submission probability of LSP_{-k,t_1} at t_1 is the probability that conditions (8) and (9) both hold:

$$\begin{aligned} & Pr(x^{LS}_{-k,t_1}|\Xi, L_{t_0}) \\ & = Pr\Big[(P^{cum.LS}_{-k} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x^{LS}_{-k}}|\Xi, L_{t_0}) > 0, \\ & \qquad (P^{cum,LS}_{-k} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x^{LS}_{-k}}|\Xi, L_{t_0}) > (P^{cum,LS}_{\sim -k} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x^{LS}_{\sim -k}}|\Xi, L_{t_0}), \forall \sim k\Big] \end{aligned} \tag{10}$$

Access pricing Ξ in our model is set either by an exchange or, alternatively, by a Social Planner. In doing so, both take into account optimal investor trading behavior given the access pricing Ξ. An exchange chooses Ξ to maximize its expected payoff from completed transactions:

$$\max_{\substack{MF,TF \\ -\tau<MF,TF<+\tau}} \pi^{Ex}_{t_1}(MF,TF) = \left[\sum_{x_{t_1}\in X^L} Pr(x_{t_1}, \theta_{t_z}^{x_{t_1}}|\Xi)\right](MF+TF) \tag{11}$$

where $Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}}|S, \tau, \Xi)$ are transaction probabilities induced by the exchange Ξ fees and the optimal investor order–submission strategies from (2), which are products of the submission probabilities of different limit orders $x_{t_z} \in X^L$ and their execution probabilities summed over all possible states of the book $L_{t_{z-1}}$

$$Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}}|\Xi) = \sum_{L_{t_{z-1}}} Pr(L_{t_{z-1}}|\Xi)\, Pr(x_{t_z}|\Xi, L_{t_{z-1}})\, Pr(\theta_{t_z}^{x_{t_z}}|\Xi, L_{t_{z-1}}) \tag{12}$$

The formula in (12) reflects the fact that, in a limit order market, transactions only occur when limit orders are first submitted and then later executed. The exchange has non-negative profits since $TF = MF = 0$ is feasible and gives zero profits.

A Social Planner chooses fees to maximize the total welfare of all market participants:

$$\max_{\substack{MF,TF \\ -\tau<MF,TF<+\tau \\ MF+TF\geq 0}} \sum_{t_z\in\{t_1,t_2\}} \left(W^{INV}_{t_z}(MF,TF) + \pi^{Ex}_{t_z}(MF,TF)\right) \tag{13}$$

$$= \sum_{x_{t_1} \in X^L} \left(W_{t_1}^{INV}(x_{t_1} | \Xi, L_{t_0}) + W_{t_2}^{INV}(\tilde{x}_{t_2}(x_{t_1}) | \Xi, L_{t_0}) + Pr(x_{t_1}, \theta_{t_1}^{x_{t_1}} | \Xi)\,(MF + TF) \right)$$

given that limit orders x_{t_1} are only submitted at t_1 and lead to investor welfare:

$$W_{t_1}^{INV}(x_{t_1} | \Xi, L_{t_0}) = \int_{\beta_{t_1} \in B_{t_1}(x_{t_1}, \Xi, L_{t_0})} I_{t_1} \times [\beta_{t_1} - P(x_{t_1}) - MF)] f(\beta_{t_1}) d\beta_{t_1} \times Pr(\theta_{t_1}^{x_{t_1}} | \Xi, L_{t_0}) \tag{14}$$

where I_{t_z} is an indicator variable equal to 1 (-1) for buy (sell) orders at t_z, $\mathrm{B}_{t_1}(x_{t_1}, \Xi, L_{t_0})$ is the interval of the β_{t_1} realizations for which a given limit order x_{t_1} is optimal at t_1, and where market orders $\tilde{x}_{t_2}(x_{t_1})$ at t_2 executing earlier limit orders x_{t_1} from t_1 lead to investor welfare:

$$W_{t_2}^{INV}(\tilde{x}_{t_2}(x_{t_1}) | \Xi, L_{t_0}) = Pr(x_{t_1} | \Xi, L_{t_0}) \times \int_{\beta_{t_2} \in B_{t_2}(\tilde{x}_{t_2}(x_{t_1}), \Xi, L_{t_1})} I_{t_1} \times [P(x_{t_1}) - \beta_{t_2} - TF)] f(\beta_{t_2}) d\beta_{t_2} \tag{15}$$

given the interval $B_{t_2}(\tilde{x}_{t_2}(x_{t_1}), \Xi, L_{t_1})$ of β_{t_2} realizations for which a market order $\tilde{x}_{t_2}(x_{t_1})$ is optimal at t_2. The third term in (13) is the exchange's profit $\pi_{t_z}^{Ex}(MF, TF)$ from (11).

Given the optimization problems solved by investors and the exchange, or by the investors and the Social Planner, we can define an equilibrium:

Definition. A *Subgame Perfect Nash Equilibrium* of the trading game is a collection $\{x_{t_z}(\beta_{t_z} | \Xi, L_{t_{z-1}})$, $Pr(\theta_{t_z}^{x_{t_z}} | \Xi^*, L_{t_{z-1}})$, $\Xi^*\}$ of order-submission strategies, order-execution probabilities, and access fees such that conditions 1, 2 and 3 or conditions 1, 2 and 4 hold:

1. The equilibrium order-submission strategies $x_{t_z}(\beta_{t_z} | \Xi, L_{t_{z-1}})$ solve investors' optimization problems (2) given the equilibrium execution probabilities $Pr(\theta_{t_z}^{x_{t_z}} | \Xi^*, L_{t_{z-1}})$.

2. The order-execution probabilities $Pr(\theta_{t_z}^{x_{t_z}} | \Xi^*, L_{t_{z-1}})$ for an order x_{t_z} submitted at time t_z are consistent with the equilibrium order-submission strategies $x_{t_z}(\beta_{t_z} | \Xi, L_{t_{z-1}})$ at times $t_{z'} > t_z$.

3. The access fees Ξ^* are optimal for the exchange given its optimization problem (11).

4. The access fees Ξ^* are optimal for the Social Planner given its optimization problem (13).

As in Foucault et al. (2013), a discrete tick size guarantees traders cannot neutralize access pricing by adjusting posted limit prices to exactly offset the impact of access fees and rebates on their net cum-fee transaction prices. Our model differs from Foucault et al. (2013) in that investors in our model have random gains-from-trade and exogenous arrival/monitoring timing. Our model also differs from both Foucault et al. (2013) and Chao et al. (2018) in that the endogenous choice of limit order prices is central in our model.[8]

2.1 Results

This section solves for equilibrium fees in the two-period $\{t_1, t_2\}$ model in closed-form using results and proofs in Appendix B. Our analysis then examines the relation between access pricing and the investor valuation support $S = [\underline{\beta}, \bar{\beta}]$, the tick size τ, and investor trading behavior. In particular, we consider access pricing in a market with a large tick size (Section 2.1.1) and then in a market with a small tick size (Section 2.1.2). In the large-tick market (LTM), the tick size is normalized to $\tau = 1$, and the price grid has four possible price levels, $P_k = \{P_{-2}, P_{-1}, P_1, P_2\}$, centered around the mean investor valuation v with $P_{-2} < P_{-1} < v < P_1 < P_2$. The outside quotes are $P_2 = v - \frac{3}{2}\tau$ and $P_{-2} = v + \frac{3}{2}\tau$, and the inside quotes are $P_1 = v - \frac{1}{2}\tau$ and $P_{-1} = v + \frac{1}{2}\tau$. In a small-tick market (STM), the tick size is set to $\frac{\tau}{3}$ relative to the LTM tick size. The resulting STM price grid has ten price levels $p_j = \{p_{-5}, p_{-4}, p_{-3}, p_{-2}, p_{-1}, p_1, p_2, p_3, p_4, p_5\}$, with $p_{-1} < v < p_1$ where the furthest outside STM quotes coincide with the outside LTM quotes with $p_{-5} = P_{-2} = v - \frac{3}{2}\tau$ and $p_5 = P_2 = v + \frac{3}{2}\tau$.

The Subgame Perfect Nash equilibrium has qualitatively different access pricing and trading strategies

[8] Foucault et al. (2013) has an extension in which the limit price is determined by Nash Bargaining. In contrast, limit price choice in our model is a decision of the limit order submitter. Chao et al. (2018) includes one example of endogenous limit order choice, but otherwise their analysis is focused more on exchange competition than on investor order choice.

for different investor valuation support widths Δ given a tick size τ. The largest support we consider in the LTM, $[v-\frac{5}{2}\tau, v+\frac{5}{2}\tau]$, has a width $\Delta = 5\,\tau$, which corresponds to a market populated by very heterogeneous traders, some of whom have strong trading demands (who prefer to take available liquidity) and others with weaker trading demands (who tend to supply liquidity). The specific rationale for our largest support is that it is the largest support such that in equilibrium traders never want to post limit orders beyond the outside quotes $\{P_{-2}, P_2\}$ in the LTM or $\{p_{-5}, p_5\}$ in the STM. Smaller valuation support widths Δ correspond to market environments in which arriving traders are ex ante more predisposed to supply liquidity because individual potential gains-from-trade are small. Our analysis does not depend on the mean investor valuation, since v just centers the price grid and valuation support S.

2.1.1 Optimal access pricing for an exchange in a Large Tick Market

The exchange solves problem (11) to set its access pricing Ξ to maximize its expected profit given the trading behavior Ξ induces in the investor trading subgame. Appendix B presents the equilibrium construction and shows that the equilibrium fees and rebates for the exchange are given by the simple close-form expressions in Theorem 1 for supports S with widths Δ in different ranges.

Theorem 1. *When the valuation support width is $\Delta \leq 3\tau$, the equilibrium access pricing for a profit-maximizing exchange in the two-period LTM where $\tau = 1$ is either Taker-Maker with fees and rebates*

$$MF^* = \frac{\Delta+3}{6} \qquad TF^* = \frac{\Delta-3}{6} < 0, \tag{16}$$

or Maker-Taker with fees and rebates

$$MF^* = \frac{\Delta-3}{6} < 0 \qquad TF^* = \frac{\Delta+3}{6}. \tag{17}$$

When the valuation support with is $\Delta \in [3\tau, 5\tau]$, the equilibrium access fees are strictly positive and unique

$$MF^{*}\&TF^{*} = \begin{cases} 1 \ \& \ \frac{1}{2}(\Delta - 3) & \text{if } 3\tau < \Delta \leq 4\tau \\ 1 \ \& \ \frac{1}{4(\Delta-2)}\left(\Delta^2 - 5\Delta + 8\right) & \text{if } 4\tau < \Delta \leq 4.7\tau \\ \frac{1}{2} \ \& \ 1 & \text{if } 4.7\tau < \Delta \leq 5\tau \end{cases} \tag{18}$$

The different access pricing in the different ranges of Δ are due to the fact that, as the valuation support widens, the set of ex ante feasible prices expands to include more possible limit prices, and the execution probabilities of limit orders at more extreme limit prices increases.

Figure 1 shows equilibrium fees and rebates from Theorem 1. The upper plot in Figure 1 shows the equilibrium MF (solid lines) and TF (dashed lines) chosen by the exchange in the LTM for different valuation support widths Δ on the horizontal axis. For support widths $\Delta \leq 3\tau$, there are two symmetric Maker-Taker (blue lines) and Taker-Maker (orange) equilibria given in (16) and (17) in Theorem 1. For support widths $\Delta \in (3\tau, 5\tau]$, equilibrium access fees are non-negative (i.e., with no rebates) and unique as given in (18).

Table 1 provides additional details about equilibrium trading strategies and market properties for the LTM. For different valuation support widths Δ, it reports the equilibrium access pricing, the buyer's trading strategies, the cum-fee buy and sell transaction prices $P_k^{cum,LB}$ and $P_k^{cum,MS}$, the equilibrium probabilities of buy limit order submission, $Pr(x_{t_z}|\Xi, L_{t_{z-1}})$, and execution, $Pr(\theta_{t_z}^{x_{t_z}}|\Xi, L_{t_{z-1}})$, the equilibrium transaction probability, $\sum_{x_{t_1} \in X^L} Pr(x_{t_1}, \theta_{t_z}^{x_{t_1}}|\Xi)$, and the equilibrium expected exchange profit, $\pi^{Ex}(MF,TF)$. When there are pairs of symmetric Maker-Taker and Taker-Make equilibria — in which the cum-fee prices and the exchange's net profit per transaction are the same — they are shown in two rows of fees for the associated support Δ.[9] When there is just one row of fees for a given Δ, the profit-maximizing access pricing is unique.

[9]To economize space, Table 1 does not report the equilibrium strategies of the seller arriving at t_2 as they can be inferred from the buyer's equilibrium strategies at t_1. For example, if a limit buy is posted at t_1, $x_{t_1}^{LB}$, the equilibrium strategy of the seller taking liquidity at t_2 will be a market sell, $x_{t_2}^{MS}$. In addition, Table 1 does not report the probability of No Trade as it is the complement

The results are symmetric for limit sells at time t_1.

Proposition 1 presents a property of the equilibrium fees, which is illustrated quantitatively in Table 1 and follows immediately from the fees and rebates in (16) and (17) in Theorem 1.

Proposition 1. *The sum of the make and take fees is one third of the support width, $MF+TF=\Delta/3$ for all support widths $\Delta < 3\,\tau$ in the two-period LTM.*

The key part of the proof is that the exchange's expected profit can be expressed as

$$\pi^{Ex}(MF,TF) = 2\max\left\{0, \frac{\overline{\beta}-P_{-1}^{cum,LB}}{\Delta}\right\}(MF+TF)\max\left\{0, \frac{P_{-1}^{cum,MS}-\underline{\beta}}{\Delta}\right\} \tag{19}$$

which is the product of the relevant limit-order submission probability at time t_1, the net fee, and the relevant market-order submission probability at time t_2. The specific functional form of (19) follows from there just being two periods and from the uniform valuation distribution assumption and symmetry between the buy and sell sides of the market. The three components $\overline{\beta}-P_{-1}^{cum,LB}$, $MF+TF=P_{-1}^{cum,LB}-P_{-1}^{cum,MS}$, and $P_{-1}^{cum,MS}-\underline{\beta}$ in (19), when they are positive, sum to the valuation support width Δ. Proposition 1 shows that the product in (19) is maximized by the exchange choosing MF and TF to set these three components equal to each other, which implies that $MF+TF=\Delta/3$. The next proposition highlights other two specific properties of the equilibrium fees and rebates in Theorem 1.

Proposition 2.

1 If investor valuation dispersion is low (i.e., the valuation support width is $\Delta \le 3\tau$), symmetric Maker-Taker and Taker-Maker equilibria exist in the two-period LTM. If valuation dispersion is higher (i.e., the support width is $\Delta \in [3\tau, 5\,\tau]$), the equilibrium TF and MF are jointly positive and unique.

to 0.5 of the probability of order submission on one side of the market. For example, for the support $[9.8333, 10.1667]$ with the smallest width 0.33τ, the probability of No Trade at t_1 is $0.5-0.333=0.167$.

2 *When an exchange optimally uses Maker-Taker or Taker-Maker access pricing in the two-period LTM, then rebates are decreasing and fees are increasing as the valuation support width Δ increases.*

The profit-maximizing access pricing depends on the relationship between the support of traders' valuation and the tick size. Here we hold the tick size fixed at $\tau = 1$ and vary the support width Δ. Proposition 2 states that, given a tick size τ, optimal access pricing depends on the support of traders' valuations and therefore on the types of traders populating the market. As the support of investor valuations increases, the ex ante potential gains-from-trade increase, which increases investor trading demand. As a result, the exchange has less need to incentivize trading. In equilibrium, the exchange exploits investors' greater ex ante gains-from-trade by increasing fees and reducing rebates. When the investor valuation support is small (and trading demand is low), there is a pair of symmetric Taker-Maker and Maker-Taker equilibria. This is the same as in Chao et al. (2018). The intuition for the multiplicity of equilibria in these parameterizations is that the same cum-fee prices can be achieved either by subsidizing liquidity supply (which leads to more aggressive posted limit prices) or by subsidizing liquidity demand (which leads to less aggressive posted prices). In contrast, when Δ is larger (and trading demand is greater), then the positive-fee equilibrium is unique. This is a new result. Proposition 2 leads to an empirical prediction:

Empirical Prediction: *Exchanges in markets with low investor-valuation dispersion optimally use rebate-based Taker-Maker and Maker-Taker access pricing. Conversely, exchanges in markets with high valuation dispersion optimally have a unique optimal positive-fee access pricing.*

The relation between investor valuation heterogeneity and when an exchange optimally uses rebate-based pricing vs positive fees from Proposition 2 is clear in Figure 1. With a very narrow trader-valuation support width $0.33\,\tau$, the LTM has a pair of symmetric Maker-Taker and Taker-Maker equilibria. Since this valuation support is within the inside LTM quotes, P_{-1} and P_1, there are no prices at which buyers and sellers would mutually transact in the absence of rebates. Thus, a rebate on either limit orders or market orders is necessary for trading. Consider a potential buyer with a high personal valuation $\beta_{t_1} > v$ who arrives at t_1. (The case of a potential seller with a low valuation at t_1 is symmetric). With Maker-Taker pricing ($TF = 0.556$ and $MF = -0.444$), the exchange offers a rebate on liquidity-making via limit orders such that the buyer is willing to use an aggressive LBP_1 limit order at t_1 to offer to buy at a quoted price P_1 above his valuation ($\beta_{t_1} \leq \bar{\beta} < P_1$) to earn the make rebate. An investor with a low valuation β_{t_2} arriving at t_2 can then sell at P_1 above his valuation ($\beta_{t_2} \leq \bar{\beta} < P_1$) but must also pay a take fee. In this case, Maker-Taker pricing generates trading by cross-subsidizing liquidity-making via limit orders at aggressive posted prices at t_1 and imposing fees on liquidity-taking via market orders at t_2 (which benefit from the aggressive limit prices). The converse logic applies to the Taker-Maker equilibrium pricing ($MF = 0.556$ and $TF = -0.444$). Now investors with high valuations β_{t_1} at t_1 use LBP_{-1} limit orders to buy at P_{-1}, and investors at t_2 then either use MSP_{-1} market orders to sell at P_{-1} and receive the take rebate, or they do not trade. In each case, investor trading decisions depend on the cum-fee prices they pay or receive net of market access fees and rebates (rather than on quoted prices alone), and the exchange uses its access pricing to affect the cum-fee prices.

The relationship between access pricing and the support width in Proposition 2 is also evident in Figure 1 and Table 1. Starting from the smallest valuation support, $0.33\,\tau$, the exchange monotonically increases both MF and TF as the support width Δ increases up until the point that the regulatory cap on fees binds. For example, when the support width reaches $2\,\tau$, the buyer still buys either at P_{-1} or at P_1 and the exchange sets the symmetric Taker-Maker and Maker-Taker pricing with a positive fee of 0.883 and rebate of -0.167. Taker-Maker and Maker-Taker access pricing persists until, holding the LTM tick size fixed at τ, the investor valuation support reaches the outside quotes P_{-2} and P_2 with a width $\Delta = 3\,\tau$.

Proposition 2 and Figure 1 show three things happen once $\Delta > 3\,\tau$: First, investor trading demand is sufficiently strong that the exchange ceases giving rebates to incentivize trade and switches to strictly positive-fee access pricing. Second, the regulatory cap on fees is reached on one side of the market. Third, optimal access pricing becomes unique. As a result, the exchange starts charging the highest possible make fee on limit orders given the regulatory cap, $MF = 1.000$, and also charges a positive take fee for market orders. For example, when the support width is $3.1\,\tau$, the optimal take fee is $TF = 0.050$. In these parameterizations, low-valuation investors still profitably sell at the low price P_{-1}. In equilibrium, a high-valuation investor arriving at t_1 knows that, given the wide valuation support and the relatively low TF, there is a sufficiently high probability of a seller arriving in t_2 willing to demand liquidity at the lower price P_{-1}.

As in Foucault et al. (2013) and Chao et al. (2018), access fees by a profit-maximizing exchange have two effects: First, they can help a market mitigate the discrete tick-size trading friction. This aspect of access pricing is related to asymmetries between the make and take fees, which adjust the gains from trade between liquidity makers and takers. We measure this asymmetry by the absolute difference $|MF^* - TF^*|$.[10] Second, the net fee $MF^* + TF^*$ determines (along with the probability of a transaction) the exchange's expected profit in (11). These quantities, which can be computed analytically for the LTM from Theorem 1, are illustrated for the LTM in columns 5 and 6 of Table 1. We see that the access pricing asymmetry is decreasing and the net fee is increasing as the valuation support width Δ increases up to a support width of 4.7τ. This is consistent with trading frictions becoming less important as ex ante investor gains from trade increase. As the ex ante demand for trade increases, mitigation of trading frictions becomes less important for the exchange than profiting from both sides of transactions. Interestingly, the relation of access pricing and the support width can have jumps, as can be seen when Δ exceeds 4.7τ. The reason is that profit-maximizing fees and optimal investor trading are all endogenously determined in equilibrium. Thus, we note that optimal strategies for investors also change at 4.7τ (i.e., rather than buying at P_{-1} and P_{-2} now investors buy at P_{-1} and P_1). However, the exchange's expected profit in Table 1 is still increasing in investor trading demand, as measured by Δ. Although the net fee per transaction is now lower, the greater willingness

[10] We thank Thierry Foucault for helpful insights on the relation of fee asymmetries and net fees.

to trade increases the probability of completed transactions.

We note two differences about optimal access pricing in our model relative to Chao et al. (2018). One new feature is that equilibrium fees in our model are jointly positive for some parameterizations, whereas Chao et al. (2018) only have rebate-based access pricing. The reason for this difference is that Chao et al. (2018) constrain access pricing so that jointly positive fees are not allowed. A second new feature in our model is that make and take fees are not equivalent, unlike in Chao et al. (2018). While there are pairs of symmetric optimal fees for some market parameterizations, optimal fees are unique for other parameterizations in our model. In particular, the exchange's optimal fees are unique in the two-period market once the investor valuation support Δ is larger than 3τ, which is where the exchange starts using strictly positive take and make fees. For example, when $\Delta = 3.1\tau$, the unique equilibrium fees are $MF = 1.000$ and $TF = 0.050$. In particular, the symmetric fees $MF = 0.050$ and $TF = 1.000$ lead to lower exchange expected profits. The reason is a conceptual asymmetry between make and take fees. In our two-period market, this asymmetry takes the following form: Market orders in the two-period model are only used at time t_2 and, thus, take fees only affect the willingness of the investor at time t_2 to trade with whatever limit order happens to be in the book. In contrast, limit-order submitters at time t_1 have a decision about *which* price to post a limit order. When the investor valuation support is small, only one limit buy (or sell) price is ex ante feasible (i.e., has an positive expected profit), but when the valuation support is larger, multiple prices are ex ante feasible, and the limit-price decision becomes non-trivial. As a result, with a large valuation support, make fees affect a more complicated decision of limit-order submitters about where to post limit order (and can be used by the exchange to deter the submission of orders with low execution probabilities but private gains from favorable execution prices) as opposed to the simpler trade/no-trade decision of market-order submitters.[11] Section 3

[11] For example, in the $\Delta = 3.1\tau$ market discussed above, the hypothetical symmetric fees are suboptimal because, with $MF = 0.050$, buy limit orders at P_{-1} and P_1 both can have positive expected profits for some realized βs. Thus, given a low make fee (like 0.050) and a sufficiently wide investor valuation support (like $\Delta = 3.1\tau$) — such that there is a sufficient probability of investors at t_2 with very low private valuations who would be willing to sell at a low cum-fee price of $P_{-1} - TF$, — some investors at t_1 (with valuations only slightly above v) would post buy limit orders at P_{-1} rather than at P_1. Since such orders have lower execution

shows a related asymmetry in make and take fees in our three-period market.

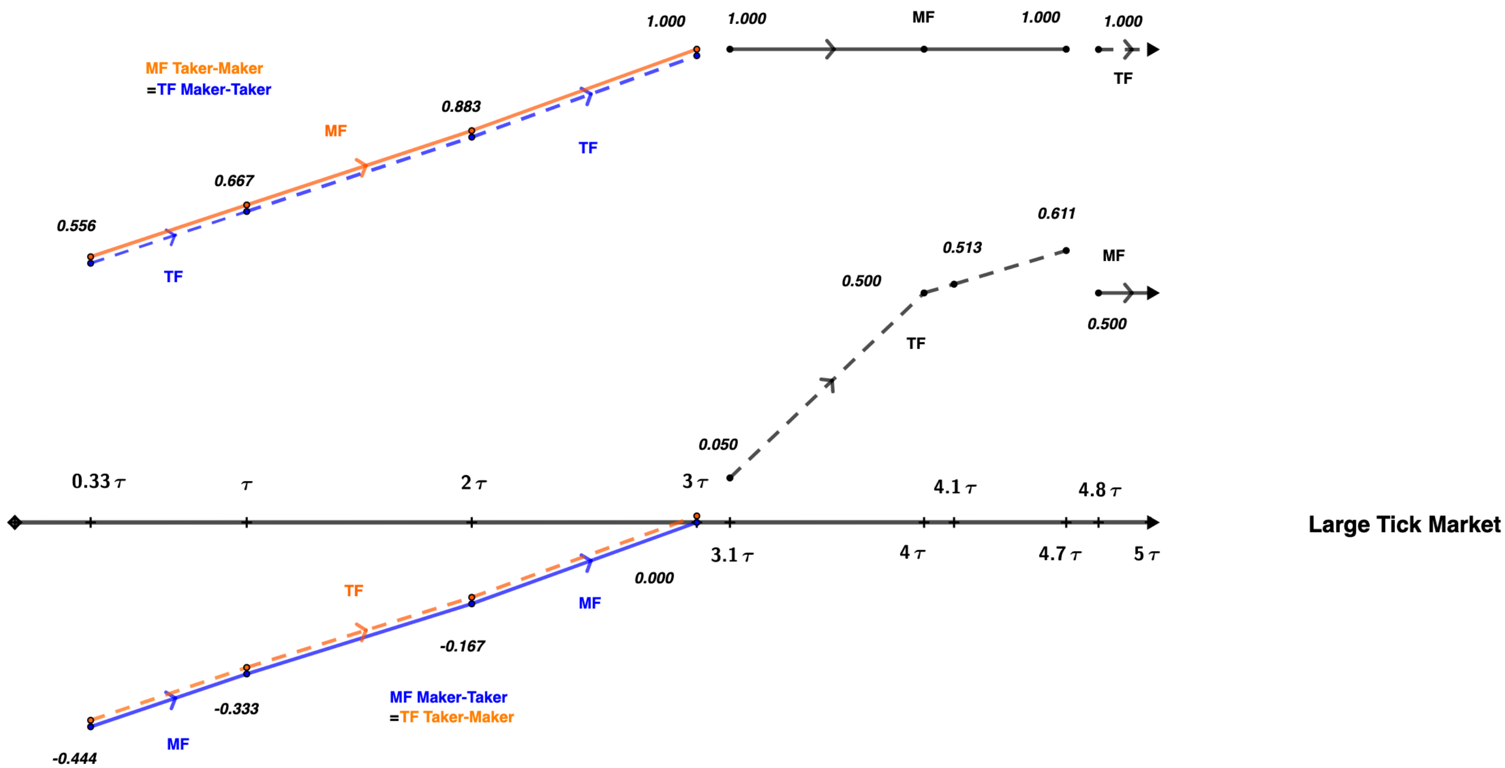
Qualitative changes in the equilibrium investor strategy at t_1 generally coincide with discrete changes in the exchange's optimal fees. We see this clearly in Table 1 and Figure 1. As the valuation support width Δ increases beyond $4\,\tau$ (in the region with strictly positive fees), buyers start using two possible different limit orders at t_1 — i.e., they now buy at P_{-1} or at P_{-2} — for two different intervals of β_{t_1}. When $\Delta = 4\,\tau$, the buyer has no incentive to post a limit buy at P_{-2} (i.e., a seller with the minimum possible valuation of 8 would not sell at the cum fee sell price $P_{-2}^{cum,MS} = 8.5 - 0.5 = 8$), but with a wider support, e.g., $\Delta = 4.1\tau$, the buyer does have an incentive to post limit buys at 8.5 as the incoming seller even with the minimum valuation 7.95 would be willing to sell at $P_{-2}^{cum,MS} = 8.5 - 0.513 = 7.987$. The exchange exploits the larger ex ante gains-from-trade of sellers at t_2 by setting a higher TF, and keeps charging buyers at t_1 the maximum $MF = 1.000$ up until the support width reaches $\Delta = 4.8\,\tau$. Once $\Delta \geq 4.8\,\tau$, the buyer switches from using LBP_{-2} and LBP_{-1} to using LBP_{-1} and LBP_1, and the exchange halves the MF to 0.500 and increases the TF to 1.000. This increases the transaction probabilities, which increases the exchange's profit.

2.1.2 Small tick market

A general issue of interest is the relation between access pricing and both the support S of investor private valuations and the tick size τ. Section 2.1.1 shows how optimal access pricing is driven by the relative size of the valuation support width Δ holding the tick size τ fixed. This section shows how optimal access pricing is also driven by the absolute tick size τ given that the regulatory cap on fees is tied to the absolute tick size. We change the tick size τ and compare LTMs and STMs given the same range of valuation supports. The comparative static for τ depends on both the relative tick-size channel and also on an absolute tick-size channel.

probabilities than limit orders at P_1, this reduces the exchange's expected profits relative to the equilibrium fees, thereby making the hypothetical symmetric fees $MF = 0.050$ and $TF = 1.000$ suboptimal.

Figure 1: Make Fees and Take Fees in a 2-Period Market with Profit-Maximizing Exchange. This figure reports the equilibrium make fees (MF & mf) and take fees (TF & tf) in the Large Tick Market (LTM) (upper panel) and Small Tick Market (STM) (lower panel) corresponding to different investor valuation supports with widths ranging from $0.33\,\tau$ to $5\,\tau$ on the horizontal axes (where $\tau = 1$ is the tick size in the LTM). The figure reports in blue (orange) italics the equilibrium fees MF (TF). The Taker-Maker and Maker-Taker pricing structures are optimal and symmetric for a support with widths ranging from $0.33\,\tau$ to $3\,\tau$ in the large tick market, and for a support with widths ranging from $0.33\,\tau$ to τ in the small tick market.



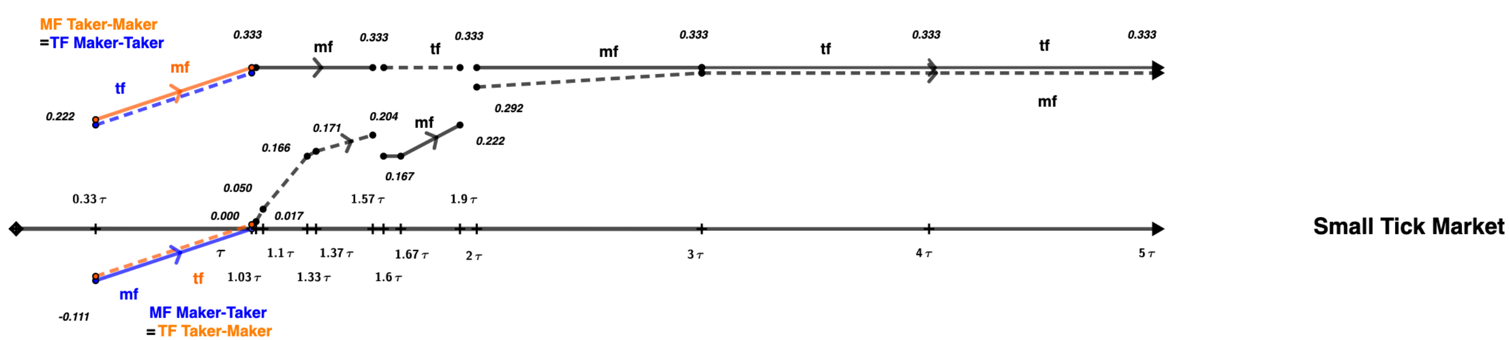


Table 1: 2-Period Large Tick Market (LTM) with Profit-Maximizing Exchange: Equilibrium Fees and Trading Strategies. This table reports for different investor valuation support width, $\Delta = \overline{\beta} - \underline{\beta}$ expressed in terms of the LTM tick size, τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fees, MF^* and TF^* (column 3 and 4), the sum and the absolute difference of the equilibrium MF^* and TF^* (column 5 and 6), the buyer's equilibrium trading strategies at t_1, x_{t_1} other than No Trade (column 7) and the associated probability of submission at t_1, $Pr(x_{t_1}|\Xi^*, L_{t_0})$ (column 8). The table also shows the cum-fee buy and sell prices $P_k^{cum,LB}$ and $P_k^{cum,MS}$ (column 9 and 10), the equilibrium probability of execution of the buyer's order posted at t_1, $Pr(\theta_{t_1}^{x_{t_1}}|\Xi^*, L_{t_0})$, which correspond to the unconditional probability of MS at t_2 (column 11), the equilibrium transaction probability $Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}}|S, \tau, \Xi)$ (column 12), and the exchange expected profit from both buyers and sellers, $\pi^{Ex}(MF^*, TF^*)$ (column 13). When the equilibrium pricing is rebate-based for a given support, we report the Taker-Maker fees on the first row and then the Maker-Taker fees on the second row. Results are rounded to the third decimal.

Support width $\Delta = \overline{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	MF^*	TF^*	MF^*+TF^*	$\lvert MF^*$-$TF^*\rvert$	Eq.Strategy x_{t_1} at t_1	Pr. Submission $Pr(x_{t_1}\lvert\Xi^*, L_{t_0})$	$P_k^{cum,LB}$	$P_k^{cum,MS}$	Pr. Execution $Pr(\theta_{t_1}^{x_{t_1}}\lvert\Xi^*, L_{t_0})$	Pr.Trans	Exchange E[Profit] $\pi^{Ex}(MF^*, TF^*)$
$0.33\,\tau$	9.833, 10.167	0.556	-0.444	0.111	1.000	LBP_{-1}	0.333	10.056	9.944	0.333	0.222	0.025
		-0.444	0.556	0.111	1.000	LBP_1	0.333	10.056	9.944	0.333	0.222	0.025
τ	9.500, 10.500	0.667	-0.333	0.333	1.000	LBP_{-1}	0.333	10.167	9.833	0.333	0.222	0.074
		-0.333	0.667	0.333	1.000	LBP_1	0.333	10.167	9.833	0.333	0.222	0.074
$2\,\tau$	9.000, 11.000	0.833	-0.167	0.666	1.000	LBP_{-1}	0.333	10.333	9.667	0.333	0.222	0.148
		-0.167	0.833	0.666	1.000	LBP_1	0.333	10.333	9.667	0.333	0.222	0.148
$3\,\tau$	8.500, 11.500	1.000	0.000	1.000	1.000	LBP_{-1}	0.333	10.500	9.500	0.333	0.222	0.222
		0.000	1.000	1.000	1.000	LBP_1	0.333	10.500	9.500	0.333	0.222	0.222
$3.1\,\tau$	8.450, 11.550	1.000	0.050	1.050	0.950	LBP_{-1}	0.339	10.500	9.450	0.323	0.219	0.229
$4\,\tau$	8.000, 12.000	1.000	0.500	1.500	0.500	LBP_{-1}	0.375	10.500	9.000	0.250	0.188	0.281
$4.1\,\tau$	7.950, 12.050	1.000	0.513	1.513	0.487	LBP_{-1} , LBP_{-2}	0.369, 0.131	10.500, 9.500	8.987, 7.987	0.253, 0.009	0.189	0.286
$4.7\,\tau$	7.650, 12.350	1.000	0.611	1.611	0.389	LBP_{-1} , LBP_{-2}	0.342 , 0.157	10.500 , 9.500	8.889 , 7.889	0.264 , 0.051	0.197	0.317
$4.8\,\tau$	7.600, 12.400	0.500	1.000	1.500	0.500	LBP_1 , LBP_{-1}	0.104, 0.396	11.000, 10.000	9.500, 8.500	0.396, 0.188	0.231	0.346
$5\,\tau$	7.500, 12.500	0.500	1.000	1.500	0.500	LBP_1 , LBP_{-1}	0.100 , 0.400	11.000 , 10.000	9.500 , 8.500	0.400 , 0.200	0.240	0.360

Theorem 2. *The equilibrium access pricing Ξ^* in the two-period STM with a tick size τ^{STM} equal to a fraction $\frac{\tau^{STM}}{\tau}$ of the tick size τ of the LTM, for $0 < \Delta \leq 5\tau$, is given by:*

$$\Xi^* = \{mf^*, tf^*\} = \{\frac{\tau^{STM}}{\tau} MF^*, \frac{\tau^{STM}}{\tau} TF^*\} \tag{20}$$

where MF^* and TF^* are the optimal LTM access pricing given the tick size τ in Theorem 1.

The STM and LTM equilibrium access pricing are isomorphic in that optimal access pricing scales linearly in the tick size τ. This follows because τ is a normalization (see proof of Theorem 2 in Appendix B.3). Thus, all else equal, when regulation caps fees to be smaller than the tick size (in Section 2.1.4 we relax this assumption), a smaller tick size leads the exchange to set weakly smaller fees and rebates.

Figure 1 (lower panel) (and Table D1 in Online Appendix D.2) illustrate Theorem 2 for a STM with a tick size $\frac{1}{3}\tau$, which is $\frac{1}{3}$ of the LTM tick size $\tau = 1$.[12] The optimal access pricing in this STM is $\frac{1}{3}$ of the pricing in a LTM market with the same relative valuation support ratio Δ/τ. For example, consider a LTM with a valuation support $\Delta = \tau = 1$. Theorem 1 gives the optimal LTM Taker-Maker and Maker-Taker fees to be $\{MF^*, TF^*\} = \{\frac{2}{3}, -\frac{1}{3}\}$ and $\{-\frac{1}{3}, \frac{2}{3}\}$ respectively. From Theorem 2, the optimal STM fees for a support $\frac{1}{3}\tau$ are therefore $\{\mathrm{mf}^*, \mathrm{tf}^*\} = \{\frac{2}{9}, -\frac{1}{9}\}$ and $\{-\frac{1}{9}, \frac{2}{9}\}$, which are one third smaller.

Given Theorem 2, all the results for the LTM also hold in the STM. For example, we see in Figure 1 (lower panel) that, as the valuation support increases, the STM exchange still has an incentive to increase its fees and reduce its rebates. The STM access pricing changes are qualitatively the same as in the LTM but rescaled. In addition, the region of valuation supports in Figure 1 consistent with the STM exchange profitably using Taker-Maker or Maker-Taker pricing is narrower, and the fees themselves are smaller in absolute values. Not surprisingly, STM access pricing reaches the threshold when both fees are positive earlier since the regulatory cap on fees (which is tied to the tick size) binds sooner in absolute terms (although exactly the same in proportional terms relative to the tick size).[13] Thus, the exchange's optimal access pricing Ξ^* depends on both the absolute tick size (given the regulatory cap on fees tied to the tick size) and the relative size of the investor valuation support compared to the tick size.

Empirical Prediction: *When, holding the trading population constant, the tick size increases (decreases),*

[12] The exchange profit functions and their maximizers are qualitatively similar in the STM to the Figure D1 for the LTM and are available from the authors upon request.

[13] To ease the comparison between the STM and the LTM, we provide finer numerical detail for the STM in the regions of the valuation support where there are discontinuities in optimal access pricing. These correspond to support widths in the LTM where there are discontinuities in access pricing divided by three.

the exchange has an incentive to offer greater (smaller) fees and rebates.

It is not just the absolute value of the tick size that matters when determining the optimal fee structure but also the relative tick size given the valuation support width. More precisely, when the tick size is smaller, the exchange has less freedom in setting the trading fees.

Our empirical prediction can be tested by investigating how a change in the tick size alters the incentive for the exchange to offer rebates. Our model predicts that when, all else equal, the tick size increases, the exchange, to attract volume, should increase the rebates offered to the same population of market participants. However, with competition, if the exchange does not adjust the rebates to the new tick size, it runs the risk of seeing orders migrating to other more profitable venues. Comerton-Forde et al. (2019) investigate the effects of an increase in the tick size within the U.S. tick size pilot program started in October 2016 and, interestingly, find that following the increase in the U.S. tick size from 1 penny to 5 pennies a substantial amount of orders migrated from the maker-taker to the taker-maker inverted fees platforms. This finding is consistent with our model's prediction that following an increase in the tick size the exchange should offer greater rebates to ensure that volume is maximized within a trading platform.

2.1.3 Optimal fees for a Social Planner

This section considers optimal fees in the two-period model set by a Social Planner rather than by a profit-maximizing exchange. Figure 2 shows the welfare-maximizing MF and TF for different investor valuation support widths Δ for the large tick market (upper part) and small tick market (lower part). Tables 2 reports additional detail about equilibrium strategies, cum fee buy and sell prices, and the welfare of both limit ($W_{t_1}^{INV}$) and market orders ($W_{t_2}^{INV}$) in the LTM. (See Table D2 in Online Appendix D.2 for the STM.) From the same logic as Theorem 2, optimal Social Planner fees for the STM and LTM are isomorphic, because the tick size τ is a normalization.

The Social Planner fees and rebates in the LTM are symmetric and equal to 0.5 up to the 2τ support width. As the support grows beyond 2τ, the welfare-maximizing fees become asymmetric with the TF rebate and the MF fee decreasing in the Taker-Maker regime down to ± 0.250 and increasing in the Maker-Taker regime up to ± 0.750. The reason the absolute value of the fees decreases in the Taker-Maker and instead increases in the Maker-Taker is that, when there are two — rather than a unique — equilibrium sets of fees, the cum-fee buy and sell prices for the Taker-Maker have to be the same as those for the Maker-Taker so that traders' payoffs are the same under the two regimes. Therefore, the equilibrium Taker-Maker cum-fee limit buy price at P_{-1} (i.e., $v - \frac{1}{2}\tau + MF_{TM}$) has to be equal to the Maker-Taker cum-fee limit buy price at P_1 (i.e., $v + \frac{1}{2}\tau + MF_{MT}$). For example, when the valuation support is 3τ, the Taker-Maker rebate is 0.250, whereas Maker-Taker rebate is 0.750, so that Taker-Maker $P_{-1}^{cum,LB}$ = Maker-Taker $P_{1}^{cum,LB}$. Beyond the 3τ support width, the TF rebate reverts reaching ± 0.750 when the support is 5τ.

The pattern in the Social Planner's welfare-maximizing fees follows from four intuitions: The first is that when $\Delta \leq 2\tau$, the Social Planner can use fees and rebates to allow trading at a mid-quote equal to v. The second intuition is that, as the valuation support width Δ widens, mid-quote trading is no longer possible. In particular, investors at time t_2 eventually become potentially willing to hit limit orders with worse limit prices. This creates an incentive for investors at t_1 with valuations β_{t_1} close to v to submit worse limit orders, which give them a private gain from price improvement, but which lower total welfare due to their lower execution probability. To maximize total welfare, the Social Planner adjusts access prices to prevent this for a range of Δs. For example, for Δ between 2τ and 3τ, the Social Planner in the Taker-Maker equilibrium in the LTM reduces the take rebate to deter execution of limit buys at P_{-2} at t_2 so that potential buyers at t_1 will not submit them and instead continue to submitting limit buys at P_{-1}. The third intuition is that expected welfare on submitted limit orders is maximized if the probability of limit order submission and the probability of submission of market orders that execute standing limit orders are equal. However, deterring latent limit orders at worse prices skews these probabilities away from equality. Thus, there is a trade-off between these two effects. For example, in the LTM, once Δ exceeds 3τ, the probability distortion is so large, that the Social Planner switches and begins to adjust fees to reduce the probability distortion. As a result, potential buyers at t_1 with valuation above but close to v start submitting limit buys at P_{-2} (while investors at t_1 with higher valuations β_{t_1} continue to submit limit orders at P_{-1} for the higher order-execution probability). The fourth intuition is that another effect of a wider support Δ is that there are more investors at t_1 with extreme valuations who value higher execution probability more than price improvement. The Social Planner also uses access pricing to increase the endogenous number of such investors. For example, that is why the Social Planner continues increasing the take rebate above 0.5 when Δ is between 4τ and 5τ. Interestingly, the lower part of Figure 2 for the STM shows that this pattern in optimal Social Planner access pricing repeats for even larger relative support widths Δ as additional prices successively further from v become ex ante feasible limit prices. Given the isomorphism between the LTM and STM, this implies the equilibrium MF and TF in the LTM fluctuate between ± 0.250 and ± 0.750 when the support widens further beyond 5τ.

2.1.4 Regulatory regimes

Regulatory restrictions can have a major impact in equilibrium on optimal access pricing by an exchange. To show the impact, we compare two alternative regulatory regimes: The first regime is one in which the exchange can freely choose its access pricing. We call this the *“No Restrictions"* regime. The second is the regime in our model, which assumes the trading platform cannot set trading fees that exceed the tick size, $-\tau \leq MF \leq \tau$ and $-\tau \leq TF \leq \tau$. We call this the *RRS Regulatory Restrictions* regime. As discussed in Section 1, this is qualitatively similar to US regulation (and also with Foucault et al. (2013)).[14]

[14]Using a cap of 0.3 of the tick size from Reg NMS makes our results tighter. In Europe, there is no formal regulatory cap but informal regulatory understandings and industry norms following US markets lead exchanges usually to set fees less than one tick.

Figure 2: Make Fees and Take Fees in a 2-period Model with Social Planner This figure reports the equilibrium make fees (MF & mf) and take fees (TF & tf) in the Large Tick Market (LTM) (upper panel) and Small Tick Market (STM) (lower panel) corresponding to different investor valuation supports with widths ranging from $0.33\,\tau$ to $5\,\tau$ on the horizontal axes (where $\tau = 1$ is the tick size in the LTM). The figure reports in blue (orange) italics the equilibrium fees MF (TF). The Taker-Maker and Maker-Taker pricing structures are optimal and asymmetric.

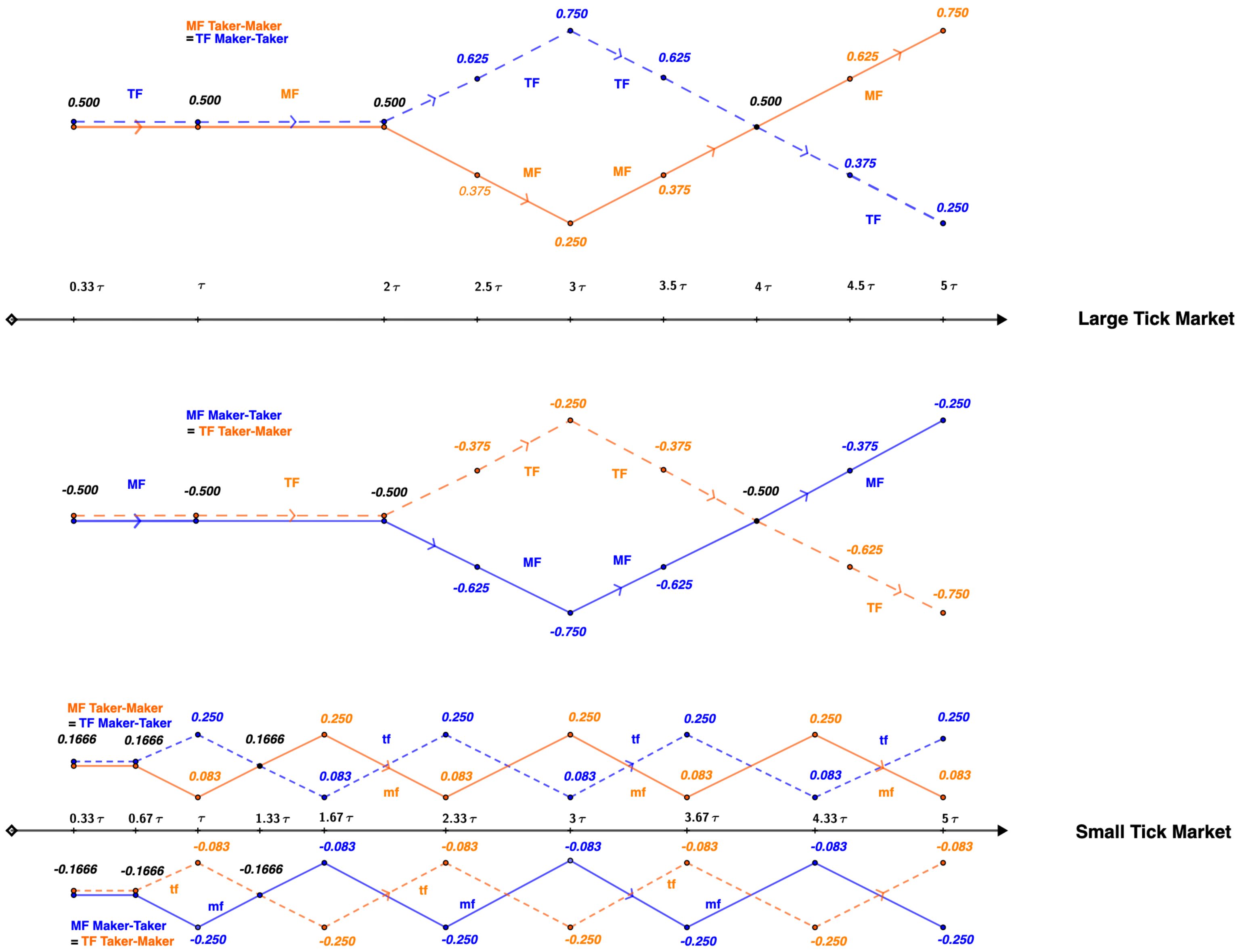


Table 2: 2-Period Large Tick Market (LTM) with Social Planner: Equilibrium Fees and Trading Strategies. This table reports for different investor valuation support width, $\Delta = \overline{\beta} - \underline{\beta}$ expressed in terms of the LTM tick size, τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fees, MF and TF (column 3 and 4), the sum and the absolute difference of the equilibrium MF^* and TF^* (column 5 and 6), the buyer's equilibrium trading strategies at t_1, x_{t_1} other than No Trade (column 7) and the associated the cum-fee buy and sell prices $P_k^{cum,LB}$ and $P_k^{cum,MS}$ (column 8 and 9). It also reports the equilibrium welfare of the limit order submitted at t_1, $W_{t_1}^{x_{t_1}}$, and the welfare associated with the market order posted at t_2, $W_{t_2}^{x_{t_2}}$, as well as the sum of limit and market orders $W_{t_1}^{x_{t_1}} + W_{t_2}^{x_{t_2}}$ (columns 10, 11 and 12). When the equilibrium pricing is rebate-based for a given support, we report the Taker-Maker fees on the first row and then the Maker-Taker fees on the second row. Results are rounded to the third decimal.

Support width $\Delta = \overline{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	MF	TF	|MF-TF|	Eq.Strategy x_{t_1}	$P_k^{cum,LB}$	$P_k^{cum,MS}$	W. x_{t_1}	W. x_{t_2}	W. $x_{t_1} + x_{t_2}$
τ	9.500, 10.500	0.500	-0.500	1.000	LBP_{-1}	10.000	10.000	0.125	0.125	0.250
		-0.500	0.500	1.000	LBP_1	10.000	10.000	0.125	0.125	0.250
2τ	9.000, 11.000	0.500	-0.500	1.000	LBP_{-1}	10.000	10.000	0.250	0.250	0.500
		-0.500	0.500	1.000	LBP_1	10.000	10.000	0.250	0.250	0.500
2.5τ	8.750, 11.250	0.375	-0.375	0.750	LBP_{-1}	9.875	9.875	0.338	0.253	0.591
		-0.625	0.625	1.250	LBP_1	9.875	9.875	0.338	0.253	0.591
3τ	8.500, 11.500	0.250	-0.250	0.500	LBP_{-1}	9.750	9.750	0.417	0.260	0.677
		-0.750	0.750	1.500	LBP_1	9.750	9.750	0.417	0.260	0.677
3.5τ	8.250, 11.750	0.375	-0.375	0.750	LBP_{-1}, LBP_{-2}	9.875, 8.875	9.875, 8.875	0.485	0.285	0.770
		-0.625	0.625	1.250	LBP_1, LBP_{-1}	9.875, 8.875	9.875, 8.875	0.485	0.285	0.770
4τ	8.000, 12.000	0.500	-0.500	1.000	LBP_{-1}, LBP_{-2}	10.000, 9.000	10.000, 9.000	0.562	0.313	0.875
		-0.500	0.500	1.000	LBP_1, LBP_{-1}	10.000, 9.000	10.000, 9.000	0.562	0.313	0.875
4.5τ	7.750, 12.250	0.625	-0.625	1.250	LBP_{-1}, LBP_{-2}	10.125, 9.125	10.125, 9.125	0.639	0.349	0.988
		-0.375	0.375	0.750	LBP_1, LBP_{-1}	10.125, 9.125	10.125, 9.125	0.639	0.349	0.988
5τ	7.500, 12.500	0.750	-0.750	1.500	LBP_{-1}, LBP_{-2}	10.250, 9.250	10.250, 9.250	0.710	0.396	1.106
		-0.250	0.250	0.500	LBP_1, LBP_{-1}	10.250, 9.250	10.250, 9.250	0.710	0.396	1.106

To compare the effects of these two regulatory restrictions on optimal access pricing, we also solve our two-period model in the No Restrictions regime for different supports and different tick sizes. Table 3 illustrates the differences between the two regimes for two different support widths, $\Delta = \tau$ and $\Delta = 2\tau$, and two different tick sizes, τ and $\frac{\tau}{3}$. With *"No Restrictions"* on access pricing, the Taker-Maker (shown) and symmetric Maker-Taker (not shown) pricing structures are both optimal in equilibrium. The exchange optimally charges the same positive fee MF and pays same rebate TF irrespective of the tick size. Under this regime, when we hold the support constant and change the tick size, the exchange holds the optimal fees constant and forces investors to trade at the outside quotes. When instead, still under the *"No Restrictions"* regime, we hold the tick size constant either at $Tick = \tau$ or at $\frac{\tau}{3}$ and increase the support width Δ, the exchange changes its optimal fees to exploit the larger gains from trade by increasing the positive charge and reducing the rebate. This result comes from the same logic as the first part of Theorem 1 when the tick-size constraint is not binding. It follows then that the total fee increases proportionally with the support.

When access pricing is subject to the *RRS Regulatory Restrictions* regime, the results differ. For the parameterizations considered in Table 3, we see the Taker-Maker pricing and symmetrically the Maker-Taker pricing prevail when the tick size is equal to τ. When we hold the support constant and change the tick size, as in Table 3, we also change the regulatory restriction. For a smaller tick size (e.g., $\frac{\tau}{3}$) both the positive charge on MF and the rebate on TF are smaller. Once again, the net fee is one third of the valuation support width. Intuitively, under the RRS Regulatory Restrictions, the exchange, not being allowed to impose extreme trading fees given the cap, maximizes profits by imposing the symmetric Taker-Maker or Maker-Taker pricing when the support is equal to the tick size. When instead, still under the *RRS Regulatory Restrictions* regime, we hold the tick size constant and increase the support, as in Table 3, the exchange exploits the larger gains-from-trade and sets strictly positive fees. This result holds both with the *"No Restrictions"* and with the *RRS Regulatory Restrictions*.

2.1.5 Exchange Competition in Trading Fees

Having examined access pricing with a single monopolistic exchange, we next consider duopolistic competition between exchanges. A striking result in Chao et al. (2018) is the absence of pure-strategy equilibria with

Table 3: Optimal Trading Fees and Restrictions This table reports the equilibrium optimal make (MF) and take fee (TF), Exchange Expected Profit, equilibrium strategies, cum-fee buy and sell prices ($P_k^{cum,LB}$ and $P_k^{cum,MS}$) for a support with width $\Delta = \tau$ and $\Delta = 2\tau$ for markets with two different tick size specifications (τ, $\frac{\tau}{3}$) and under two different regulatory regimes. The "RRS Regulatory Restrictions" are $-\tau \leq MF, TF \leq \tau$; the "No Restrictions" protocol imposes no restrictions on MF and TF fees.

		Support 1 τ		Support 2 τ	
		Tick=τ	Tick=$\frac{\tau}{3}$	Tick=τ	Tick=$\frac{\tau}{3}$
	MF	0.667	0.667	0.833	0.833
"No Restrictions"	TF	-0.333	-0.333	-0.167	-0.167
	Exchange E[Profit]	0.074	0.074	0.148	0.148
$-\Delta \leq MF \leq \Delta$	Eq.Strategies x_{t_1}	$LB_{9.500}$	$LB_{9.500}$	$LB_{9.500}$	$LB_{9.500}$ & $LB_{9.167}$
$-\Delta \leq TF \leq \Delta$	$P_k^{cum,LB}$	10.167	10.167	10.333	10.333 & 10.000
	$P_k^{cum,MS}$	9.833	9.833	9.667	9.667 & 9.333
	MF	0.667	0.333	0.833	0.333
"RRS Regulatory	TF	-0.333	0	-0.167	0.292
Restrictions"	Exchange E[Profit]	0.074	0.074	0.148	0.130
$-\tau \leq MF \leq \tau$	Eq.Strategies x_{t_1}	$LB_{9.500}$	$LB_{9.833}$	$LB_{9.500}$	$LB_{9.500}$ & $LB_{9.833}$
$-\tau \leq TF \leq \tau$	$P_k^{cum,LB}$	10.167	10.167	10.333	10.167 & 9.833
	$P_k^{cum,MS}$	9.833	9.833	9.667	9.541 & 9.208

competing exchanges. As a result, Chao et al. (2018) derive mixed-strategy equilibria. However, a critical assumption in Chao et al. (2018) is that competing exchanges move simultaneously in setting their access pricing schedules. We revisit exchange competition with one significant change: We consider a sequential-move Nash bargaining equilibrium with a series of alternating access-pricing "moves" by the exchanges. These pricing moves consist of announcements of proposed access pricing schedules possibly bundled with threats of how an exchange will respond if its proposal is not accepted. The sequential moves could, in principle, continue for an arbitrary number of multiple rounds, but, we consider here a simple game with only three moves by two exchanges to illustrate our main points. We show, in particular, that important features of the Chao et al. (2018) analysis still hold in a sequential-move equilibrium. In particular, our equilibria also do not converge to zero-profit Bertrand competition. This is, in part, due to the same two-sided nature of the market (Rochet and Tirole (2006)) that drives Chao et al. (2018) competition analysis. In particular, investors have heterogeneous preferences over both order execution probability and execution price given their realized valuations β_{t_z}.

The change from simultaneous moves to finite sequential-move bargaining means, most importantly, that now pure-strategy equilibria are possible. Indeed, multiple pure-strategy equilibria are possible depending on the exchanges' precommitment ability. With sequential bargaining, exchanges observe each others' access pricing schedules and then set their access pricing as best responses to the pricing and threats of the other exchanges. In particular, exchanges can take into account how other exchanges threaten to respond to proposed pricing schedules. In contrast, in a simultaneous-move equilibrium, exchanges set their fees given beliefs about other exchanges' fees but are unable to respond to other exchanges' actual fees or follow through on contingent threats if an exchange's access-pricing proposal is countered in a way that triggers a threatened response. Sequential-move Nash is realistic in that exchanges in actuality can observe each others' pricing schedules when setting their access pricing. For example, on September 2012 (two years since its last change in fees on November 2010), Turquoise reduced its rebate-based access pricing from TF=0.30 bps and MF=-0.20 bps to TF=0.30 bps and MF=-0.14 bps. On January 2013 (four months later), BATS Europe reacted by reducing the rebate-based pricing on CXE from TF=0.30 bps and MF=-0.20 bps to TF=0.30 bps and MF=-0.15 bps.[15] This timing seems more consistent with sequential moves than simultaneous moves.

Consider a fee-setting game between two exchanges, denoted as exchange A and exchange B, in which A moves first, B responds, and then A has the final move. The nature of the equilibrium depends on how much pre-commitment power exchange A has. We specifically consider a variation on a grim trigger equilibrium in which exchange A moves first and sets access prices $\{MF^A, TF^A\}$ and stipulates some schedule

[15] See, for example, the access price list for Turquoise: https://www.lseg.com/sites/default/files/content/documents/Turquoise%20tariff%20schedule%20-%207.8.9%20%28Eff.%2001%20July%202020%29_0.pdf
and for CXE: https://cdn.cboe.com/resources/participant_resources/CboeEurope_TradingPricing.pdf

$\{MF^B, TF^B\}$ that exchange B should use so as not to trigger a threatened counterresponse $\{\hat{MF}^A, \hat{TF}^A\}$ by exchange A. Our analysis here focuses on the threatened counterresponse. One possible threat is for exchange A to counter any unacceptable positive-profit fee schedule $\{\overline{MF}^B, \overline{TF}^B\} \neq \{MF^B, TF^B\}$ by exchange B by matching the take rebate/fee $\hat{TF}^A = \overline{TF}^B$ (so as to keep market-order submission incentives and, thus limit-order execution probabilities, unchanged relative to exchange B) but improve the attractiveness for liquidity makers by lowering the make fee/raising the make rebate so that $\hat{MF}^A < \overline{MF}^B$ with $\hat{MF}^A + \hat{TF}^A > 0$. This is possible since the schedule $\{\overline{MF}^B, \overline{TF}^B\}$ has positive profits by hypothesis. In this case, the exchange A's threat schedule would attract all limit orders at time t_1 since it is has lower make costs than $\{\overline{MF}^B, \overline{TF}^B\}$ and equally good order execution probabilities at t_2. We call this a *MF-undercutting/TF-matching* schedule. It is a specific type of grim trigger threat. Credible threats often require precommittment. However, it is reasonable to suppose that precommitment difficulty is increasing in the ex post suboptimality of a threat. Hence, the MF-undercutting/TF-matching threat should have relatively low precommitment difficulty since exchange A still earns positive expected profits, rather than a loss, if it follows through on the threat.

Different pure-strategy equilibria can be sustained given the particular commitment mechanisms available to exchanges to make credible threats. We illustrate several possible forms of credible threats and the associated sequential-move bargaining equilibria. Our insight is that standard results about sequential bargaining with precommitment apply to the access pricing game between the duopolistic exchanges. Thus, pure-strategy equilibria for the exchanges seem plausible. In addition, we show that, as in Chao et al. (2018), Bertrand competition between the exchanges does not lead to zero-profits here.

Theorem 3. *(i) Monopolistic access pricing can be implemented as a grim trigger sequential-move equilibrium of the exchange-competition subgame in which exchange A sets access prices using the monopolistic pricing schedule and exchange B does not compete. (ii) Monopolistic access pricing can also be implemented in a grim trigger sequential-move equilibrium in which exchanges A and B both set access prices to the* *identical monopolistic schedule, and investors mix between them in submitting their limit orders at time t_1.*[16]

It is also possible to implement monopolistic equilibria with price discrimination with competing exchanges. For example, Chao et al. (2018) construct an equilibrium for a single exchange operator that runs multiple exchanges that price discriminate between investors with high realized gains-from-trade and those with low realized gains-from-trade.

Theorem 4. *(i) A price-discrimination equilibrium for a single exchange operator with two exchanges can be implemented in a grim trigger sequential-move equilibrium with competing exchange operators if operator A operates two exchanges. (ii) A price discriminating equilibrium can also be implemented in a grim trigger sequential-move equilibrium by exchanges A and B both using one of the two discriminating pricing schedules.*

Other pure-strategy equilibria can be implemented without grim trigger precommitment. For example, the opposite extreme is no precommitment.

Theorem 5. *With no precommitment and two rounds of trade and no precommitment, a Stackelberg equilibrium can be implemented by exchanges A and B.*

Both exchanges in the Stackelberg equilibrium can have trading volume and positive profits given that this is a two-sided market in which limit order submitters care about both limit order submission costs (which are affected by makes fees) and limit order execution probabilities (which are affected by take fees). Thus, this key insight about exchange competition in Chao et al. (2018) is still robust with sequential moves, but now mixed strategies are not needed for the exchanges.

Chao et al. (2018) cite changing fee schedules as evidence consistent with their simultaneous-move mixed-strategy equilibria. Our analysis of exchange competition is consistent with a different interpretation

[16]The equilibrium in part (ii) has pure strategies for the two exchanges although it does involve mixed strategies for investors.

of these same empirical facts. First, the time lag between when one exchange changes its fees and another exchange changes its fees is measured in months (e.g., as the previously mentioned Turquoise-CXE sequential moves show). Thus, it is reasonable to think exchanges set their fees knowing other exchanges' currently prevailing fees and also with some forecast of how other exchanges may respond to changes in a exchange's fees. Second, exchanges in our model change access pricing, not as draws from a distribution over possible pricing schedules that they are indifferent about, but rather as optimal responses to changing market conditions and changing exchange market power (i.e., which may affect the credibility of their threats and bargaining power). In particular, if the distribution of private values of arriving investors become more or less heterogeneous (i.e., if the investor valuation support widens or contracts), our analysis shows exchanges have an incentive to change their fees. In addition, we predict exchanges change their fees in response to both changing public information about the probability distribution of arriving investors and also in response to private information about investors trading on their exchange and given inferences about the distribution of investors on other exchanges. Third, equilibrium with unrestricted exchange entry with simultaneous moves in Chao et al. (2018) involves an infinite number of exchanges, whereas with sequential-move bargaining, our model can implement pure-strategy equilibria with a small number of competing exchanges. This seems consistent with the fact that, in practice, a relatively small number of exchanges and exchange operators account for a substantial share of trading volume. For example, the LSE and Turquoise exchanges (run by LSE) and Cboe BXE (the former BATS Europe) and Cboe CXE (former Chi-X Europe) exchanges (run by BATS) account for approximately 90% of trading volume in UK stocks.

The insight here extends to other trading subgames such as, for example, a model with more than two periods, such as our 3-period extension in Section 3, and markets with high frequency trading in Section 4. With HFTs, exchanges set their fees in response not only to changes in the distribution of arriving regular investors, but also in order to attract HFTs that may also have market power and that may have changing cost structures that may imply different break-even fees and rebates (i.e., break-even net of HFT fixed and marginal operational costs may be positive, rather than 0 as in our analysis, and may change over time). Once again, there exist threats that, if credible, allow monopolistic or other oligopolistic profit-sharing rules to be implemented. Thus, the main conclusions in this section are that pure-strategy equilibria can exist with sequential Nash bargaining but that, as in Chao et al. (2018), exchange competition still need not lead to zero equilibrium exchange profits, both with and without precommitment.

3 Three-Period model

Our analysis is readily extended to a richer market environment with an arbitrary number N of investor arrivals at times $t_z \in \{t_1, \ldots, t_N\}$. This extension lets us describe the effect of increased trading activity on access pricing. In particular, trading activity can refer either to potentially longer trading horizons or to more frequent investor arrival over a fixed horizon (e.g. over a trading day). From a modeling viewpoint, there are two new elements: First, the limit order book can accumulate depth at a given price or at different prices in the multiperiod market whereas there is at most only one limit order in the book in the two-period model. In particular, the arrival of new limit and market orders augments or reduces the depth of the limit order book respectively, leading to dynamics:

$$L_{t_z} = L_{t_{z-1}} + Q_{t_z} \quad z = 1, \ldots, N \tag{21}$$

where $Q_{t_z} = [Q_{t_z}^{P_k}]$ is a vector of changes in the limit order book due to an arriving investor's action x_{t_z} at t_z. The change $Q_{t_z}^{P_k}$ in depth at price P_k is "+1" when an arriving limit order adds an additional share and "-1" when a market order executes a limit order where P_k is the best bid or offer (BBO), and otherwise is zero (at other prices unaffected by arriving orders). The changes $Q_{t_z}^{P_k}$ are all zero if no order is submitted.

Second, investors arriving after t_1 and before t_N have a non-trivial choice between market and limit orders. Once again, an arriving investor at t_1 still only chooses between different limit orders at different possible limit prices and NT, and the investor in the final time t_N still chooses between buy and sell market orders and NT. For tractability, we assume limit orders cannot be modified or cancelled after submission and that investors can only send one order of unitary size at a time.[17]

The objectives for access pricing with N periods are analogous to those in the two-period model. An exchange chooses its fees, Ξ, to maximize its expected payoff from completed transactions:

$$\max_{\substack{MF,TF \\ -\tau<MF,TF<+\tau}} \sum_{t_z \in \{t_1,\ldots,t_N\}} \pi_{t_z}^{Ex}(MF,TF) = \left[\sum_{t_z \in \{t_1,\ldots,t_N\}} \sum_{x_{t_z} \in X^L} Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}} | \Xi) \right] (MF+TF) \tag{22}$$

given the transaction probabilities $Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}} | \Xi)$ defined in (12) but where now the limit-order execution probabilities $Pr(\theta_{t_z}^{x_{t_z}} | \Xi, L_{t_{z-1}})$ need to take into account the fact that a limit order submitted at time t_z can potentially be executed at multiple possible dates in the future.

A Social Planner maximizes the total welfare, which generalizes to the N-period model to:

$$\begin{aligned} & \max_{\substack{MF,TF \\ -\tau<MF,TF<+\tau \\ MF+TF\geq 0}} \sum_{t_z \in \{t_1,\ldots,t_N\}} \Big(W_{t_z}^{INV}(MF,TF) + \pi_{t_z}^{Ex}(MF,TF) \Big) \qquad (23) \\ = & \sum_{t_z \in \{t_1,\ldots,t_N\}} \sum_{\forall L_{t_{z-1}}} Pr(L_{t_{z-1}} | \Xi) \times \Bigg[\sum_{x_{t_z} \in X^L} \Bigg(\int_{\beta_{t_z} \in B_{t_z}(x_{t_z}, \Xi, L_{t_{z-1}})} I_{t_z} \times [\beta_{t_z} - P(x_{t_z}) - MF)] f(\beta_{t_z}) d\beta_{t_z} \times Pr(\theta_{t_z}^{x_{t_z}} | \Xi, L_{t_{z-1}}) \\ & + Pr(x_{t_z} | \Xi, L_{t_{z-1}}) \times \sum_{t_n \in \{t_{z+1},\ldots,t_N\}} \sum_{\forall L_{t_{n-1}}} Pr(L_{t_{n-1}} \cap \mathcal{N}_{t_{n-1}}^{x_{t_z}} | x_{t_z}, L_{t_{z-1}}, \Xi) \int_{\beta_{t_n} \in B_{t_n}(\tilde{x}_{t_n}(x_{t_z}), \Xi, L_{t_{n-1}})} I_{t_z} \times [P(x_{t_z}) - \beta_{t_n} - TF)] f(\beta_{t_n}) d\beta_{t_n} \\ & + Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}} | \Xi) (MF+TF) \Bigg) \Bigg] \end{aligned}$$

where $W_{t_z}^{INV}(MF,TF)$ is the expected welfare of arriving investors at each date t_z and $\pi_{t_z}^{Ex}(MF,TF)$ is the exchange's expected profit from (22) from limit orders submitted at dates t_z and subsequently executed at

[17] As noted in Parlour and Seppi (2008), such limit orders are essentially "take it or leave it" offers of liquidity.

later dates. The first term on the right is the welfare of investors who submitted different possible limit orders $x_{t_z} \in X^L$ at all dates t_z where $B_{t_z}(x_{t_z}, \Xi, L_{t_{z-1}})$ is the interval of private value realizations β_{t_z} at time t_z for which a limit order x_{t_z} is optimal given the book $L_{t_{z-1}}$ and fees Ξ. The second term on the right is the welfare of investors who subsequently submit market orders $\tilde{x}_{t_n}(x_{t_z})$ at later dates $t_n > t_z$ that execute earlier limit orders x_{t_z}, where $B_{t_n}(\tilde{x}_{t_n}(x_{t_z}), \Xi, L_{t_{n-1}})$ is the corresponding interval of private value realizations β_{t_n} at times $t_n > t_z$ for which the market order $\tilde{x}_{t_n}(x_{t_z})$ is optimal at date t_n. Investor private valuations at different dates (i.e., β_{t_z} or β_{t_n} at t_z or t_n) are i.i.d. random variables with uniform distributions $U[\underline{\beta}, \overline{\beta}]$. The indicator function I_{t_z} denotes limit buys ($I_{t_z} = +1$) and sells ($I_{t_z} = -1$) at t_z. The conditioning information at date t_n includes both the incoming book $L_{t_{n-1}}$ and also the fact that the limit order from t_z is still unexecuted as of date t_{n-1}. In particular, $\mathscr{N}_{t_{n-1}}^{x_{t_z}}$ denotes the set of states in which the limit order x_{t_z} from t_z is not executed before time t_n. The optimization in (23) is subject to a non-negative net fee constraint (individual rationality) for the exchange ($MF + TF \geq 0$) and a regulatory tick-size constraint on fees. Using first principles, we have the following existence result for a general N-period model:

Theorem 6. *The equilibrium of a trading game with N periods and a price grid with a fixed number of prices exists and can be constructed analytically via backward induction.*

Proofs for general N-period models are in Appendix A. The functional forms of both the exchange profit function and the Social Planner total welfare function can be complex as the number of periods grows and as the number of possible limit prices increases — i.e., as more limit orders become a priori feasible as larger investor valuation supports encompass more prices. Therefore, rather than explicitly differentiating the analytic exchange expected profit function or the analytic Social Planner total welfare, we report results using a search algorithm to solve the first-order conditions for Ξ^*. Section E in the Online Appendix presents the Simulated Annealing Algorithm (SA) and Grid Search Algorithm (GS) we use to find numerical results.[18]

[18]As a robustness check, we confirmed that optimal fees computed using the SA and GS algorithms in the two-period model

Although we obtain optimal fees of the three-period model numerically, Appendices F.1 and F.2 show how to obtain a closed-form solution for the three-period benchmark model without and with HFTs.

To illustrate optimal access pricing and trading in a multi-period market, we consider a three-period market with investor arrival dates $\{t_1, t_2, t_3\}$. Two key intuitions drive our results: First, in the two-period model in Section 2, investors in the first period are monopolists in supplying liquidity since there is no opportunity for later traders to compete against the first-period trader's limit orders. In particular, investors at t_2 can only accept or decline liquidity offered by a limit order posted at time t_1 since the game ends after t_2. With more than two periods, the first-period liquidity supply is no longer monopolistic, and some amount of intertemporal competition in liquidity supply is possible. Second and relatedly, a higher level of market activity with more rounds of investor-arrival increases the opportunities for limit order execution.

Figure 3 shows equilibrium access pricing for the three-period model for different investor valuation supports.[19] Many of the results for the three-period model are similar to the two-period model. There is still a region of valuation supports with both Taker-Maker and Maker-Taker equilibria and, again, as the valuation support width Δ increases, the exchange optimally increases both MF and TF subject to the regulatory cap, and eventually there is a unique equilibrium with strictly positive fees. However, there are also some differences. To help explain these differences, Table 4 reports the equilibrium strategies, order-execution probabilities and other information for the three-period LTM.

Proposition 3. *The set of valuation supports associated with rebates is smaller in the three-period model, and fees are larger, and rebates are smaller.*

Comparing Figures 1 and 3 shows that the region with rebate-based access pricing (Maker-Taker or Taker-Maker) is smaller in the three-period market. The largest valuation support width associated with

agree with the analytic ones.

[19] Once again, the 3-period exchange profit functions look qualitatively similar to those for the two-period exchange modulo the asymmetry discussed below.

rebate-based pricing is $2.3\,\tau$ in the three-period market vs. $3\,\tau$ in the two-period framework. In addition, because trading volume is higher in the three-period model, exchange profits are higher. The levels of fees (rebates) in the three-period model are also larger (smaller). The intuition for the effect of the number of trading periods on the use of rebates and the level of access pricing is the following: Holding everything fixed, the probability limit orders are executed increases because there are more opportunities for investors with complementary reasons to trade to arrive and trade with each other. As a result, the exchange can set larger fees and has less of an incentive to offer rebates.

Proposition 4. *Maker-Taker and Taker-Maker pricing is asymmetric in the three-period model with smaller rebates in the Maker-Taker equilibrium than in the Taker-Maker equilibrium.*

This asymmetry in rebate-based access pricing is new and is in contrast to the symmetry in our two-period model and also in Chao et al. (2018). The equilibrium fees are asymmetric because in the three-period model the investor at time t_1 is no longer a monopolist in liquidity provision. An arriving investor at time t_2 can compete with the t_1 limit order (by submitting a limit order in the same direction as the t_1 limit order with a better price) or may seek price improvement (by submitting a limit order in the opposite direction of the t_1 limit order rather than hitting it with a market order).

Consider, for example, the equilibrium strategies in Row 1 of Table 4 for a support width $\Delta = 0.33\,\tau$. In the Taker-Maker equilibrium, when the investor in period t_1 limit buys (LBP_{-1}) at the price P_{-1}, an incoming seller in period t_2 has the option of either market selling (MSP_{-1}) at P_{-1} or limit selling (LSP_1) at the higher price P_1. In contrast, in the Maker-Taker equilibrium, the investor at t_1 limit buys (LBP_1) at P_1 (because of the rebate $MF = -0.428$), which consequently means a seller at t_2 has no other trading option than market selling (MSP_1) at the high price P_1 — since limit selling at P_{-1} is not allowed given the pre-existing limit buy at P_1 in order to prevent a locked market — and therefore will be charged a positive fee $TF = 0.557$.[20]

[20]The state of the book when the seller arrives at t_2 has a limit order at P_1, hence the seller does not compete for the provision of

This theoretical result about the connection between access pricing and locked markets is new.[21]

Table 4 shows that in the Taker-Maker equilibrium the seller at t_2 opts only to market sell at P_{-1}. This choice is driven by the higher TF rebate (-0.443) that encourages market orders at t_2 — and, thus transactions — over limit orders at t_2 in the Taker-Maker equilibrium. The two grey rows 3 and 4 in Table 4 show that if, off equilibrium, the exchange hypothetically were to use symmetric fees — i.e., the equilibrium Taker-Maker TF and MF are flipped for the Maker-Taker MF and TF, or if the equilibrium Maker-Taker TF and MF are flipped and used for the Taker-Maker MF and TF — an incoming seller would opt for either market selling (MSP_{-1}) or limit selling (LSP_1), and exchange profits would be smaller. This explains why, in equilibrium, the exchange offers a larger TF rebate than the MF rebate.

Observation: *The minimum rebate (in absolute value) in the three-period Taker-Maker equilibrium, $|TF|$, is strictly positive, whereas it is 0 in the Maker-Taker equilibrium, $|MF|$, because the regulatory cap on the Taker-Maker TF binds for smaller support widths than in the Maker-Taker equilibrium.*

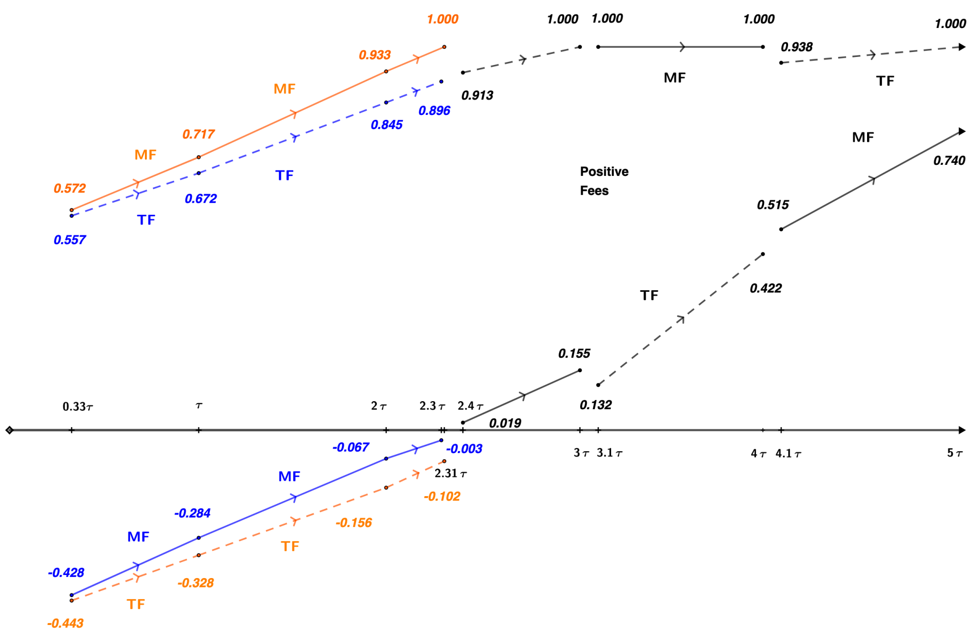
This discontinuity can be seen in Figure 3. When Δ is just larger than $2.3\,\tau$, the minimum Taker-Maker rebate $|TF|$ on the left is close to $|-1.04|$ but the minimum Maker-Taker rebate $|MF|$ on the right is 0.

The comparison between the two-period and three-period markets shows how optimal access fees differ for stocks with different rates of trading activity. In particular, high investor arrival is associated with a reduced need for rebates to encourage trading. However, empirically, exchanges use rebate-based access pricing for actively traded stocks. That is another inconsistency with the predictions of basic price-friction models of access pricing. In contrast, HFT trading in the next section provides a potential explanation.

liquidity as a limit sell order at P_{-1} is dominated by the market sell order at P_1.

[21] We thank Mao Ye for insights on this point.

Figure 3: Make Fees and Take Fees in 3-Period Market. This figure reports the equilibrium make fees (MF) and take fees (TF) in the Large Tick Market (LTM) corresponding to different investor valuation supports ranging from $0.33\,\tau$ and $5\,\tau$, (where τ is the tick size in the LTM) on the horizontal black axes. The left (right) part of the figure reports the equilibrium trading fees consistent with the Taker-Maker (Maker-Taker) fee structure. The figure reports in blue (orange) italic the equilibrium MF (TF) set by the exchange.



4 High frequency trading and access pricing

Our previous results show that an exchange's optimal access pricing depends crucially on the mix of investors in the market. In particular, the more traders have concentrated personal valuations, the greater is the exchange's incentive to offer rebates. In real markets, one important type of traders whose personal valuations typically do not differ from the fundamental asset value are high frequency traders (HFTs).[22] This section extends our analysis to include HFT firms.

[22]There are many types of HFT trading strategies ranging from latency arbitrage to order anticipation. We focus here on HFTs engaging in a type of reactive liquidity provision.

Table 4: 3-Period Large Tick Market (LTM). Equilibrium Fees and Trading Strategies. This table reports for different investor valuation support width, $\Delta = \overline{\beta} - \underline{\beta}$ expressed in terms of the LTM tick size τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fees (MF^* and TF^*) (column 3 and 4), the sum and the absolute difference of the equilibrium MF^* and TF^* (column 5 and 6), the equilibrium orders x_{t_1} at t_1 other than No Trade (column 7) and the equilibrium orders x_{t_2} at t_2, conditional on the trading strategy indicated at t_1 (column 8). The table also shows the associated probabilities of submission, $Pr(x_{t_1}|\Xi^*, L_{t_0})$ and $Pr(x_{t_2}|\Xi^*, L_{t_1})$, (column 9 and 10), as well the cum-fee buy and sell prices $P_k^{cum,LB}$ and $P_k^{cum,MS}$ (column 11 and 12), the probability of execution of the order posted at t_1, $Pr(\theta_{t_1}^{x_{t_1}}|\Xi^*, L_{t_0})$, (column 13), the equilibrium transaction probability $Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}}|S, \tau, \Xi)$ (column 14), and the Exchange Expected Profit, $\pi^{Ex}(MF^*, TF^*)$ (column 15). The third and fourth gray rows report results (marked with a *) for off-equilibrium fees that symmetrically flip the corresponding equilibrium fees. When the equilibrium pricing is rebate-based for a given support, we report the Taker-Maker fees on the first row and then the Maker-Taker fees on the second row. When, for a given support and set of fees, there are multiple optimal orders given different valuations β_{t_1} for the investor at t_1, these orders are shown on different rows along with the optimal potential responses at t_2.

Support width $\Delta = \overline{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	MF^*	TF^*	MF^*+TF^*	$\|MF^*$-$TF^*\|$	Eq.Strategies x_{t_z}		Pr. Submission $Pr(x_{t_z}\|\Xi^*, L_{t_{z-1}})$		$P_k^{cum,LB}$	$P_k^{cum,MS}$	Pr. Execution $Pr(\theta_{t_1}^{x_{t_1}}\|\Xi^*, L_{t_0})$	Pr. Trans	Exchange E[Profit] $\pi^{Ex}(MF^*, TF^*)$
						t_1	t_2	t_1	t_2					
0.33 τ	9.833, 10.167	0.572	-0.443	0.129	1.015	LBP_{-1}	MSP_{-1}	0.284	0.328	10.072	9.943	0.548	0.391	0.051
		-0.428	0.557	0.129	0.985	LBP_1	MSP_1	0.284	0.328	10.072	9.943	0.548	0.391	0.051
0.33 τ*	9.833, 10.167	0.557*	-0.428*	0.129	0.985	LBP_{-1}*	MSP_{-1}* & LSP_1*	0.328*	0.266* & 0.062*	10.057*	9.927*& 9.943	0.475*	0.388*	0.050*
		-0.443*	0.572*	0.129	1.015	LBP_1*	MSP_1*	0.328*	0.284*	10.057*	9.927*	0.487*	0.383*	0.050*
τ	9.500, 10.500	0.716	-0.328	0.388	1.044	LBP_{-1}	MSP_{-1}	0.284	0.328	10.216	9.828	0.548	0.392	0.152
		-0.284	0.672	0.388	0.956	LBP_1	MSP_1	0.284	0.328	10.216	9.828	0.548	0.392	0.152
2 τ	9.000, 11.000	0.933	-0.156	0.777	1.089	LBP_{-1}	MSP_{-1}	0.284	0.328	10.433	9.656	0.548	0.392	0.304
		-0.067	0.845	0.778	0.912	LBP_1	MSP_1	0.284	0.328	10.433	9.655	0.548	0.392	0.304
2.31 τ	8.850, 11.150	1.000	-0.102	0.898	1.102	LBP_{-1}	MSP_{-1}	0.284	0.328	10.500	9.602	0.548	0.392	0.351
		-0.001	0.898	0.897	0.899	LBP_1	MSP_1	0.284	0.328	10.499	9.602	0.548	0.392	0.351
2.4 τ	8.800, 11.200	0.019	0.913	0.932	0.894	LBP_1	MSP_1	0.284	0.328	10.519	9.587	0.548	0.392	0.365
3 τ	8.500, 11.500	0.155	1.000	1.155	0.845	LBP_1	MSP_1	0.282	0.333	10.655	9.500	0.556	0.395	0.456
3.1 τ	8.450, 11.550	1.000	0.132	1.132	0.868	LBP_{-1}	LBP_1, MSP_{-1}, LSP_1	0.339	0.016, 0.278, 0.060	10.500, 11.500	9.368, 9.500	0.487	0.414	0.468
4 τ	8.000, 12.000	1.000	0.422	1.422	0.578	LBP_{-1}	LBP_1, MSP_{-1}, LSP_1, LSP_2	0.367	0.125, 0.231, 0.125, 0.270	10.500, 11.500	9.078, 9.500, 10.500	0.404	0.430	0.612
						LBP_{-2}	LBP_{-1}, MSP_{-2}, LSP_1, LSP_2	0.133	0.375, 0.000, 0.356, 0.270	9.500, 10.500	8.078, 10.500, 11.500	0.012		
4.1 τ	7.950, 12.050	0.515	0.938	1.453	0.423	LBP_1	LBP_2, MSP_1	0.149	0.009, 0.393	11.015, 12.015	9.562	0.628	0.432	0.627
						LBP_{-1}	LBP_1, MSP_{-1}, LSP_1	0.347	0.252, 0.088, 0,408	10.015, 11.015	8.562, 9.985	0.187		
5 τ	7.500, 12.500	0.740	1.000	1.740	0.260	LBP_1	LBP_2, MSP_1	0.108	0.052, 0.400	11.240, 12.240	9.500	0.619	0.451	0.785
						LBP_{-1}	LBP_1, MSP_{-1}, LSP_1	0.344	0.252, 0.137, 0.315	10.240, 11.240	8.500, 9.760	0.259		

Our main result is that the constant presence of HFT liquidity providers increases the incentive for rebate-based access pricing by profit-maximizing exchanges. This theoretical linkage is consistent with the empirical prevalence of both rebate-based access pricing and HFTs. We also demonstrate a third non-equivalence of Maker-Taker vs Taker-Maker pricing with HFTs. This non-equivalence is due to asymmetries in how Maker-Taker and Taker-Maker pricing differentially incentivize posted liquidity in the limit order book vs latent liquidity from HFT orders. HFT firms differ from regular investors (INV) in four ways in our model: First, rather than having stochastic private valuations, HFT firms have the same non-random valuation equal to the mean INV valuation v. Second, rather than arriving sequentially, our HFTs are continually present. Unlike in Foucault et al. (2013), our HFTs have no monitoring costs. Thus, they can react quickly to market events (i.e., order submissions by INVs) before the next INV arrives. Third, as in Li et al. (2019), our HFT firms are opportunistic liquidity providers who take advantage of any profitable trading opportunities in limit orders from arriving regular INVs.

For example, if an INV posts an aggressive buy (sell) limit order in period t_z such that the associated cum-fee price for a sell (buy) market order is above (below) the HFT valuation v, an HFT firm quickly submits a sell (buy) market order within the same period t_z to take the other side of the profitable trade. We call these *fast market orders*. If more than one HFT submits a fast market order, then one is randomly selected for execution, and the rest are cancelled. Fourth, maintaining standing limit order positions is costly for HFTs.

Budish, Cramton, and Shim (2015) endogenize limit order submission costs by showing there is a natural bid-ask spread for HFT limit orders given endogenous picking-off costs for stale orders.[23] For simplicity, however, we just assume directly that HFTs are unwilling to provide ex ante liquidity via limit orders (as a

[23] Allowing for the possibility that HFTs might sometimes use limit orders when they are unwilling to use fast market orders given a hypothetical exchange access pricing structure would simply complicate the analysis. In Budish et al. (2015), the break-even condition in a limit order book such that HFT firms supply liquidity is that the payoff from market making is at least equal to the costs of being sniped by other competing HFT firms. Thus, our assumption of no HFT limit orders is simply a convenient reduced-form for picking-off risks for a smart trading crowd as first suggested in Seppi (1997).

reduced-form for picking-off costs), but that HFTs are willing to provide liquidity ex post to regular INVs via fast market orders. This friction in HFT liquidity-provision simply means there is not a tight (i.e, one tick) standing bid-ask spread. This creates space for the trading dynamics in our model. Our analysis shows how rebate-based access pricing can counteract this friction.

Previous research on HFTs often draws a sharp dichotomy between two types of HFT firms: HFT snipers who consume liquidity via market orders and HFT market makers who trade via limit orders (as in, e.g., Budish et al. (2015)). Our modeling of HFTs blurs this distinction. In our model, aggressive INV orders can be interpreted as "requests for liquidity," and our HFTs provide *reactive liquidity* in the form of transaction consummation. Effectively, they function in a limit order market like dealers in that they respond ex post to arriving orders. This is consistent with empirical evidence in Latza, Marsh, and Payne (2014) and Hendershott and Riordan (2013).[24]

The HFT optimization problem is as follows: HFTs have the ability to react and submit orders in period t_z after any orders submitted by the INV in period t_z. Thus the HFT action set, $X_{t_z}^{HFT} = \{MBP_{k(L_{t_z})}, MSP_{k(L_{t_z})}, NT\}$, consists of fast buy and sell market orders — where $k(L_{t_z})$ denotes the BBO price at which the HFT hits INV limit orders at time t_z — and also the no-trade alternative.

In each period t_z, HFT firms choose their order $x_{t_z}^{HFT}$ to maximize their expected payoff depending on whether or not there is an aggressive limit order in the book L_{t_z} it would be profitable to trade with:

$$\max_{x_{t_z}^{HFT} \in X_{t_z}^{HFT}} \pi_{t_z}^{HFT}(x_{t_z}^{HFT} | \Xi, L_{t_z}) = \begin{cases} [v - P(x_{t_z}^{HFT}) - \xi(x_{t_z}^{HFT})] & \text{if } x_{t_z}^{HFT} \text{ is a buy \& there is a limit sell in } L_{t_z} \\ [P(x_{t_z}^{HFT}) - v - \xi(x_{t_z}^{HFT})] & \text{if } x_{t_z}^{HFT} \text{ is a sell \& there is a limit buy in } L_{t_z} \\ 0 & \text{if } x_{t_z}^{HFT} \text{ is } NT. \end{cases} \tag{24}$$

The execution probability for a fast market order is 1 if it is submitted. The key intuition is that HFTs provide latent liquidity: HFTs are continuously present and willing to immediately take the opposite side of

[24]In today's markets, HFT firms can act both as passive (e.g., Quantlab and Jump Trading) and as aggressive liquidity providers (Hudson River Trading and Citadel Securities). In addition, HFT firms are sensitive to rebate based pricing and can easily adjust their dynamic trading strategies to take advantage of profitable trading fees.

limit orders posted by INVs provided that the posted limit prices are sufficiently aggressive. In particular, the corresponding cum-fee price must higher than v for a MS and lower than v for a MB.

The INV optimization problem is formally the same as in (2) with the same action set $X_{t_z}^{INV}$ as without HFTs. However, the specifics of the order-execution probabilities $Pr(\theta_{t_z}^{x_{t_z}^{INV}} | \Xi, L_{t_{z-1}})$ with HFTs differ from those without HFTs due to the possibility of immediate execution of aggressive INV limit orders by HFT fast orders. For example, limit orders are now possible in equilibrium for INVs in the final trading time t_3 due to the possibility of execution by the HFTs. However, if access pricing is such that HFTs choose not to trade, then the market reduces to a market without HFTs.

Competition by the HFTs simplifies the structure of equilibrium. Since HFTs are always willing to buy and sell at v, the exchange can set the fees and rebates Ξ so that in equilibrium the cum-fee prices paid and received by the HFTs is their break-even valuation v. Considering our regulatory constraint that fees cannot exceed the tick size, this has the implication that HFTs only trade at either P_{-1} or at P_1. In addition, when the exchange set Taker-Maker or Maker-Taker pricing, limit buys at prices below P_{-1} and limit sells at prices above P_1 are never used in equilibrium. HFTs and INVs know such limit orders would always be undercut by future HFTs who will be willing to trade via fast market orders at their break-even cum-fee prices. Therefore, the INVs choose between submitting limit orders at P_{-1} and P_1, market orders (if there are any pre-existing limit orders in the book at P_{-1} and P_1), and NT.

The exchange sets its access fees and rebates to maximize its expected payoff given the INV and HFT behavior induced by its access pricing Ξ. Formally, this problem is the same as in (22) but now with the specific forms of order-submission and order-execution probabilities associate with HFTs.

A Subgame Perfect Nash Equilibrium consists of: i) Order-submission strategies $x_{t_z}^{INV}$ and $x_{t_z}^{HFT}$ for INVs and HFTs that maximize their expected trading profits given the order-execution probabilities they induce and ii) access fees Ξ that maximize the exchange's expected profit.

Theorem 7. *The equilibrium of an N-period model with HFTs exists and can be constructed using an analytic recursion.*

The proof of Theorem 7 is in Appendix A. Given the market access fees, the trading subgame can be solved in closed-form to determine the equilibrium order-submission strategies in each period of the trading game. The equilibrium execution probabilities, the equilibrium thresholds and the equilibrium order submission strategies, can be derived using an analytic recursion as described in Appendix F.2. However, solving for profit-maximizing access pricing in the three-period model with HFTs can be complicated, so we again solve for optimal exchange fees using the SA and GS numerical algorithms in section E of the Online Appendix.

An important consideration in optimal access pricing by the exchange is that HFTs are willing to trade at cum-fee prices equal to their break-even valuation v, which is ex ante common knowledge. This fact allows take fees and rebates to be set to reduce HFT trading profits to zero and to use these savings to increase the exchange's profit per trade or to reduce INV trading costs and thereby increase trading and, thus, increase exchange profits or total welfare.

Figure 4 shows on the left (right) the equilibrium fees and rebates in the three-period market with HFTs for different INV valuation supports for the exchange (social planner). Table 5 shows cum-fee prices, optimal orders, and other details. Comparing the results with HFTs for the exchange in Figure 4 with the previous three-period model without HFTs in Figure 3, there are several points to make: First, rebate-based access pricing is optimal for a wider range of INV valuation supports with HFTs than in markets without HFTs. In particular, the region denoting Taker-Maker access pricing extends up to a maximum valuation support width of $4\,\tau$ with HFTs, whereas it is only $2.3\,\tau$ in the three-period market without HFTs. These results lead to our fourth empirical prediction:

Empirical Prediction 4: *Markets with HFTs are more likely to have rebate-based access pricing.*

This result is a new explanation for widespread rebate-based access pricing in U.S. markets since Reg NMS. In particular, it is consistent with empirical findings of Cardella et al. (2015) that most U.S. exchanges adopted a rebate-based pricing in 2008 following Reg NMS and the concurrent growth of HFT trading.[25]

Second, Taker-Maker and Maker-Taker access pricing both maximize the exchange's expected profit when investor valuations are similar to each other (i.e., the valuation support is small). The corresponding fees and rebates are asymmetric, but they result in identical cum-fee prices and transaction outcomes — in terms of probabilities of execution and submission — since investors undo differences in the two access pricing schedules by submitting limit orders at different endogenous posted prices. Figure 4 and Table 5 show that the exchange sets either the MF or TF to attract HFTs. For example, when the valuation support width is $0.33\,\tau$, the exchange in the Taker-Maker regime pays a take rebate equal to half a tick ($TF = -0.50\tau$) so that HFT firms have an incentive to submit fast market sells to execute INV aggressive limit buys at a posted price of P_{-1} and, thus, receive a cum-fee MS price of $P_{-1} + TF = v$. Similarly, in the Maker-Taker regime the exchange charges a take fee equal to half a tick ($TF = 0.50\tau$), so that HFTs have an incentive to submit fast market sells to take the other side of aggressive INV limit buys with a posted price at P_1 and, thus, receive a cum-fee MS price of $P_1 - TF = v$. Since HFTs break-even at v, the exchange has no incentive to offer a rebate greater than half a tick in the Taker-Maker regime. In the Maker-Taker regime instead the exchange cannot charge the HFTs more than half a tick.

Third, once INV valuations are sufficiently heterogeneous ex ante, the equivalence between Taker-Maker and Maker-Taker pricing breaks down. With a valuation support width Δ between 2τ and 4τ, Taker-Maker pricing with the maximum possible make fee $MF = 1$ and a take rebate $TF = -0.5$ is the unique optimal access pricing for the exchange. The TF rebate gives HFTs their minimum acceptable expected trading profit, and the INV investors are paying the maximum possible make fee MF given the regulatory constraint.

[25] Similar results hold also in the small tick market (STM). In results available from the Authors upon request, we show that when, all else equal, the tick size is smaller the exchange has a a smaller incentive to use rebate-based access pricing.

Notice there is no equivalent Maker-Taker pricing regime. For support widths Δ over the interval $[2\tau, 4\tau]$, the Maker-Taker pricing regime cannot reproduce the same cum-fee prices and transaction outcomes as in the Taker-Maker pricing regime. The reason is as follows: i) It is not profitable for an HFT seller to take the other side of a limit buy at P_{-1} using a fast market sell (and receive a cum-fee MS price of $P_{-1} - TF < P_{-1} < v$), and so a LBP_{-1} from an INV remains standing in the book into later periods. ii) A standing limit buy at P_{-1} from an INV in an earlier period preempts liquidity provision by HFT buyers in later periods. In particular, a later INV seller cannot submit an aggressive limit sell at P_{-1} to trade with a later HFT buyer because, with a standing LB in the book at P_{-1}, the later INV's aggressive LSP_{-1} would be automatically converted into a MSP_{-1} and crossed with the standing LBP_{-1} (i.e., the market cannot be locked with standing limit buy and limit sell orders both at P_{-1}). However, the cum-fee MS price when an INV seller takes the other side of a standing LBP_{-1} is $P_{-1} - TF < P_{-1}$, which is worse than the cum-fee LS price $P_{-1} - MF > P_{-1}$ on an aggressive limit sell to trade with an HFT.[26] Given the worse terms of trade when INVs are blocked from accessing HFT latent liquidity by standing limit orders, there is less trading by INV investors. iii) When the investor valuation support widens sufficiently, the execution probability of a ”blocking” limit buy at P_{-1} becomes large enough such that some INVs prefer the probability of price improvement by posting a less aggressive limit buy LBP_{-1} over certain execution by trading with an aggressive LBP_1 with an HFT.

This analysis points out a fundamental incentive-compatibility difference between Maker-Taker and Taker-Maker pricing with HFTs. Taker-Maker pricing directly subsidizes HFT liquidity provision (making HFTs willing to take the other side of INV limit orders with worse prices for the HFTs) and penalizes limit order submission (which shrinks the ex ante feasible limit order action set for INVs to a single posted price, P_{-1}, for limit buys and a single posted price, P_1, for limit sells). In contrast, Maker-Taker pricing subsidizes INV limit orders and penalizes HFT market orders. Maker-Taker pricing can implement the same

[26] With Maker-Taker $MF < 0$.

equilibrium outcomes as with Taker-Maker pricing provide the INVs post limit orders at more aggressive prices (i.e., which are less attractive for INVs but more attractive for the HFTs) so as to pass through the subsidy to the HFTs. However, if the INV investor valuation support is too wide, less aggressive limit orders also become ex ante feasible. In order to implement the Taker-Maker equilibrium, it must be incentive compatible for INVs to choose to post the more aggressively priced limit order than the less aggressive order. As our analysis shows, this is incentive compatible when the valuation support width Δ is less than 2τ (for which the probability of less aggressive limit order execution is low), but not when Δ is greater than 2τ (and the probability of less aggressive limit order execution is higher). Once again, this incentive compatibility issue is not present in the Taker-Maker regime because HFTs are subsidized directly and because there is only one ex ante feasible LB and LS price respectively.

Fourth, when INVs become even more heterogeneous, and the investor valuation support in Figure 4 widens beyond 4τ, the exchange exploits the high ex ante gains-from-trade by setting strictly positive make and take fees for all market participants. In particular, when the support width Δ reaches 4τ, optimal access pricing changes discontinuously, and the exchange optimally sets the take fee to $TF = 0.500$ (so HFT firms will submit fast market sells at P_1) and sets the make fee to $MF = 0.520$ (so selling at P_{-1} will be profitable for enough INVs). However, now, as seen in Table 5, some INVs submit less aggressive limit buys LBP_{-1} (for price improvement) rather than aggressive limit buys LBP_1 to trade with an HFT. Figure 4 shows that, as the support widens further beyond 4τ, the exchange holds the TF constant at 0.500 and gradually increases the MF to take advantage of the larger INVs' gains from trade.

To summarize, our model shows that exchange profits increase when HFTs are active in the market. The reason is that HFTs are willing to both buy or sell, unlike regular INVs who have directional trading demands. Thus, participation by HFT firms can generate greater volume than INVs alone. Exchanges, therefore, use rebate-based access pricing to cross-subsidize HFTs trading. This is the reason the region associated with optimal take rebates is larger with HFTs than in the three-period model without HFTs. Empirically, rebate-based pricing via Maker-Taker pricing is more common than rebate-based Taker-Maker pricing. In our model, however, Taker-Maker rebate pricing is optimal over a wider set of market parameterizations than Maker-Taker pricing. This is because of the assumption that our HFT firms trade via fast market orders. In a model in which the assumed HFT trading friction is less, HFTs might also provide liquidity through limit orders and, thus, subsidizing HFT limit orders via Maker-Taker pricing is likely to become more viable. Thus, our key point here is the role of rebate-based access pricing as a means to subsidize HFT liquidity provision. Our HFT results are also consistent with Foucault et al. (2013), who show that the fee breakdown matters when the tick size is positive. Holding the total fee constant, Foucault et al. (2013) show that when the gains-from-trade to market takers increase relative to the gains-from-trade to the market makers, the optimal trading fees become larger.

Social Planner and HFTs: To assess further the welfare impact of HFT firms, we consider access pricing by a Social Planner in the three-period market with HFTs:

$$
\begin{aligned}
&\max_{\substack{MF,TF \\ -\tau<MF,TF<+\tau \\ MF+TF\geq 0}} \sum_{t_z\in\{t_1,t_2,t_3\}} \left(W^{INV}_{x_{t_z}}(MF,TF) + W^{HFT}_{x_{t_z}}(MF,TF) + \pi^{Ex}_{t_z}(MF,TF) \right) \qquad (25) \\
= \quad & \sum_{t_z\in\{t_1,t_2\}} \sum_{x_{t_z}\in X^L} Pr(L_{t_{z-1}}|\Xi) \Bigg(\int_{\beta_{t_z}\in B_{t_z}(x_{t_z},\Xi,L_{t_{z-1}})} I_{t_z}\times[\beta_{t_z} - P(x_{t_z}) - MF)] f(\beta_{t_z}) d\beta_{t_z} \times Pr(\theta^{x_{t_z}}_{t_z}|\Xi, L_{t_{z-1}}) \\
& + Pr(x_{t_z}|\Xi, L_{t_{z-1}}) H_{t_z} I_{t_z} \times [P(x_{t_z}) - v - TF] \\
& + Pr(x_{t_z}|\Xi, L_{t_{z-1}}) (1-H_{t_z}) \sum_{x_{t_n}\in\{x_{t_2},x_{t_3}\}} \sum_{\forall L_{t_{n-1}}} Pr(L_{t_{n-1}} \cap \mathcal{N}^{x_{t_z}}_{t_{n-1}} | x_{t_z}, L_{t_{z-1}}, \Xi) \int_{\beta_{t_n}\in B_{t_n}(\tilde{x}_{t_n}(x_{t_z}),\Xi,L_{t_{n-1}})} I_{t_z} \times [P(x_{t_z}) - \beta_{t_n} - TF)] f(\beta_{t_n}) d\beta_{t_n} \\
& + \left[Pr(x_{t_z}, \theta^{x_{t_z}}_{t_z} |\Xi) \right] (MF+TF) \Bigg).
\end{aligned}
$$

The expression for total welfare with HFTs in (25) and the associated logic is parallel to the three-period of total welfare in (23) but with the addition of HFT welfare. A limit order submitted at t_z that is immediately executed by an HFT at date t_z leads to the HFT welfare in line 3, but if the limit order is not executed by an HFT and is instead executed later by an arriving regular investor at some date $t_n > t_x$, then the INV submitting that market order has the welfare in line 4. The indicator function H_{t_z} keeps track of whether a limit order at time t_z is executed by the HFT at time t_z ($H_{t_z} = 1$) or not ($H_{t_z} = 0$).

The Subgame Perfect Equilibrium with a Social Planner and HFTs, takes the following form: With valuation support widths of up to 2τ, total welfare is maximized by access pricing:

$$\Xi^* = \{MF^*, TF^*\} = \begin{cases} MF^* = -0.5\tau \;\&\; TF^* = 0.5\tau & \text{if Maker-Taker} \\ \\ MF^* = 0.5\tau \;\&\; TF^* = -0.5\tau & \text{if Taker-Maker} \end{cases} \tag{26}$$

Figure 4 on the right, shows the equilibrium MF (solid blue line) and TF (dashed orange line) set by a Social Planner. The equilibrium is symmetric for $S \leq 2\tau$, whereas it is unique Taker-Maker for $2\tau < S \leq 5\tau$.

In the Maker-Taker equilibrium, the INV trading strategy is stationary. In each period, INVs trade by posting limit buy orders at P_1 (and limit sell orders at P_{-1}) and competition drives HFTs to immediately take these orders by trading at their break-even cum-fee price that includes a positive take fee $TF = 0.5\tau$. Symmetrically, in the Taker-Maker equilibrium, the INV trading strategies are also stationary. INVs trade by posting limit buy orders at P_{-1} (and limit sell orders at P_1), and HFTs trade at the break-even cum-fee price that include the rebate of half a tick, $TF = -0.5\tau$. Note that the Social Planner's take fees/rebates are the same as those set by a profit-maximizing exchange. However, the Social Planner sets a larger MF rebate with Maker-Taker pricing and a smaller positive MF fee in Taker-Maker pricing until the exchange just breaks even in order to encourage more INV limit orders.

Once the valuation support exceeds 2τ, Maker-Taker pricing is suboptimal. This is because INVs with private valuations β_t close to v would start posting limit buys at P_{-1} rather than at P_1 (and the reverse for limit sells). Although individually optimal for these investors, it reduces total welfare. Thus, the unique

Social Planner optimal pricing is Taker-Maker for a valuation support greater than 2τ.

To summarize, since HFTs are continuously willing to buy or sell, the Social Planner maximizes total welfare by setting access fees and rebates to maximize total volume subject to constraints that trading must be individually rational for the INVs, the HFTs, and the exchange. Once again, since HFTs trade exclusively via fast market orders in our model, Taker-Maker pricing subsidizes HFTs directly whereas as with the profit-maximizing exchange market, there is an incentive compatibility complication that limits the usefulness of Maker-Taker pricing. Intuitively we expect again that if HFTs could post limit orders, then Maker-Taker pricing might become even more robust as a way to maximize total welfare.

5 Welfare

Access pricing that maximizes exchange profits does not necessarily improve the overall welfare of other market participants. This section revisits the four markets discussed in the previous sections — the two-period LTM, the two-period STM and three-period with and without HFTs — and investigates how access pricing affects the welfare of market participants.

We compare equilibrium welfare in our four market settings under three different regimes in which access pricing is set by an exchange; by a Social Planner; and a "benchmark" case with no fees or rebates (i.e., $MF = TF = 0$). Figures 5 and 6 show our welfare results for different investor valuation supports Δ. Total welfare is computed for all investors (INV) and the exchange (Ex) at all dates:

$$TW = \sum_{t_z \in \{t_1, \ldots, t_N\}} \left(W_{t_z}^{INV}(MF, TF) + \pi_{t_z}^{Ex}(MF, TF) \right) \tag{27}$$

Figure 4: Pattern of Make Fees and Take Fees: 3-Period Model with High Frequency Traders (HFT) This figure shows the equilibrium make fees (MF) and take fees (TF) in the Large Tick Market (LTM) corresponding to different investor valuation supports ranging from $0.33\,\tau$ to $5\,\tau$ (where τ is the tick size in the LTM) on the horizontal black axes. The left (right) part of the figure reports the equilibrium trading fees consistent with the Taker-Maker (Maker-Taker) fee structure. The left figure reports in blue (orange) solid (dashed) line the equilibrium MF (TF) set by the exchange. The right figure reports in blue (orange) solid (dashed) line the equilibrium MF (TF) set by the Social Planner.

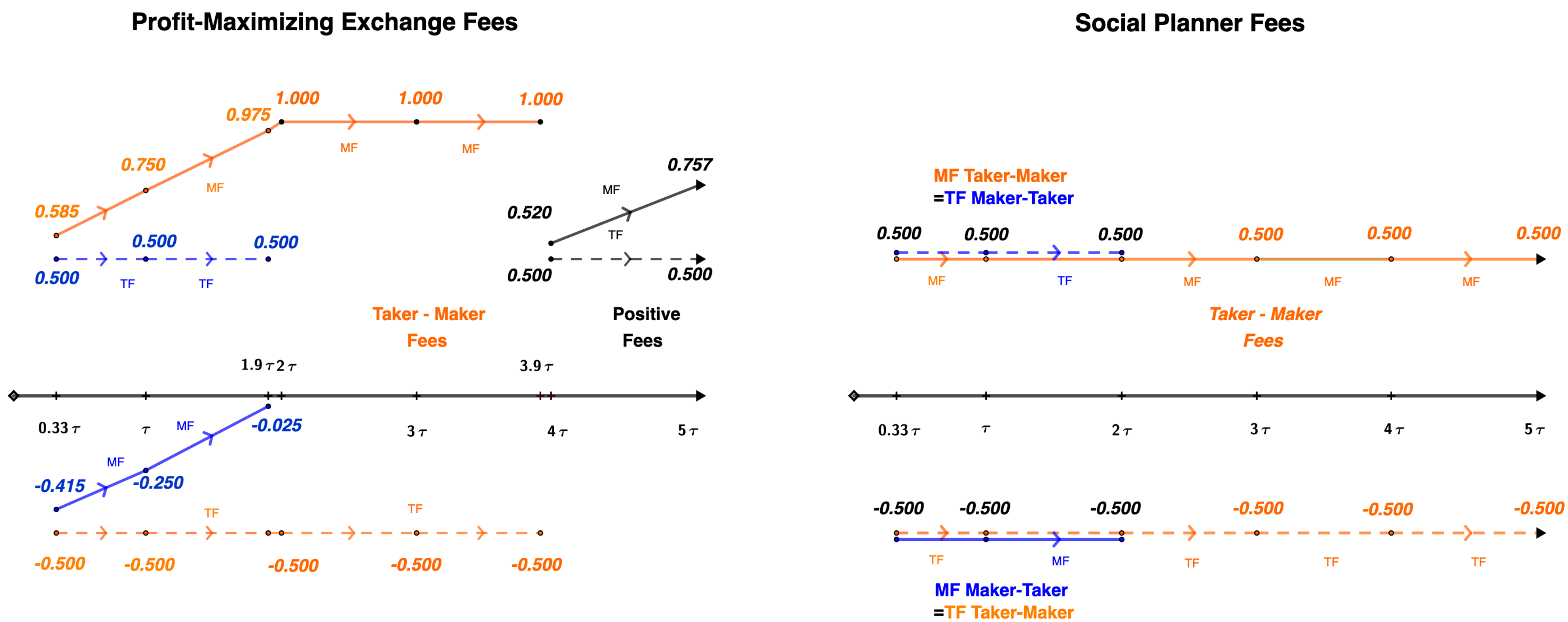


Table 5: 3-Period Large Tick Market with HFTs: Equilibrium Fees and Trading Strategies. This table reports for different investor valuation support width $\Delta = \overline{\beta} - \underline{\beta}$ are expressed in terms of the LTM tick size τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fee, MF^* and TF^*, (column 3 and 4), the sum and the absolute difference of the equilibrium MF^* and TF^* (column 5 and 6), the equilibrium orders x_{t_1} at t_1, other than No Trade (column 7) and the equilibrium orders x_{t_2} at t_2 conditional on the trading strategy indicated at t_1 (column 8). The table also shows the associated probabilities of submission, $Pr(x_{t_1} | \Xi^*, L_{t_0})$ and $Pr(x_{t_2} | \Xi^*, L_{t_1})$ (column 9 and 10), as well the cum-fee buy and sell price, $P_k^{cum,LB}$ and $P_k^{cum,MS}$, (column 11 and 12), the probability of execution of the order posted at t_1, $Pr(\theta_{t_1}^{x_{t_1}} | \Xi^*, L_{t_0})$, (column 13) and the exchange expected profit $\pi^{Ex}(MF^*, TF^*)$ (column 14). When the equilibrium pricing is rebate-based for a given support, we report the Taker-Maker fees on the first row and then the Maker-Taker fees on the second row. When, for a given support and set of fees, there are multiple optimal orders given different valuations β_{t_1} for the investor at t_1, these orders are shown on different rows along with the optimal potential responses at t_2. We do not report the order-submission probabilities for HFT fast market orders (e.g. MSP_{-1}^{HFT}) after aggressive limit orders because in equilibrium they are always equal to 1. We only report the Pr. Execution of the limit order posted at t_1.

Support width $\Delta = \overline{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	MF^*	TF^*	MF^*+TF^*	$\|MF^*$-$TF^*\|$	Eq. Orders x_{t_z}		Pr. Submission $Pr(x_{t_z} \| \Xi^*, L_{t_{z-1}})$		$P_k^{cum,LB}$	$P_k^{cum,MS}$	Pr. Execution $Pr(\theta_{t_1}^{x_{t_1}} \| \Xi^*, L_{t_0})$	Exchange E[Profit] $\pi^{Ex}(MF^*, TF^*)$
						t_1	t_2	t_1	t_2				
$0.33\,\tau$	9.833, 10.167	0.585	-0.500	0.085	1.085	LBP_{-1}, MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.245	0.245	10.085	10.000	1	0.127
		-0.415	0.500	0.085	0.915	LBP_1, MSP_1^{HFT}	LBP_1, MSP_1^{HFT}	0.245	0.245	10.085	10.000	1	0.127
τ	9.500, 10.500	0.750	-0.500	0.250	1.250	LBP_{-1}, MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.250	0.250	10.250	10.000	1	0.375
		-0.250	0.500	0.250	0.750	LBP_1, MSP_1^{HFT}	LBP_1, MSP_1^{HFT}	0.250	0.250	10.250	10.000	1	0.375
$1.9\,\tau$	9.050, 10.950	0.975	-0.500	0.475	1.475	LBP_{-1},MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.250	0.250	10.475	10.000	1	0.712
		-0.025	0.500	0.475	0.525	LBP_1, MSP_1^{HFT}	LBP_1, MSP_1^{HFT}	0.250	0.250	10.475	10.000	1	0.712
$2\,\tau$	9.000, 11.000	1.000	-0.500	0.500	1.500	LBP_{-1}, MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.250	0.250	10.500	10.000	1	0.750
$3\,\tau$	8.500, 11.500	1.000	-0.500	0.500	1.500	LBP_{-1}, MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.333	0.333	10.500	10.000	1	1.000
$3.9\,\tau$	8.050, 11.950	1.000	-0.500	0.500	1.500	LBP_{-1}, MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.372	0.372	10.500	10.000	1	1.115
$4\,\tau$	8.000, 12.000	0.520	0.500	1.020	0.020	LBP_1, MSP_1^{HFT}	LBP_{-1}; LBP_1, MSP_1^{HFT}	0.130	0.333, 0.161	11.020, 10.020, 11.020	10.000, 10.000	1	1.167
						LBP_{-1}	LBP_1, MSP_1^{HFT}; MSP_{-1}, LSP_1	0.365	0.245, 0.168, 0.327	10.020, 11.020	9.000, 10.000, 9.980	0.315	
$5\,\tau$	7.500, 12.500	0.757	0.500	1.257	0.257	LBP_1, MSP_1^{HFT}	LBP_{-1}; LBP_1, MSP_1^{HFT}	0.120	0.286, 0.162	11.257, 10.257, 11.257	10.000, 10.000	1	1.522
						LBP_{-1}	LBP_1, MSP_1^{HFT}; MSP_{-1}, LSP_1	0.328	0.259, 0.236, 0.212	10.257, 11.257	9.000, 10.000, 9.743	0.391	

in three cases: First, $ExchangeTW$ is total welfare given the optimal trading fees set by the profit-maximizing exchange. Second, *BenchmarkTW* is computed using zero fees and rebates $\{MF^{\dagger}=0, TF^{\dagger}=0\}$. Third, $SocialPlannerTW$ is total welfare given optimal fees $\{MF^{*}, TF^{*}\}$ chosen by the Social Planner. The figures also show the welfare breakdown for investors (INV), HFT firms (HFT) where relevant, and the exchange (EXCH) for different investor valuation supports. Finally, the figures show three different regions: The *PIW* region in which optimal fees by the profit-maximizing exchange are Pareto-improving with all market participants better off relative to the no-fee benchmark. The *RW* region in which optimal access pricing by an exchange increases total welfare, but reallocations of welfare are needed from the exchange to investors for investors to be better off. The *DL* region in which total welfare is lower due to dead weight losses, but the exchange is better off. Our findings are consistent across all four market settings.

- The PIW region (in which the exchange's profit-maximizing fees improve welfare for all market participants) happens for small valuation supports Δ. This is expected, since, when the support is small relative to the tick size, there is no-trade without a take or make rebate. For example, in the LTM when the support equals the tick size (i.e., $\underline{\beta}=P_{-1}$ and $\overline{\beta}=P_1$) even though some investors are willing to limit buy (sell) at P_{-1} (P_1), no investors will market sell (buy) at P_{-1} (P_1) without a rebate.

  In general, the reason why rebate based pricings Pareto improve welfare, even when there are gains from trade, is an externality in investor behavior: Individual investors care about both the probability of order execution (which increases total welfare) and also about their execution price (which affects their personal payoff but is neutral for total welfare). When the valuation support is small relative to the tick size, there are many investors for whom the favorable execution-price externality makes them unwilling to trade at available prices. An exchange can increase its expected profit and simultaneously improve total welfare by setting fees and rebates to reduce the price-improvement externality and, thereby, increase the order-execution probability.

- The RW region (in which total welfare increases but investors are worse off unless there are Pareto transfers from the exchange to investors) occurs for somewhat larger valuation supports. As the valuation support widens, a growing share of arriving investors have sufficiently strong trading demands (extreme private valuations) that rebates are not needed for trading. However, there is also an externality in exchange behavior. Exchange expected profits and, thus, their access pricing depend on their net fee (which reduces total welfare) as well as on the order-execution probability. When valuation dispersion becomes larger relative to the tick size, exchanges set larger net fees to increase their expected profit although this reduces order-execution probabilities. For a range of support widths Δ, the net effect of rebate-based exchange fees is to increase total welfare relative to the no-fee-and-rebate equilibrium, but with the exchange capturing a growing share of the gains-from-trade at the expense of investors. This leads to the RW region.

- The PIW and RW regions shrink when the tick size is smaller and when the number of trading periods is larger. In particular: the PIW (RW) region extends up to supports of $1.27\,\tau$ ($1.88\,\tau$) for the LTM but only to $0.42\,\tau$ ($0.63\,\tau$) for the STM; and the PIW (RW) region only extends to supports of $1.2\,\tau$ ($1.62\,\tau$) for the three-period LTM down from $1.27\,\tau$ ($1.88\,\tau$) for the two-period LTM. The effect is stronger when decreasing the tick size,[27] but the intuition is similar. The frictions associated with the price grid discreteness are alleviated both when the tick size is smaller and when the number of trading period is larger. Consequently, the positive effect of rebates on investors' welfare is also smaller.

- Finally, the DL region (where the exchange is better off but there are total dead weight losses relative to the no-fee equilibrium because the welfare gains for the exchange are smaller than the investors losses) happens when the investor valuation support is even larger. For example, for the two-period market, the DL region extends beyond 1.88τ, for the three-period beyond 1.62τ, and for the three-

[27] We only increase the number of periods by just one trading round, so the increase in trading activity is positive but small.

period with HFTs it extends beyond 3.90τ.[28] Once the dispersion in investor valuations is large relative to the tick size, the exchange's profit-maximizing net fee becomes so large that it reduces total welfare. The shaded areas reported in Figures 5 and 6 show the DL region due to rebate-based pricing — as opposed to positive pricing — set by a profit-maximizing exchange.[29] As the incentive for the exchange to set rebate-based pricing decreases with tick size and trading activity, the dark shaded DL region also decreases when the tick size is smaller and the trading game is longer.

Interestingly, in the framework with 3-period and HFTs the dark shaded DL region disappears and there are no dead weight losses associated with rebate based pricing by the exchange. In contrast, the RW region increases substantially with HFTs. The reason being that the rebates set by the exchange subsidize an increase in trading opportunities for all investors thus strongly increasing trading opportunities for investors and consequently enhancing exchange profits, with the result that the increase in exchange profits induced by rebate-based pricings is larger than the increase in investors losses.

Figures 5 and 6 also show the welfare improvement by the Social Planner. The setting that benefits the most from the Social Planner is the market with HFT firms. The framework that benefits the least is the small tick market. This is intuitive. With a small tick size, the benefit from rebate-based pricing is the smallest. Note that in all DL regions in which an exchange uses rebate-based pricing, a Social Planner also sets rebate-based pricing. Thus, the dead weight loss is not due to rebates per se, but to the fact that the exchange sets rebates to maximize its own profits. When setting rebates, the exchange faces a trade-off. The smaller the investors gains from trade, the larger the rebates necessary to induce them to participate and the smaller the exchange net revenue from each trade. Hence, in equilibrium the exchange does not subsidise investors with smaller gains from trade. The Social Planner pricing instead aims to maximize all investors welfare and therefore it also subsidises traders with smaller gains from trade. This is the reason why in

[28] In the 2-period frameworks the DL region starts earlier, as the STM is isomorphic to the LTM framework - see Theorem 2.

[29] See Tables 1, D1, 4 and 5

Figures 5 and 6 *SocialPlannerTW* is always greater or equal than *BenchmarkTW* so that in correspondence of the DL region, the Social Planner rebate-based pricing leads to an improvement in total welfare.

Our welfare results have policy implications. In particular, rebate-based pricing is not detrimental per se to investors given price frictions. It is how exchanges set fees in combination with rebates that can generate dead weight losses. Our result, therefore, suggests a positive potential role for regulation limiting the ability of exchanges to set fees that are too large. Our results also have policy implications for the regulatory cap on access fees.

Proposition 5. *Optimal access pricing by an exchange: When the valuation support* Δ *is sufficiently large such that the exchange optimally sets strictly positive fees, a smaller regulatory cap on fees can increase total welfare. However, when* Δ *is sufficiently small such that welfare is increasing in rebate-based pricing, a tighter regulatory cap on fees can possibly reduce total welfare.*
Optimal access pricing by the Social Planner: Tighter regulatory caps on fees can potentially reduce total welfare.

For optimal access pricing by an exchange, this proposition follows immediately from the existence of welfare-increasing and deadweight loss regions with rebate-based pricing. The intuition is that, when rebates are needed to encourage trading, a cap on fees potentially reduces the possible rebates the exchange can afford. In contrast, when the exchange uses strictly positive fees, which increases its profits but which leads to deadweight welfare losses, a tighter cap on its fees can alleviate this problem. For access pricing by the Social Planner, the intuition follows from the fact that the Social Planner always optimally needs rebate-based access pricing to improve total welfare.

Figure 5: Welfare: 2-period LTM and STM This figure shows how the welfare of the Exchange (EXCH) - dashed line, Investors, (INV - dotted line) and Total Welfare (INV + EXCH - dashed-dotted line) change with the investors' support (S) in the large tick market (LTM) on the left and in the small tick market (STM) on the right. Both figures also report the welfare of investors under the Benchmark regime (solid line) with no trading fees (MF=TF=0). The support is expressed in large tick unit of measure (τ). Both figures show the results for three regions: Pareto Improvement Welfare (PIW), Redistribution Welfare (RW) and Deadweight Loss (DL).

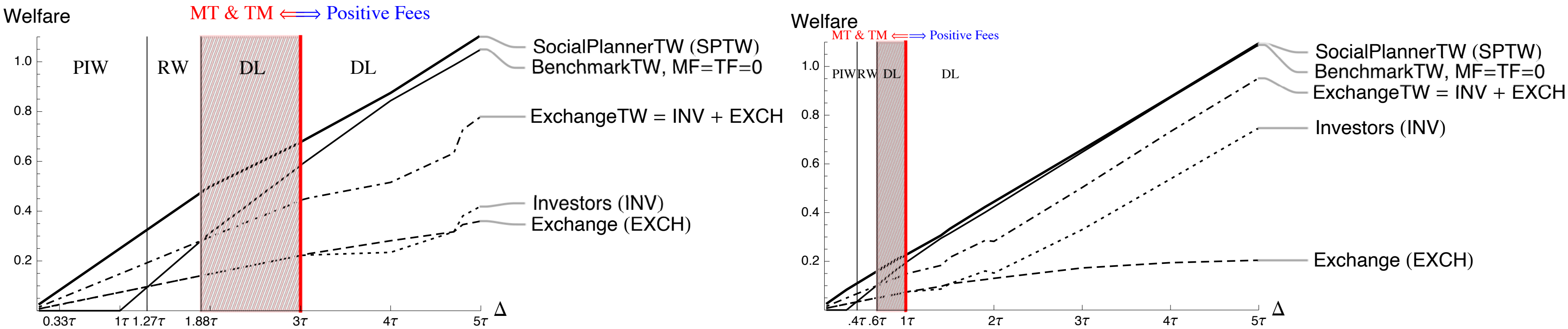
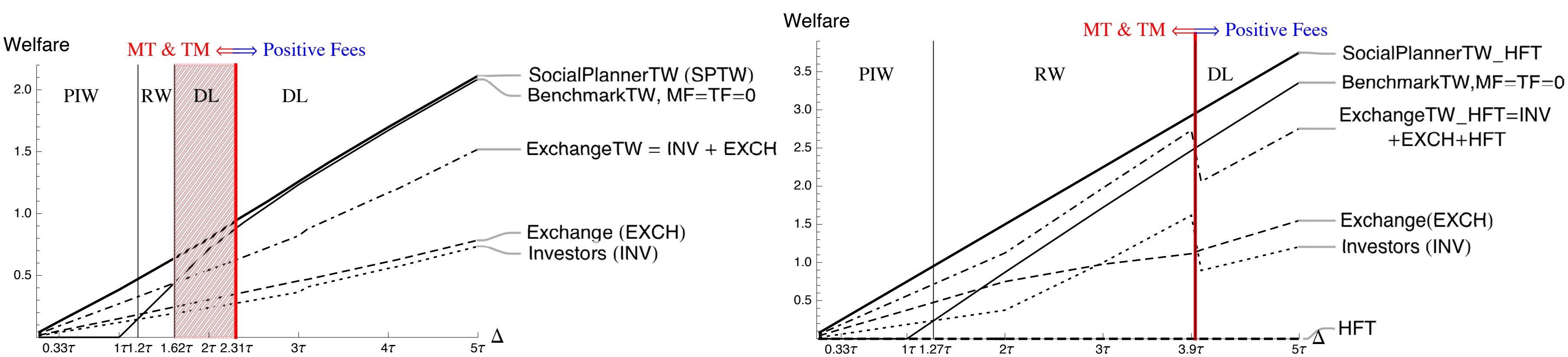


Figure 6: Welfare: 3-period LTM and 3-period LTM with HFT This figure shows on the left how the welfare of the Exchange (EXCH) - dashed line, Investors, (INV - dotted line) and Total Welfare (INV + EXCH - dashed-dotted line) change with the investors' support (S) in the large tick market (LTM); and on the right it shows how the welfare of the Exchange (EXCH) - dashed line, Investors and High Frequency Traders (INV&HFT - dotted line) and Total Welfare of Investors, HFTs and the Exchange (INV&HFT+EXCH - dashed-dotted line) change with the investors' support (S). Both figures also report the welfare of investors under the Benchmark regime (solid line) with no trading fees (MF=TF=0). The support is expressed in large tick unit measure (τ). Both figures show the results for three regions: Pareto Improvement Welfare (PIW), Redistribution Welfare (RW) and Deadweight Loss (DL). The shaded area indicates the DL region with rebate-based pricing.



6 Conclusions

This paper models optimal access pricing for an exchange or Social Planner and gives new insights about access pricing, its drivers and welfare effects. Our analysis shows investor valuation dispersion is a key driver of optimal access pricing. When the market is mainly populated by investors with valuations close together, the equilibrium access pricing by a profit-maximizing exchange is rebate-based. When the market is populated by long-term investors with ex ante disperse valuations, the exchange chooses jointly positive make and take fees. The Social Planner instead sets rebate-based pricing so the net fee is zero.

Access pricing in our model can alleviate trading frictions generated by price discreteness. With regulatory constraints on access fees, the tick size also affects optimal access pricing. Optimal access pricing scales linearly with the tick size and depends on both the absolute tick size and on the tick size relative to investor-valuation dispersion. When the tick size is small, the region of investor supports consistent with a rebate-based pricing is small, and exchanges have a lower/smaller degree of freedom in setting access fees.

From a policy perspective, regulation crucially affects access pricing. When there is no cap, the exchange maximizes its expected profit using rebate-based access pricing that induces market participants to trade at the outside quotes. When access fees are instead capped by the tick size, the exchange uses rebate-based pricing when the trader valuation support is small and investors need to be cross-subsidized to trade at the inside quotes. In this case, a tighter regulatory cap on fees can reduce total welfare. When traders have dispersed valuations, the exchange sets positive make and take fees. In this second case, a tighter cap can increase total welfare.

Importantly for regulators, our analysis also shows that rebate-based pricing is not welfare reducing per se, but rather that the welfare effects of rebate-based pricing depend on the incentives of who sets access fees and rebates and on the magnitude of trading frictions relative to investor ex ante trading demand. In particular, we show that a Social Planner always uses rebate-based access pricing to increase total welfare. In contrast, welfare with optimal access pricing by a profit-maximizing exchange is more nuanced with three different possible welfare effects: First, when the market is populated by investors with small gains-from-trade, and frictions from price discreteness are severe, we find rebate-based pricing by the exchange reduces pricing frictions and Pareto improves total welfare for both investors and the exchange. Second, when investors have large gains-from-trade, and the tick size friction is less severe, optimal exchange access pricing — without and sometimes even with rebates — can lead to total deadweight losses as increased profits for the exchange are less than welfare losses for investors. Third, when investor gains-from-trade are in between, rebate-based optimal access pricing by an exchange increases total welfare with higher exchange expected profits but reduces welfare for investors without Pareto transfers.

Our model analyzes two other important determinants of optimal access pricing: The amount of trading activity and HFTs. In a basic price-friction model, an exchange's incentive for rebate-based pricing is weak for active stocks with high trading activity and large gains-from-trade. For such stocks, there is less need to subsidize trading via rebate-based pricing and therefore the potential welfare improvement from rebate-based pricing is low. This appears inconsistent with current market practices with widespread rebate-based pricing for liquid stocks. However, when we extend our model to include HFTs, we can explain this seeming inconsistency. An exchange optimally uses rebate-base pricing to attract HFT firms, and the range of investor valuation supports associated with a rebate-based pricing is wider than without HFTs. This helps explain why rebate-based access pricing became predominant, since Reg NMS led to a sharp increase in HFT activity. The increase in HFT led to increased trading opportunities for regular investors and, thus, increased volume, and, hence, also to increased exchange profits. As a consequence, exchanges have an incentive to cross-subsidize investors by offering rebates to HFTs. Rebate-based pricing allows the compensation paid to HFTs for the provision of liquidity to be minimized thus leading to an increase in both trading activity and total welfare. When gains-from-trade are small, rebate-based pricing with HFTs leads to a small Pareto improvement in welfare — thus making investors overall better off compared to the framework without HFTs. However, the most important effect of rebate-based pricing with HFTs is the increase in the "Redistribution Welfare" region and disappearance of the DL region. By subsidizing HFTs, rebates increase trading activity, thus increasing the exchange's welfare at the expenses of regular investors in a way that increases total welfare. This means that HFTs increase a regulator's incentive to implement Pareto transfers from the exchange to regular investors. It is also the reason a Social Planner always sets access fees — conditional on any support — to attract HFT firms. The Social Planner also sets the exchange's net profits to zero and redistributes welfare from the exchange to regular investors, which leads to Pareto improvement in welfare. Thus, rebate-based access pricing and HFT liquidity provision potentially can have a positive welfare impact given appropriate regulation.

Our model has a number of other "firsts" from a modeling perspective. Our model is the first to provide a complete analysis of the effect of endogenous limit price choice and market/limit order choice on access pricing. Our model also is the first to consider more than two periods. This extension lets us show that access pricing changes with greater market activity. We also are the first to formally model the Social Planner.

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Appendices

A General proofs

Lemma 1. *If access pricing fees are capped at one tick by regulation, then an exchange never sets rebates larger than one tick in equilibrium.*

Proof for Lemma 1 If the access rebate is larger than the access fee, then the exchange's net profit per transaction is negative. However, an exchange can always earn a zero net profit per transaction by setting its rebates equal to its fee. Thus seeting rebates larger than fees does not maximize exchange profits (if profit-maximizing exchanges set access pricing) and violates incentive compatibility (if a welfare-maximizing Social Planner sets access pricing). Q.E.D.

Lemma 2. *Given access pricing such that the exchange has a non-negative profit $MF+TF \geq 0$ per transaction, a priori arriving investors only consider submitting limit buys (sells) at posted prices P_k that are sufficiently low (high) such that the associated cum-fee prices for market sells (buys) satisfies $P_k^{cum,MS} < \overline{\beta}$ ($P_k^{cum,MB} > \underline{\beta}$).*

Proof for Lemma 2: : This follows immediately from the fact that submitting a limit buy is only profitable for an investor if the cum-fee limit-buy price $P_k^{cum,LB} = P_k + MF \leq \overline{\beta}$ and the fact that $P_k^{cum,MS} = P_k - TF = P_k + MF - (MF+TF)$ where $MF+TF \geq 0$ for an exchange with a non-negative profit per transaction and, thus, the cum-fee market sell price satisfies $P_k^{cum,MS} \leq \overline{\beta}$. The argument for limit sells is symmetric. Q.E.D.

Lemma 3. *If the standing limit order book is symmetric at a time t_z, then investors with $\beta_{t_z} > v$ are potential buyers at time t_z (i.e., they either submit limit buy orders or NT but they never submit limit sell orders). Similarly, investors with $\beta_{t_z} < v$ are potential sellers at time t_z.*

Proof of Lemma 3: This result follows from the fact that the investor expected profit functions from limit buy and sell orders are symmetric and increasing in the distance of β_{t_z} from the posted limit prices. Q.E.D.

Comment about Lemma 3: In particular, Lemma 3 applies at time t_1 since the initial book is empty.

Lemma 4. *i) In a Taker-Maker regime with $-1 \leq TF < 0 \leq MF \leq 1$, limit orders are never posted at prices P_k outside of the interval $[\underline{\beta}-1, \bar{\beta}]$ for limit buys or outside of the interval $[\underline{\beta}, \bar{\beta}+1]$ for limit sells. ii) In a Maker-Taker regime with $-1 \leq MF < 0 \leq TF \leq 1$, limit orders are never posted at prices P_k outside of the interval $[\underline{\beta}, \bar{\beta}+1]$ for limit buys or outside of the interval $[\underline{\beta}-1, > \bar{\beta}]$ for limit sells. iii) In a positive-fee regime with $0 \leq TF \leq 1$ and $0 \leq MF \leq 1$, limit buy and sell orders are never posted at prices P_k outside of the interval $[\underline{\beta}, \bar{\beta}]$ for both limit buys or limit sells.*

Proof of Lemma 4: In a Taker-Maker regime, the highest possible cum price $P_k^{cum,MS} = P_k - TF$ for a market sell given a limit buy at a posted price $P_k < \underline{\beta} - 1$ is $P_k + 1 < \underline{\beta}$ given the bounded take rebate $-TF \leq 1$. Thus, no investor arriving at t_2 will be willing to submit a market sell given a limit buy at posted prices $P_k < \underline{\beta} - 1$. Similarly, the lowest possible cum price $P_k^{cum,LB} = P_k + MF$ for a limit buy at a posted price $P_k > \bar{\beta}$ is $P_k > \bar{\beta}$ given the non-negative fee $MF > 0$ in a Taker-Maker regime. As a result, no investor arriving at t_1 will be willing to post a limit buy at prices $P_k > \bar{\beta}$. A similar logic applies for the result for potential posted limit prices in the Maker-Taker regime and the positive-fee regime. Q.E.D.

Proof for Theorem 6: The proof strategy is standard for finite sequential games and consists of three steps: The recursion step for deriving analytic investor strategies is the following: Given access pricing fees Ξ, the

Table A1: Trading Strategies and Payoffs This table reports the trading strategies and associate payoffs available to investors .

Action	Payoff
Market Order to Sell: $x_{t_z}^{MS} = MSP_{t_z}$	$P(x_{t_z}) - \beta_{t_z} - TF$
Limit Order to Sell: $x_{t_z}^{LS} = LSP_{t_z}$	$[P(x_{t_z}) - \beta_{t_z} - MF]\, Pr(\theta_{t_z}^{x_{t_z}} \mid \Xi, L_{t_{z-1}})$
No Trade: NT_{t_z}	0
Limit Order to Buy: $x_{t_z}^{LB} = LBP_{t_z}$	$[\beta_{t_z} - P(x_{t_z}) - MF]\, Pr(\theta_{t_z}^{x_{t_z}} \mid \Xi, L_{t_{z-1}})$
Market Order to Buy: $x_{t_z}^{MB} = MBP_{t_z}$	$\beta_{t_z} - P(x_{t_z}) - TF$

order-execution probabilities $Pr(\theta_{t_z}^{x_{t_z}} \mid \Xi, L_{t_{z-1}})$ for computing the investor expected profit for each possible order $x_{t_z} \in X_{t_z}$ at any time t_z in the investor maximization problem (2) are either 1 for market orders at the BBO or are determined recursively for limit orders from the order-submission probabilities $Pr(x_{t_z} \mid \Xi, L_{t_{z-1}})$ at later dates. The upper envelope of the expected investor payoffs for the different possible actions at a generic time t_z determines the optimal investor actions at t_z and, given the distribution over the investor valuation β_{t_z} the associated order-submission probabilities for the optimal actions in terms of intervals on the investor valuation support S for any incoming book $L_{t_{z-1}}$. Given the assumptions of a discrete number of possible investor actions and discrete tine, the set of possible incoming books is finite.

The initiation step starts the recursion at the terminal period t_N, at which time the order-execution probabilities take a simple form: They are zero for new limit orders (since the game ends after time t_N) and one for market orders (which can only be submitted if the book is non-empty). Thus, investor expected profit for different orders, the upper envelope, the optimal orders, and the order-submission probabilities at time t_N can be derived directly.

The exchange profit optimization step is then as follows: The order-submission and order-execution probabilities from the first two steps can then be used to construct the exchange's expected profit in (22) analytically given arbitrary fees Ξ. Given the analytic exchange expected profit function, the profit-maximizing fees Ξ^* can then be found analytically since the set of possible fees and rebates is compact given the regulatory cap on access fees. Q.E.D.

Proof of Theorem 7: The proof structure is the same as for Theorem 6 with the addition that INVs and HFTs investors arrive sequentially. First, the recursion step again involves characterizing analytic optimal order submissions and order-submission probabilities in term of intervals of valuations β_{t+z} along the support S associated with the analytic upper envelope of the payoffs of all of the possible investor actions. Again, there are a finite number of possible investor actions with linear payoff and a finite number of periods and, thus, at each point in time t_z, a finite set of possible prior histories L_{tz-1}. Analytic order-execution probabilties can then be computed from the analytic order-submission probabilities. Second, the initiation step at time N involves optimization with only market orders for which the a priori execution probability is one. Third, the exchange profit optimization step is logically similar to the same step in Theorem 6. Q.E.D.

B Equilibrium of Two-Period Model and Proof of Theorem 1

This Appendix solves for the profit-maximizing exchange's optimal access pricing in the two-period model given the regulatory constraint and given optimal trading by investors. Since the investors' optimal orders are solutions to discrete choice problems, their trading strategies and the exchange's optimal fees change qualitatively in different regions of the parameter space in terms of the size of the investor valuation support Δ relative to the tick size τ, which is normalized to 1 in the LTM.

B.1 Case 1: $0 < \Delta \leq 3\tau$

Our analysis of this first case shows that, when $\Delta \leq 3\tau$, optimal access pricing by the exchange in equilibrium takes the functional forms in (16) for Taker-Maker pricing and (17) for Maker-Taker pricing.

Figure B1: Taker-Maker Pricing: $\Xi_{TM} = \{0 \leq MF \leq 1, -1 \leq TF \leq 0\}$ This Figure provides a graphical representation of how to obtain the equilibrium probabilities of order submission and execution for the Taker-Maker pricing structure and the support $\Delta \in [\underline{\beta}, \bar{\beta}]$. P_2 and P_{-2} are the outside quotes of the LTM, whereas P_1 and P_{-1} are the inside quotes of the LTM. $P_{-1}^{cum,LB}$ and $P_{-1}^{cum,MS}$ are the cum-fee buy and sell prices, respectively. LBP_{-1,t_1} is a limit buy order posted at P_{-1} at t_1, and MSP_{-1,t_2} is a market sell order posted at P_{-1} at t_2.

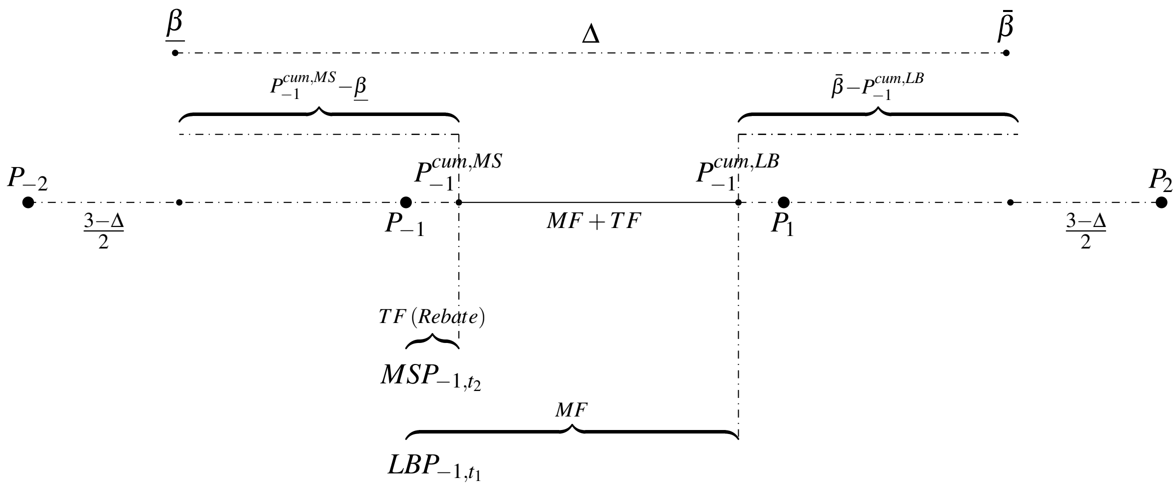


Figure B2: Maker-Taker Pricing: $\Xi_{MT} = \{-1 \leq MF \leq 0, 0 \leq TF \leq 1\}$ This Figure provides a graphical representation of how to obtain the equilibrium probabilities of order submission and execution for the Maker-Taker pricing structure and the support $\Delta \in [\underline{\beta}, \bar{\beta}]$. P_2 and P_{-2} are the outside quotes of the LTM, whereas P_1 and P_{-1} are the inside quotes of the LTM. $P_1^{cum,LB}$ and $P_1^{cum,MS}$ are the cum-fee buy and sell prices, respectively. LBP_{1,t_1} is a limit buy order posted at P_1 at t_1, and MSP_{1,t_2} is a market sell order posted P_1 at t_2.

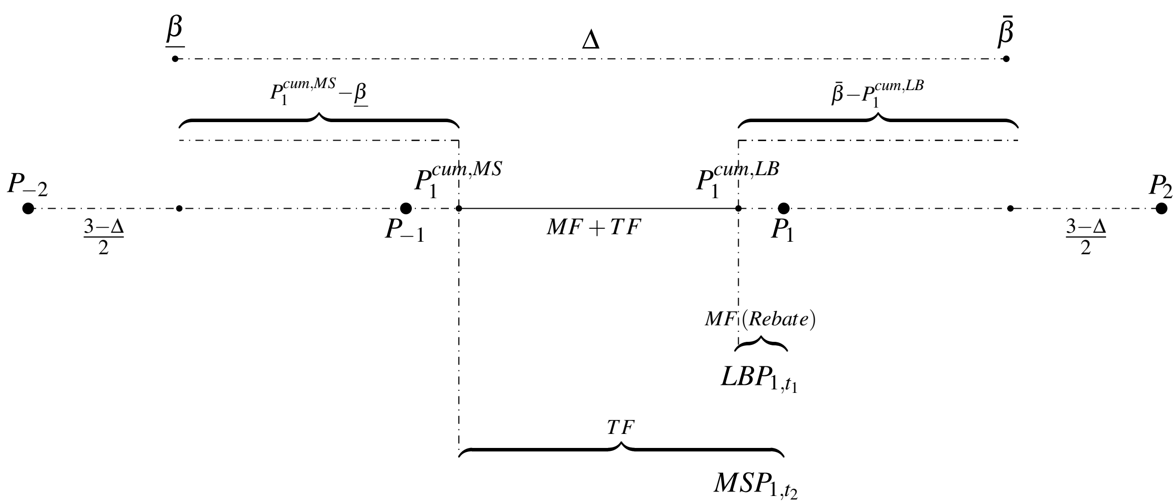


Table B1: **Submission and Execution Probability.** This table reports the price levels on the LTM price grid (column 1) and the associated probabilities $Pr(\beta_{t_1} > P_k^{cum,LB}) = \max\{0, \frac{\bar{\beta}-P_k^{cum,LB}}{\Delta}\}$ and $Pr(P_{-k}^{cum,LS} > \beta_{t_1}) = \max\{0, \frac{P_{-k}^{cum,LS}-\underline{\beta}}{\Delta}\}$, which, in equilibrium, correspond to the submission probabilities for limit orders posted at P_k and at P_{-k} at t_1 (columns 2 and 3). In addition, the table reports the associated limit order execution probabilities, $Pr(\theta_{t_1}^{x_k^{LB}}|\Xi, L_{t_0}) = Pr(x_{k,t_2}^{MS}|\Xi, L_{t_0}) = \max\{0, \frac{P_k^{cum,MS}-\underline{\beta}}{\Delta}\}$ and $Pr(\theta_{t_1}^{x_{-k}^{LS}}|\Xi, L_{t_0}) = Pr(x_{-k,t_2}^{MB}|, \Xi, L_{t_0}) = \max\{0, \frac{\bar{\beta}-P_{-k}^{cum,MB}}{\Delta}\}$ (columns 4 and 5).

P_k	$Pr(\beta_{t_1} > P_k^{cum,LB})$	$Pr(P_{-k}^{cum,LS} > \beta_{t_1})$	$Pr(\theta_{t_1}^{x_k^{LB}}\|\Xi, L_{t_0})$	$Pr(\theta_{t_1}^{x_{-k}^{LS}}\|\Xi, L_{t_0})$
P_3	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} + TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} + TF]\right\}$
P_2	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - TF]\right\}$
P_1	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - TF]\right\}$
P_{-1}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - TF]\right\}$
P_{-2}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - TF]\right\}$
P_{-3}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - TF]\right\}$

Taker-Maker: We first consider Taker-Maker pricing $\Xi_{TM} = \{0 \leq MF \leq 1, -1 \leq TF \leq 0\}$ with a take rebate and a positive make fee. Given $\Delta \leq 3$, the lower investor-valuation bound in this case is $\underline{\beta} = P_{-2} + \frac{3-\Delta}{2}$, and the upper bound is $\overline{\beta} = P_2 - \frac{3-\Delta}{2}$, as illustrated in Figures B1 and B2. Consider first a potential buyer arriving at t_1 with $\beta_{t_1} > v$. The logic for a potential seller arriving at t_1 is symmetric.

Order-submission probabilities for each possible market order at t_2 can be computed using (3) and (4) given the valuation-support restriction $\Delta \leq 3$ and Taker-Maker pricing. Columns 4 and 5 in Table B1 report the market order submission probabilities for the price levels in Column 1:

$$Pr(x_{k,t_2}^{MS}|\Xi, L_{t_1}) = \max\left\{0, \frac{P_k - TF - \underline{\beta}}{\Delta}\right\} = \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} + \frac{P_k - P_{-k}}{2} - TF\right]\right\} \tag{28}$$

$$Pr(x_{-k,t_2}^{MB}|\Xi, L_{t_1}) = \max\left\{0, \frac{\bar{\beta} - P_{-k} - TF}{\Delta}\right\} = \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} - \frac{P_k - P_{-k}}{2} - TF\right]\right\} \tag{29}$$

For example, Row 5 in Column 4 and Row 5 in Column 5 in Table B1 gives the order-submission probability at t_2 of a market sell at P_{-1}, which is equal to the order-submission probability of a market buy at P_1

$$\begin{aligned} Pr(x^{MS}_{-1,t_2}|\Xi,L_{t_1}) &= \max\left\{0, \frac{P^{cum,MS}_{-1} - \underline{\beta}}{\Delta}\right\} \\ &= Pr(x^{MB}_{1,t_2}|\Xi,L_{t_1}) = \max\left\{0, \frac{\bar{\beta} - P^{cum,MB}_{1}}{\Delta}\right\} = \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} - \frac{1}{2} - TF\right]\right\}. \end{aligned} \tag{30}$$

To understand the intuition in the last term in (30), note from Figure B1 that only traders with β_{t_2} in the interval $[\underline{\beta}, P^{cum,MS}_{-1}]$ with width $\frac{\Delta}{2} - \frac{1}{2} - TF$ are willing to use a market order to sell at a posted price P_{-1}. This interval is equal to half of the support minus half the tick size, hence $\frac{1}{2}$, given $\tau = 1$, which is the distance from the fundamental asset value v to P_{-1}, minus TF (negative in the Taker-Maker regime), which increases the interval of the support including βs belonging to sellers. This interval is strictly positive for $\Delta \geq 1$, which means that $Pr(x^{MS}_{-1,t_2}|\Xi,L_{t_1}) > 0$ for $\Delta \geq 1$.

The market-order submission probabilities at t_2 are, in turn, respectively the corresponding order-execution probabilities of limit orders posted at t_1. Thus, we can consider the expected profits for all possible limit orders that a potential buyer and symmetrically a potential seller can post at t_1. We verify the conditions under which (5) and (6) hold — and symmetrically (8) and (9) — and finally compute the limit order submission probabilities at t_1 consistent with both (7) and (10) .

To check that conditions (5) and (8) hold, we compute $Pr(\beta_{t_1} > P^{cum,LB}_{k})$ and $Pr(P^{cum,LS}_{-k} > \beta_{t_1})$ for each order in Columns 2 and 3 of Table B1. For example, for a limit buy at P_{-1} and limit sell at P_1 we have:

$$\begin{aligned} Pr(\beta_{t_1} > P^{cum,LB}_{-1}) &= \max\left\{0, \frac{\bar{\beta} > P^{cum,LB}_{-1}}{\Delta}\right\} \\ &= Pr(P^{cum,LS}_{1} > \beta_{t_1}) = \max\left\{0, \frac{P^{cum,LS}_{1} > \underline{\beta}}{\Delta}\right\} = \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} + \frac{1}{2} - MF\right]\right\}. \end{aligned} \tag{31}$$

To understand the intuition for the final term in (31), notice, for example, from Figure B1 that only traders with a β_{t_1} in the interval $[P^{cum,LB}_{-1}, \overline{\beta}_{t_1}]$ with width $\frac{\Delta}{2} + \frac{1}{2} - MF$ will be willing to buy at the quoted price P_{-1}.

This interval is equal to half of the investor valuation support (consistent with Lemma 3 only traders with a personal evaluation larger than the fundamental value v will be buying) plus half the tick size (the distance between the mid-point of the support/fundamental asset value v and P_{-1}) minus MF, which decreases the interval of the support including βs belonging to buyers.

We also need to check whether both conditions (6) and (9) hold for each possible order at t_1:

- First, consider a limit buy at P_2 and symmetrically a limit sell at P_{-2}. Given the assumed investor valuation support with width $\Delta \leq 3$ and given the positive MF with Taker-Maker pricing, the expected payoff associated with limit orders at P_2 (P_{-2}) would be negative since the associated cum-fee buy (sell) price would be above (below) the maximum (minimum) possible trader valuation. Hence, such limit orders would never be submitted.

- Second, the expected profit $(\beta_{t_1} - P_{-1}^{cum,LB}) \times Pr(\theta_{t_1}^{x_{-1}^{LB}} | \Xi, L_{t_0})$ on a limit buy at P_{-1} for a potential buyer with $\beta_{t_1} > v$ and $(P_1^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}} | \Xi, L_{t_0})$ on a limit sell at P_1 for a potential seller with $\beta_{t_1} < v$ is:

$$\left(|\beta_{t_1} - v| + \frac{1}{2} - MF\right) \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} - \frac{1}{2} - TF\right]\right\}, \tag{32}$$

  which is positive given Take rebates ($0 \geq TF \geq -1$) and Make fees ($1 \geq MF \geq 0$).

- Third, the expected profit $(\beta_{t_1} - P_{-2}^{cum,LB}) \times Pr(\theta_{t_1}^{x_{-2}^{LB}} | \Xi, L_{t_0})$ on a limit buy at P_{-2} for a potential buyer with $\beta_{t_1} > v$ or $(P_2^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_2^{LS}} | \Xi, L_{t_0})$ on a limit sell at P_2 for a potential seller with $\beta_{t_1} < v$ is:

$$\left(|\beta_{t_1} - v| + \frac{3}{2} - MF\right) \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} - \frac{3}{2} - TF\right]\right\}, \tag{33}$$

To characterize when the expected profit on limit buys at P_{-1} (and limit sells at P_1) are greater than on limit buys at P_{-2} (and limit sells at P_2), we write the expected profits in (32) for limit buys at P_{-1} and limit sells at P_1 as $a*b$ where $a = |\beta_{t_1} - v| + \frac{1}{2} - MF$ and $b = \frac{1}{\Delta}\left[\frac{\Delta}{2} - \frac{1}{2} - TF\right]$. Given this, we derive the β threshold between a limit buy at P_{-1} and a limit buy at P_{-2} as the β values for which (34) holds

$$(\beta_{t_1} - P_{-2}^{cum,LB}) \times Pr(\theta_{t_1}^{x_{-2}^{LB}}|\Xi, L_{t_0}) - (\beta_{t_1} - P_{-1}^{cum,LB}) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|\Xi, L_{t_0}) = 0 \tag{34}$$

The β_{t_1} values which satisfy (34) are

$$\beta_{t_1}^{x_{P_{-2}}^{LB}, x_{P_{-1}}^{LB}} = \begin{cases} v + \frac{\Delta}{2} - 2 + MF - TF & TF < 0 \wedge TF < \frac{\Delta-3}{2} \\ \\ v - \frac{1}{2} + MF & \text{Otherwise} \end{cases} \tag{35}$$

We can now compute the order-submission probabilities from (7) for a limit buy at P_{-1} and at P_{-2}

$$Pr(x_{-1,t_1}^{LB}|\Xi, L_{t_0}) = \begin{cases} \frac{1}{\Delta}[\frac{\Delta+1}{2} - MF] & TF \geq 0 \vee TF \geq \frac{\Delta-3}{2} \\ \\ \frac{1}{\Delta}\left[2 - MF + TF\right] & \text{Otherwise} \end{cases} \tag{36}$$

$$Pr(x_{-2,t_1}^{LB}|\Xi, L_{t_0}) = \begin{cases} \frac{1}{\Delta}\left[\frac{\Delta-4}{2} + MF - TF\right] & (MF > \frac{1}{2} \wedge TF < 0 \wedge TF < \frac{\Delta-3}{2}) \vee \\ & \left(MF \leq \frac{1}{2} \wedge MF > TF + \frac{1}{2} \wedge MF > \frac{4-\Delta}{2} + TF\right) \\ \\ 0 & \text{Otherwise} \end{cases} \tag{37}$$

Lastly, the expected profit $(\beta_{t_1} - P_{1,LB}^{cum,LB}) \times Pr(\theta_{t_1}^{x_1^{LB}}|\Xi, L_{t_0})$ on a limit buy at P_1 for a potential buyer with $\beta_{t_1} > 0$ and $(P_{-1}^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}}|\Xi, L_{t_0})$ on a limit sell at P_{-1} for a potential seller with $\beta_{t_1} < 0$ is:

$$\left(|\beta_{t_1} - v| - \frac{1}{2} - MF\right) \max\left\{0, \frac{1}{\Delta}\left[\frac{\Delta}{2} + \frac{1}{2} - TF\right]\right\}. \tag{38}$$

Comparing (32) and (38) shows that, in Taker-Maker regimes, limit buys at P_{-1} always have higher expected profit than limit buys at P_1 and that limit sells at P_1 have higher expected profits than limit sells at P_{-1}:[30]

$$(\beta_{t_1} - P_1^{cum,LB}) \times Pr(\theta_{t_1}^{x_1^{LB}}|\Xi, L_{t_0}) - (\beta_{t_1} - P_{-1}^{cum,LB}) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|\Xi, L_{t_0}) \tag{39}$$
$$= (P_{-1}^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}}|\Xi, L_{t_0}) - (P_1^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}}|\Xi, L_{t_0})$$
$$= (|\beta_{t_1} - v| - \frac{\Delta}{2}) - MF + TF \leq 0$$

where the inequality in the third line follows because $|\beta_{t_1} - v| \leq \frac{\Delta}{2}$ by definition for all β_{t_1} and because $MF \geq 0$ and $TF \leq 0$ in the Taker-Maker regime. We can now set the optimizing function for both the exchange and the Social Planner.

Comment: The discussion above identifies which orders are possibly used in the two-period trading subgame in the LTM with $\tau = 1$. This analysis is used next to derive optimal fees in the LTM. Section B.3 generalizes the trading subgame analysis to price grides with more prices.

Exchange Problem: Taker-Maker $\Delta \in (0, 3]$ The exchange chooses MF and TF to maximize its profits given the optimal strategy for potential buyers and sellers posting limit orders LBP_{-1,t_1} and LSP_{1,t_1} and LBP_{-2,t_1} and LSP_{2,t_1} at t_1, which we have derived as a function of the trading fees MF and TF and the investor valuation-support width Δ.[31] From now onward we concentrate on the buy side, the sell side being symmetric. The exchange's expected profit is equal to the submission probability $Pr(x_{-1,t_1}^{LB}|\Xi, L_{t_0})$ of LBP_{-1,t_1} and the submission probability $Pr(x_{-2,t_1}^{LB}|\Xi, L_{t_0})$ of LBP_{-2,t_1}, times the associated execution probability $Pr(\theta_{t_1}^{x_{-1}^{LB}}|\Xi, L_{t_0})$ and $Pr(\theta_{t_1}^{x_{-2}^{LB}}|\Xi, L_{t_0})$ times the per share net fee, MF+TF. Table B1 reports the order-

[30] Using the representation for the expected profit for a limit buy at P_{-1} and limit sell at P_1 in (32) as $a * b$ where $a = |\beta_{t_1} - v| + \frac{1}{2} - MF$ and $b = \frac{1}{\Delta}\left[\frac{\Delta}{2} - \frac{1}{2} - TF\right]$, the expected profit for a limit buy LBP_1 at P_1 and limit sell LSP_{-1} at P_{-1} in (38) can be represented as $(a-1)(b+1)$. Taking the difference $a * b - (a-1)(b+1)$ and substituting in for a and b gives the third line in (39).

[31] The case of a seller posting LSP_{1,t_1} or LSP_{2,t_1} is symmetric. As in real markets, traders arrive sequentially and, hence, either a buyer or seller may arrive at t_1.

execution probabilities.

$$\max_{\substack{MF,TF\\0\leq MF\leq\tau\\TF\leq MF}} \pi^{Ex,LTM}(MF,TF) \tag{40}$$

$$= \left[Pr(x^{LB}_{-1,t_1}|\Xi,L_{t_0})\times Pr(\theta_{t_1}^{x^{LB}_{-1}}|\Xi,L_{t_0}) + Pr(x^{LB}_{-2,t_1}|\Xi,L_{t_0})\times Pr(\theta_{t_1}^{x^{LB}_{-2}}|\Xi,L_{t_0})\right]\times(MF+TF)$$

$$= \begin{cases} \dfrac{(-\Delta+2MF-1)(MF+TF)(-\Delta+2TF+1)}{4\Delta^2} & TF\geq 0 \vee 2TF+3\geq\Delta \\ -\dfrac{(MF+TF)(-(\Delta-3)\Delta+4MF+2(\Delta-2)TF-8)}{4\Delta^2} & \text{Otherwise} \end{cases}$$

The first order conditions are:

$$\begin{cases} \dfrac{(-\Delta+2TF+1)(-\Delta+4MF+2TF-1)}{4\Delta^2}=0 & TF\geq 0\vee 2TF+3\geq\Delta \\ \dfrac{-8MF+\Delta(\Delta-2TF-3)+8}{4\Delta^2}=0 & \text{Otherwise} \end{cases} \tag{41}$$

$$\begin{cases} \dfrac{(-\Delta+2MF-1)(-\Delta+2MF+4TF+1)}{4\Delta^2}=0 & TF\geq 0\vee 2TF+3\geq\Delta \\ \dfrac{\Delta(\Delta-2MF-3)-4(\Delta-2)TF+8}{4\Delta^2}=0 & \text{Otherwise} \end{cases} \tag{42}$$

From the first-order conditions, the equilibrium optimal Take-Make fees for the exchange are in (16).

The second and mixed partial derivatives $\delta_{TF,TF}$, $\delta_{MF,MF}$ and $\delta_{MF,TF}$ are

$$\delta_{TF,TF}, \delta_{MF,MF}, \delta_{MF,TF} = \begin{cases} \left\{\frac{1}{\Delta^2}\left[-\Delta-1+2MF\right],\ \frac{1}{\Delta^2}\left[-\Delta+1+2TF\right],\ \frac{1}{\Delta^2}\left[-\Delta+2MF+2TF\right]\right\} & TF\geq 0\ \vee\ TF\geq\frac{\Delta-3}{2} \\ \left\{\frac{1}{\Delta^2}\left[-\Delta+2\right],\ -\frac{2}{\Delta^2},\ -\frac{\Delta}{2}\right\} & \text{Otherwise} \end{cases} \tag{43}$$

which, together with the equilibrium fees from (16), gives the determinant

$$Det(MF^*,TF^*) = \delta_{MF,MF}(MF^*,TF^*)\times\delta_{TF,TF}(MF^*,TF^*) - (\delta_{MF,TF}(MF^*,TF^*))^2 = \frac{1}{3\Delta^2} > 0 \tag{44}$$

Since the second-order conditions for profit-maximizing fees are satisfied, and the MF and TF in (16) maximize the exchange profit. This completes our analytic construction of the Taker-Maker equilibrium for the $\Delta \leq 3\tau$ case. Table 1 illustrates optimal Take-Make fees for the $\Delta \leq 3\tau$ case. These values were computed by substituting various widths Δ into the formulas for the equilibrium fees MF^* and TF^* in (16).

Social Planner Problem: Taker-Maker $\Delta \in (0,3]$ The Social Planner sets MF and TF to maximize total welfare of market participants, which is the sum of the welfare of investors submitting limit orders at t_1 and market orders at t_2, and expected exchange profits:

$$\begin{aligned} & \max_{\substack{MF,TF \\ 0 \leq MF \leq \tau \\ MF+TF \geq 0}} \sum_{t_z \in \{t_1,t_2\}} \left(W_{t_z}^{INV}(MF,TF) + \pi_{t_z}^{Ex}(MF,TF) \right) \qquad (45) \\ = \quad & W_{t_1}^{INV}(x_{-1,t_1}^{LB} | \Xi, L_{t_0}) + W_{t_1}^{INV}(x_{-2,t_1}^{LB} | \Xi, L_{t_0}) + W_{t_2}^{INV}(x_{-1,t_2}^{MS} | \Xi, L_{t_0}) \\ & + W_{t_2}^{INV}(x_{-2,t_2}^{MS} | \Xi, L_{t_0}) + \left[Pr(x_{-1,t_1}^{LB}, \theta_{t_1}^{x_{-1}^{LB}} | \Xi) + Pr(x_{-2,t_1}^{LB}, \theta_{t_1}^{x_{-2,t_1}^{LB}} | \Xi) \right] (MF + TF) \end{aligned}$$

where the welfare of investors submitting limit buys and market sells, and of exchange profits are defined in (14) and (15), and (22) and (12). We present the welfare of the buy side (the sell side being symmetric):

$$W_{t_1}^{INV}(x_{-1,t_1}^{LB}|\Xi,L_{t_0}) = \int_{\beta_{t_1}\in B_{t_1}(x_{-1,t_1}^{LB},\Xi,L_{t_0})} [\beta_{t_1} - P(x_{-1,t_1}^{LB}) - MF)]\frac{1}{\Delta}d\beta_{t_1} \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|\Xi,L_{t_0}) \tag{46}$$

$$= \begin{cases} \frac{1}{18} & MF = 1 \wedge TF = 0 \wedge \Delta = 3 \\ \frac{1}{18}(MF-2)^2 & 0 \leq MF < 1 \wedge TF = 0 \wedge \Delta = 3 \\ \frac{1}{4\Delta^2}\left[(TF+1)(TF-\Delta+2)(2TF-\Delta+1)\right] & MF = 1 \wedge -1 < TF < 0 \wedge TF \leq \frac{\Delta-3}{2} \wedge \Delta \leq 3 \\ -\frac{1}{4\Delta^2}\left[(MF-TF-2)(2TF-\Delta+1)(MF+TF-\Delta+1)\right] & 0 \leq MF < 1 \wedge -1 < TF < 0 \wedge TF \leq \frac{\Delta-3}{2} \wedge \Delta \leq 3 \\ -\frac{1}{16\Delta^2}\left[(\Delta-1)^2(2TF-\Delta+1)\right] & MF = 1 \wedge \Delta > 1 \wedge ((TF = 0 \wedge \Delta < 3) \vee \left(TF \leq \frac{\Delta-3}{2} \wedge TF < 0\right)\big) \\ -\frac{1}{16\Delta^2}\left[(\Delta-2MF+1)^2(\Delta-2TF-1)\right] & MF < 1 \wedge MF \geq 0 \wedge \Delta > 1 \wedge \left((TF = 0 \wedge \Delta < 3) \vee (TF \leq \frac{\Delta-3}{2} \wedge TF < 0)\right) \end{cases}$$

where the region of integration is $B_{t_1}(x_{-1,t_1}^{LB},\Xi,L_{t_0}) = [\hat{\beta}_{t_1}^{x_{P_{-2}}^{LB},x_{P_{-1}}^{LB}},\overline{\beta}]$, and $\hat{\beta}_{t_1}^{x_{P_{-2}}^{LB},x_{P_{-1}}^{LB}}$ is defined in (34).

$$W_{t_1}^{INV}(x_{-2,t_1}^{LB}|\Xi,L_{t_0}) = \int_{\beta_{t_1}\in B_{t_1}(x_{-2,t_1}^{LB},\Xi,L_{t_0})} [\beta_{t_1} - P(x_{-2,t_1}^{LB}) - MF)]\frac{1}{\Delta}d\beta_{t_1} \times Pr(\theta_{t_1}^{x_{-2,t_1}^{LB}}|\Xi,L_{t_0}) \tag{47}$$

$$= \begin{cases} -\frac{(2TF-\Delta+3)(4-2MF+2TF-\Delta)(2(MF+TF-1)-\Delta)}{16\Delta^2} & (2TF+3 < \Delta \wedge ((0 < MF < \frac{1}{2} \\ & \wedge((MF = TF+\frac{1}{2} \wedge \Delta < 3) \vee \\ & (MF < TF+\frac{1}{2} \wedge TF < 0 \wedge \Delta \leq 3) \vee \\ & (MF > TF+\frac{1}{2} \wedge \\ & TF \geq -1 \wedge \Delta+2MF < 2TF+4))) \vee \\ & (-1 \leq TF < 0 \wedge \frac{1}{2} < MF \leq 1 \wedge \Delta \leq 3))) \vee \\ & (0 \leq MF \leq \frac{1}{2} \wedge TF \geq -1 \wedge \\ & \Delta \leq 3 \wedge \Delta+2MF > 2TF+4) \\ 0 & \text{Otherwise} \end{cases}$$

where the region of integration is $B_{t_1}(x_{-2,t_1}^{LB},\Xi,L_{t_0}) = [v,\hat{\beta}_{t_1}^{x_{P_{-2}}^{LB},x_{P_{-1}}^{LB}}]$, and $\hat{\beta}_{t_1}^{x_{P_{-2}}^{LB},x_{P_{-1}}^{LB}}$ is defined in (34).

$$W_{t_2}^{INV}(x_{-1,t_2}^{MS}|\Xi,L_{t_0}) = Pr(x_{-1,t_1}^{LB}|\Xi,L_{t_0}) \times \int_{\beta_{t_2}\in B_{t_2}(x_{-1,t_2}^{MS},\Xi,L_{t_1})} [P(x_{-1,t_1}^{LB}) - \beta_{t_2} - TF)]\frac{1}{\Delta}d\beta_{t_2} \tag{48}$$

$$= \frac{1}{8\Delta}\left[(\Delta-2TF-1)^2\right] \times \begin{pmatrix} \frac{1}{2} & (2MF<1\wedge(MF<TF+\frac{1}{2}\vee(MF=TF+\frac{1}{2}\wedge\Delta\leq 3)\vee \\ & (MF>TF+\frac{1}{2}\wedge 2MF+\Delta\leq 2TF+4)))\vee \\ & (2MF=1\wedge(TF=0\vee(2TF+3\geq\Delta\wedge -1<TF\leq 0))) \\ \frac{1}{3} & MF=1\wedge TF=0\wedge\Delta=3 \\ \frac{1}{3}\left[2-MF\right] & \frac{1}{2}<MF<1\wedge TF=0\wedge\Delta=3 \\ \frac{1}{\Delta}\left[TF+1\right] & MF=1\wedge -1<TF<0\wedge TF\leq\frac{1}{2}\Delta-\frac{3}{2} \\ \frac{1}{\Delta}\left[TF-MF+2\right] & (-1<TF<0\wedge\frac{1}{2}<MF<1\wedge TF\leq\frac{1}{2}\Delta-\frac{3}{2})\vee \\ & (2MF<1\wedge MF>TF+\frac{1}{2}\wedge TF\leq\frac{1}{2}\Delta-\frac{3}{2}>2TF+4) \\ \frac{1}{2\Delta}\left[2TF+3\right] & 2MF=1\wedge -1<TF<0\wedge TF<\frac{1}{2}\Delta-\frac{3}{2} \\ \frac{1}{2\Delta}\left[\Delta-1\right] & MF=1\wedge((TF=0\wedge\Delta<3)\vee(TF>\frac{1}{2}\Delta-\frac{3}{2}\wedge -1<TF<0)) \\ \frac{1}{2\Delta}\left[\Delta+1-2MF\right] & \frac{1}{2}<MF<1\wedge((TF=0\wedge\Delta<3)\vee(TF>\frac{1}{2}\Delta-\frac{3}{2}\wedge -1<TF<0)) \end{pmatrix}$$

where the region of integration is $B_{t_2}(x_{-1,t_2}^{LB},\Xi,L_{t_0}) = [\underline{\beta},P_{-1}-TF]$.

$$W_{t_2}^{INV}(x_{-2,t_2}^{MS}|\Xi,L_{t_0}) = Pr(x_{-2,t_1}^{LB}|\Xi,\beta_{t_2},L_{t_0}) \times \int_{\beta_{t_2}\in B_{t_2}(x_{-2,t_2}^{MS},\Xi,L_{t_1})} [P(x_{-2,t_1}^{LB}) - \beta_{t_2} - TF)]\frac{1}{\Delta}d\beta_{t_2} \tag{49}$$

$$= \begin{cases} \frac{1}{16\Delta^2}\left[(\Delta-2TF-3)^2(\Delta+2MF-2TF-4)\right] & TF\geq -1\wedge TF<\dfrac{\Delta-3}{2}\wedge\Delta\leq 3\wedge\left(MF>\dfrac{1}{2}\vee\left(MF>TF+\dfrac{1}{2}\wedge\Delta+2MF>2TF+4\right)\right) \\ 0 & \text{Otherwise} \end{cases}$$

where the region of integration is $B_{t_2}(x_{-2,t_2}^{MS},\Xi,L_{t_1}) = [\underline{\beta},P_{-2}-TF]$. Substituting (46), (47), (48) and (49) into the welfare function of the Social Planner, (45), we obtain a functional form whose components are subject to different boundary conditions. The Social Planner problem then simplifies to:

$$\max_{\substack{MF,TF \\ 0\leq MF\leq\tau \\ MF+TF\geq 0}} \sum_{t_z\in\{t_1,t_2\}} \left(W^{INV}_{t_z}(MF,TF) + \pi^{Ex}_{t_z}(MF,TF) \right) \tag{50}$$

$$= \begin{cases} \dfrac{(-2\Delta+2TF+1)(-\Delta+2TF+1)}{16\Delta} & 2MF=1 \wedge 0<\Delta\leq 1 \wedge \\ & \Big((\Delta+2TF+1>0 \wedge TF+1\geq 0 \wedge \\ & (2TF+1<0 \vee \\ & (TF<0 \wedge 2TF+1<\Delta))) \vee \\ & \left(-\frac{1}{2}<TF<0 \wedge 2TF+1<\Delta\right)\Big) \\ \dfrac{\Delta(\Delta-2TF-1)^2-4(MF-TF-2)(-\Delta+2TF+1)(-\Delta+MF+TF+1)}{16\Delta^2} & -1<TF<0 \wedge 1<\Delta\leq 3 \wedge \\ & \Big((2MF=1 \wedge 2TF+3=\Delta) \vee \\ & \big(0\leq MF<\frac{1}{2} \wedge 2TF+3\leq\Delta \wedge \\ & (MF\leq TF+\frac{1}{2} \vee \Delta+2MF\leq 2TF+4)\big)\Big) \end{cases}$$

The fees MF^* and TF^* in Table 2 maximize (50) for values of $0<\Delta\leq 3$ satisfying the given conditions of (50). For example, for $\Delta=2$ the second expression in (50) is maximized by $MF^*=0.5$ and $TF^*=-0.5$. Figure D2 in Section D.2 in the Online Appendix shows plots of the Social Planner's value function for the Taker-Maker case for the different values of the support ($\Delta\in\{\tau,2\tau,2.5\tau,3\tau\}$).

Comment: When $\Delta\in(0,3\tau)$, the logic of the construction of optimal Maker-Taker fees for a profit-maximizing exchange and the Social Planner is similar to the logic for Taker-Maker fees. To conserve space, the details for the Maker-Taker derivation are in Online Appendix D.1.

Table B2: **Difference in expected payoff from different orders.** This table reports the difference in the expected payoffs from different orders indicated in column 1. Column 2 reports such differences as a function of Δ, whereas columns 3 to 6 reports the same differences for different values of Δ.

	Δ	$\Delta=1$	$\Delta=2$	$\Delta=3$	$\Delta=4$
$Pr(\beta_{t_1}-P_2^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_2^{LB}}\mid\Xi,L_{t_0})-Pr(\beta_{t_1}-P_1^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_1^{LB}}\mid\Xi,L_{t_0})$ $Pr(P_1^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_1^{LS}}\mid\Xi,L_{t_0})-Pr(P_2^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_2^{LS}}\mid\Xi,L_{t_0})$	$\frac{4MF-3\Delta+2TF+5}{2\Delta^2}$	$2MF+TF+1$	$\frac{1}{8}(4MF+2TF-1)$	$\frac{1}{9}(2MF+TF-2)$	$\frac{1}{32}(4MF+2TF-7)$
$Pr(\beta_{t_1}-P_2^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_2^{LB}}\mid\Xi,L_{t_0})-Pr(\beta_{t_1}-P_{-1}^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_{-1}^{LB}}\mid\Xi,L_{t_0})$ $Pr(P_{-1}^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_{-1}^{LS}}\mid\Xi,L_{t_0})-Pr(P_2^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_2^{LS}}\mid\Xi,L_{t_0})$	$\frac{2MF-3\Delta+4TF+5}{2\Delta^2}$	$MF+2TF+1$	$\frac{1}{8}(2MF+4TF-1)$	$\frac{1}{9}(MF+2TF-2)$	$\frac{1}{32}(2MF+4TF-7)$
$Pr(\beta_{t_1}-P_2^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_2^{LB}}\mid\Xi,L_{t_0})-Pr(\beta_{t_1}-P_{-2}^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_{-2}^{LB}}\mid\Xi,L_{t_0})$ $Pr(P_{-2}^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_{-2}^{LS}}\mid\Xi,L_{t_0})-Pr(P_2^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_2^{LS}}\mid\Xi,L_{t_0})$	$\frac{-3\Delta+6TF+9}{2\Delta^2}$	$3(TF+1)$	$\frac{3}{8}(2TF+1)$	$\frac{TF}{3}$	$\frac{3}{32}(2TF-1)$
$Pr(\beta_{t_1}-P_1^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_1^{LB}}\mid\Xi,L_{t_0})-Pr(\beta_{t_1}-P_{-1}^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_{-1}^{LB}}\mid\Xi,L_{t_0})$ $Pr(P_{-1}^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_{-1}^{LS}}\mid\Xi,L_{t_0})-Pr(P_1^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_1^{LS}}\mid\Xi,L_{t_0})$	$\frac{TF-MF}{\Delta^2}$	$TF-MF$	$\frac{TF-MF}{4}$	$\frac{TF-MF}{9}$	$\frac{TF-MF}{16}$
$Pr(\beta_{t_1}-P_1^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_1^{LB}}\mid\Xi,L_{t_0})-Pr(\beta_{t_1}-P_{-2}^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_{-2}^{LB}}\mid\Xi,L_{t_0})$ $Pr(P_{-2}^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_{-2}^{LS}}\mid\Xi,L_{t_0})-Pr(P_1^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_1^{LS}}\mid\Xi,L_{t_0})$	$\frac{-2MF+2TF+2}{\Delta^2}$	$-2MF+2TF+2$	$\frac{1}{2}(-MF+TF+1)$	$-\frac{2}{9}(MF-TF-1)$	$\frac{1}{8}(-MF+TF+1)$
$Pr(\beta_{t_1}-P_{-1}^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_{-1}^{LB}}\mid\Xi,L_{t_0})-Pr(\beta_{t_1}-P_{-2}^{cum,LB}(x_{t_1}))\times Pr(\theta_{t_1}^{x_{-2}^{LB}}\mid\Xi,L_{t_0})$ $Pr(P_{-2}^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_{-2}^{LS}}\mid\Xi,L_{t_0})-Pr(P_{-1}^{cum,LS}(x_{t_1})-\beta_{t_1})\times Pr(\theta_{t_1}^{x_{-1}^{LS}}\mid\Xi,L_{t_0})$	$\frac{-MF+TF+2}{\Delta^2}$	$-MF+TF+2$	$\frac{1}{4}(-MF+TF+2)$	$\frac{1}{9}(-MF+TF+2)$	$\frac{1}{16}(-MF+TF+2)$

B.2 Case 2: $3\tau < \Delta \leq 5\tau$

We now consider different ranges of β valuations that are characterized by unique equilibrium strategies. As before we first consider the regime with a maximizing exchange and then a regime with a Social Planner setting optimal fees.

Exchange Maximizing Problem: Positive Fees $\Delta \in (3,5]$ Table 1 show that with the exchange setting optimal fees there are three β ranges characterized by different equilibrium strategies: $\Delta \in (3,4]$, $\Delta \in (4,4.7]$ and $\Delta \in (4.7,5]$. All these β ranges are characterized by strictly positive fees, $\Xi_{PF} = \{0 \leq MF \leq 1, 0 \leq TF \leq 1\}$.

Subcase $\Delta \in (3,4]$: Given $3\tau < \Delta \leq 4\tau$, traders choose among the same orders as in Case 1. Note that Table B2 shows that a limit order to buy at P_2 (sell at P_{-2}), and a limit order to buy at P_{-2} (sell at P_2) are dominated strategies for this subcase. Hence, to determine the optimal MF and TF, we maximize the exchange profits conditional on the buyer choosing LBP_{-1,t_1}, the case of the seller choosing LSP_{1,t_1} arriving at t_1 being symmetric:

$$\max_{\substack{MF,TF \\ MF \leq \tau \\ TF \leq \tau \\ 3<\Delta\leq 4}} \pi^{Ex,LTM}(MF,TF) = \left(Pr(x^{LB}_{-1,t_1}|\Xi, L_{t_0}) \times Pr(\theta_{t_1}^{x^{LB}_{-1}}|\Xi, L_{t_0})\right) \times (MF + TF)$$

$$= -\frac{(\Delta-1)(\mathrm{MF}+\mathrm{TF})\left(-\frac{\Delta}{2}+\mathrm{TF}+\frac{1}{2}\right)}{2\Delta^2} \tag{51}$$

The Kuhn-Tucker Lagrangian is:

$$L(MF,TF,\lambda_k,v_h) = \pi^{Ex,LTM}(MF,TF) - \lambda_1(-MF+1) - \lambda_2(-TF + \frac{\Delta-3}{2}) \tag{52}$$

The Kuhn-Tucker conditions are:

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF} = \lambda_1 + \frac{(\Delta-1)(\Delta-2TF-1)}{4\Delta^2} \geq 0 \;\&\; MF \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF} = 0 \tag{53}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF} = \frac{\left(\frac{1}{2}-\frac{\Delta}{2}\right)MF + (1-\Delta)TF + \Delta\left(\Delta\left(\lambda_2+\frac{1}{4}\right)-\frac{1}{2}\right)+\frac{1}{4}}{\Delta^2} \geq 0 \;\& TF \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF} = 0 \tag{54}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1} = (MF-1) \geq 0 \;\&\; \lambda_1 \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1} = 0 \tag{55}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_2} = (TF - \frac{\Delta-3}{2}) \geq 0 \;\&\; \lambda_2 \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_2} = 0 \tag{56}$$

The equilibrium MF^* and TF^* that satisfy these conditions are given in the first line of (18): By substituting a given value of Δ into MF^* and TF^* in the first line (18), we obtain the equilibrium fees in Table 1.

Table B3: Equilibrium Submission Probability This table reports the equilibrium submission probabilities for the buy side, $Pr(x^{LB}_{k,t_1}|\Xi, L_{t_0})$, conditional on the support Δ. Equilibrium submission probabilities for the sell side, $Pr(x^{LS}_{-k,t_1}|\Xi, L_{t_0})$ are symmetric.

	$0<\Delta\leq 4\tau$		$4<\Delta\leq 4.7\tau$	$4.7<\Delta\leq 5\tau$
	Taker-Maker	Maker-Taker	Positive Fees	Positive Fees
$Pr(x^{LB}_{1,t_1}\|\Xi, L_{t_0})$		$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2}-\frac{1}{2}-MF]\right\}$		$\max\left\{0, \frac{1}{\Delta}[TF-MF]\right\}$ for $\beta > \frac{\Delta}{2}+9.5$
$Pr(x^{LB}_{-1,t_1}\|\Xi, L_{t_0})$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2}+\frac{1}{2}-MF]\right\}$		$\max\left\{0, \frac{1}{\Delta}[TF+1]\right\}$ for $\beta > MF+\frac{\Delta}{2}-TF+8$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2}+\frac{1}{2}-TF]\right\}$ for $MF+9.5<\beta<MF+\frac{\Delta}{2}+9$
$Pr(x^{LB}_{-2,t_1}\|\Xi, L_{t_0})$			$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2}+MF-TF-2]\right\}$ for $10<\beta<MF+\frac{\Delta}{2}-TF+8$	

Subcase $\Delta\in(4,4.7]$: We have shown that for investor valuation supports with widths up $\Delta = 4$, there are dominant orders for potential buyers and sellers, and so the optimal order-submission strategy can be obtained by comparing the expected payoff associated with each possible order, as shown in Tables B1 and B2; in the latter we present the differences in expected payoffs conditional on different supports. However, for investor valuation supports with widths $\Delta > 4$, there are two possible equilibrium limit orders, which we report in Table B3 showing that both a limit buy order at P_{-1} and a limit buy at P_{-2} may be optimal depending on the investors' evaluation, β_{t_1}. We also report conditions on the value of β such that the equilibrium strategies hold. To determine the optimal MF and TF, the exchange maximizes its expected profits conditional on the buyer choosing either LBP_{-2,t_1}, or LBP_{-1,t_1} the case of the seller arriving at t_1 being symmetric:

$$\begin{aligned}
&\max_{\substack{MF,TF \\ MF\leq\tau \\ TF\leq\tau \\ MF+TF\geq 0 \\ 4<\Delta\leq 4.7}} \pi^{Ex,LTM}(MF,TF) \qquad (57)\\
&= \left[Pr(x^{LB}_{-1,t_1}|\Xi,L_{t_0}) \times Pr(\theta_{t_1}^{x^{LB}_{-1}}|\Xi,L_{t_0}) + Pr(x^{LB}_{-2,t_1}|\Xi,L_{t_0}) \times Pr(\theta_{t_1}^{x^{LB}_{-2}}|\Xi,L_{t_0})\right] \times (MF+TF)\\
&= \frac{(\mathrm{MF}+\mathrm{TF})\left(\left(\frac{3}{4}-\frac{\Delta}{4}\right)\Delta+\mathrm{MF}+\left(\frac{\Delta}{2}-1\right)\mathrm{TF}-2\right)}{\Delta^2}
\end{aligned}$$

The Kuhn-Tucker Lagrangian is:

$$L(MF,TF,\lambda_k,v_h) = \pi^{Ex,LTM}(MF,TF) - \lambda_1(-MF+1) - \lambda_2(-TF+\frac{4-\Delta}{2})$$

The Kuhn-Tucker conditions are:

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF} = \frac{8MFTF-4MF(1+\Delta)+\Delta(-1-2TF+\Delta-4\Delta\lambda_1)}{4\Delta^2} \geq 0 \text{ \& } MF \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF} = 0 \tag{58}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF} = \frac{4MF^2+4(1-3TF)TF-2MF\Delta+\Delta(-1+\Delta-4\Delta\lambda_2)}{4\Delta^2} \geq 0 \text{ \& } TF \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF} = 0 \tag{59}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1} = (-MF+1) \geq 0 \text{ \& } \lambda_1 \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1} = 0 \tag{60}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_2} = (-TF+\frac{4-\Delta}{2}) \geq 0 \text{ \& } \lambda_2 \times \frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_2} = 0 \tag{61}$$

The equilibrium MF^* and TF^* that satisfy these conditions are in the second line of (18): By substituting a given value of Δ into MF^* and TF^* in the second line of (18), we obtain the equilibrium fees in Table 1.

Subcase $\Delta \in (4.7, 5]$: In this case, the investor valuation support width can be as large as 5τ, which is the difference between P_3 and P_{-3}. So we also consider the investor's profit conditional on orders posted at P_3 and P_{-3}. Table B1 shows that the investor's profit is zero if he buys at P_3 or sells at P_{-3}. Table B3 shows that for this interval of the support the equilibrium strategies are either $x^{LB}_{1,t_1} = LBP_{1,t_1}$, or $x^{LB}_{-1,t_1} = LBP_{-1,t_1}$. Therefore, to determine the optimal MF and TF, we maximize the exchange profits conditional on the buyer optimally using these two strategies, the case of the seller arriving at t_1 being symmetric:

$$\max_{\substack{MF,TF \\ MF\leq\tau \\ TF\leq\tau \\ 4.7<\Delta\leq 5}} \pi^{Ex,LTM}(MF,TF) \tag{62}$$

$$= \left[Pr(x^{LB}_{1,t_1}|\Xi,L_{t_0})\times Pr(\theta^{x^{LB}_1}_{t_1}|\Xi,L_{t_0})+Pr(x^{LB}_{-1,t_1}|\Xi,L_{t_0})\times Pr(\theta^{x^{LB}_{-1}}_{t_1}|\Xi,L_{t_0})\right]\times(MF+TF)$$

$$= \frac{(MF+TF)\left((\Delta-1)\Delta-2(\Delta+1)MF+4MFTF-4TF^2+2TF\right)}{4\Delta^2}$$

The Kuhn-Tucker Lagrangian is:

$$L(MF,TF,\lambda_k,v_h)=\pi^{Ex,LTM}(MF,TF)-\lambda_1(-MF+1)$$

The Kuhn-Tucker conditions are:

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF}=\frac{8MFTF-4MF(1+\Delta)+\Delta(-1-2TF+\Delta+4\Delta\lambda_1)}{4\Delta^2}\geq 0\ \&\ MF\times\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial MF}=0 \tag{63}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF}=\frac{4MF^2+4(1-3TF)TF-2MF\Delta+(1-\Delta)\Delta}{4\Delta^2}\geq 0\ \&\ TF\times\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial TF}=0 \tag{64}$$

$$\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1}=(-MF+1)\geq 0\ \&\ \lambda_1\times\frac{\partial L(MF,TF,\lambda_k,v_h)}{\partial \lambda_1}=0 \tag{65}$$

The equilibrium MF^* and TF^* that satisfy these conditions are given in the third line of (18) and in Table 1. Q.E.D.

Social Planner Problem: TM and MT $\Delta\in(3,5]$ Table 2 shows that with the Social Planner setting optimal fees there is a unique β range characterized by both TM and MT pricing.

Subcase $\Delta\in(3,5]$: Under the Taker-Maker regime, to determine the optimal MF and TF, the Social Planner maximizes total welfare from both limit buy orders and market sell orders, and exchange profit as defined in (13), (14) and (15), as well as (22) and (12). We present the welfare of the buy side of the market (the sell side being symmetric):

$$\max_{\substack{MF,TF \\ TF\le\tau \\ MF\le\tau \\ MF+TF\ge 0 \\ 3<\Delta\le 5}} \sum_{t_z\in\{t_1,t_2\}} \left(W_{t_z}^{INV}(MF,TF) + \pi_{t_z}^{Ex}(MF,TF) \right) \tag{66}$$

$$= W_{t_1}^{INV}(x_{-1,t_1}^{LB}|\Xi,L_{t_0}) + W_{t_2}^{INV}(x_{-2,t_2}^{MS}|\Xi,L_{t_0}) + W_{t_2}^{INV}(x_{-1,t_2}^{MS}|\Xi,L_{t_0})$$

$$+W_{t_2}^{INV}(x_{-2,t_2}^{MS}|\Xi,L_{t_0}) + \left[Pr(x_{-1,t_1}^{LB},\theta_{t_1}^{x_{-1}^{LB}}|\Xi) + Pr(x_{-2,t_1}^{LB},\theta_{t_1}^{x_{-2,t_1}^{LB}}|\Xi)\right](MF+TF)$$

where the welfare from a limit buy at P_{-1} and from a limit buy at P_{-2} with $3 < \Delta \le 5$ are respectively:

$$W_{t_1}^{INV}(x_{-1,t_1}^{LB}|\Xi,L_{t_0}) = \int_{\beta_{t_1}\in B_{t_1}(x_{-1,t_1}^{LB},\Xi,L_{t_0})} [\beta_{t_1} - P(x_{-1,t_1}^{LB}) - MF)]\frac{1}{\Delta}d\beta_{t_1} \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|\Xi,L_{t_0}) \tag{67}$$

$$= \begin{cases} \frac{(TF+1)(-\Delta+TF+2)(-\Delta+2TF+1)}{4\Delta^2} & MF=1\wedge -1<TF\le 0\wedge 3<\Delta\le 5 \\ -\frac{(MF-TF-2)(-\Delta+2TF+1)(-\Delta+MF+TF+1)}{4\Delta^2} & -1\le TF\le 0\wedge 3<\Delta\le 5\wedge 0\le MF<1 \end{cases}$$

$$W_{t_2}^{INV}(x_{-2,t_2}^{MS}|\Xi,L_{t_0}) = Pr(x_{-2,t_1}^{LB}|\Xi,L_{t_0}) \times \int_{\beta_{t_2}\in B_{t_2}(x_{-2,t_2}^{MS},\Xi,L_{t_1})} [P(x_{-2,t_1}^{LB}) - \beta_{t_2} - TF)]\tfrac{1}{\Delta}d\beta_{t_2} \tag{68}$$

$$= \begin{cases} -\frac{(-\Delta+2TF+3)(-\Delta-2MF+2TF+4)(2(MF+TF-1)-\Delta)}{16\Delta^2} & \begin{array}{l}(\Delta\le 5\wedge\Delta>3\wedge TF+1\ge 0\wedge \\ ((2MF>1\wedge MF\le 1\wedge TF\le 0)\vee(MF\ge 0\wedge \\ MF\ge TF+\frac{1}{2}\wedge 2MF\le 1)))\vee \\ (MF\ge 0\wedge TF\le 0\wedge \\ ((\Delta+2MF>2TF+4\wedge MF<TF+\frac{1}{2}\wedge \\ \Delta\le 5)\vee(\Delta>3\wedge\Delta+2MF<2TF+4)))\end{array} \\ 0 & \text{Otherwise} \end{cases}$$

Whereas the welfare from a market sell at P_{-1} and from a market sell at P_{-2} when $3 < \Delta \le 5$ are respectively:

$$W_{t_2}^{INV}(x_{-1,t_2}^{MS}|\Xi,L_{t_0}) = Pr(x_{-1,t_1}^{LB}|\Xi,L_{t_0}) \times \int_{\beta_{t_2}\in B_{t_2}(x_{-1,t_2}^{MS},\Xi,L_{t_1})} [P(x_{-1,t_1}^{LB}) - \beta_{t_2} - TF)]\frac{1}{\Delta}d\beta_{t_2} \tag{69}$$

$$= \frac{(\Delta - 2TF - 1)^2}{8\Delta} \left(\begin{array}{cc} \frac{1}{2} & 2MF + \Delta \leq 2TF + 4 \wedge MF < TF + \frac{1}{2} \wedge 2MF \leq 1 \\ \frac{TF+1}{\Delta} & MF = 1 \wedge TF > -1 \\ \frac{-MF+TF+2}{\Delta} & MF < 1 \wedge \left(2MF > 1 \vee MF \geq TF + \frac{1}{2} \vee 2MF + \Delta > 2TF + 4\right) \end{array} \right)$$

$$W_{t_2}^{INV}(x_{-2,t_2}^{MS}|\Xi, L_{t_0}) = Pr(x_{-2,t_1}^{LB}|\Xi, L_{t_0}) \times \int_{\beta_{t_2} \in B_{t_2}(x_{-2,t_2}^{MS}, \Xi, L_{t_1})} [P(x_{-2,t_1}^{LB}) - \beta_{t_2} - TF)] \frac{1}{\Delta} d\beta_{t_2} \tag{70}$$
$$= \begin{cases} \frac{(\Delta - 2TF - 3)^2(\Delta + 2MF - 2TF - 4)}{16\Delta^2} & MF \geq TF + \frac{1}{2} \vee 2MF > 1 \vee \Delta + 2MF > 2TF + 4 \\ 0 & \text{Otherwise} \end{cases}$$

By substituting (67), (68), (69) and (70) into the welfare function of the Social Planner, (66), we obtain a functional form whose components are subject to different boundary conditions. The following component has the highest total welfare:

$$\max_{\substack{MF,TF \\ TF \leq \tau \\ MF \leq \tau \\ MF+TF \geq 0 \\ 3 < \Delta \leq 5}} \sum_{t_z \in \{t_1, t_2\}} \left(W_{t_z}^{INV}(MF, TF) + \pi_{t_z}^{Ex}(MF, TF) \right) \tag{71}$$
$$= \frac{2\Delta^3 - \Delta^2(4MF + 6TF + 3) + \Delta(MF(8TF + 4) + 4TF(TF + 2) + 7) + 8(MF - TF - 2)(MF + TF)}{16\Delta^2}$$

The optimal fees presented in Table 2 are determined by the boundary conditions of the different parts of the total welfare functional form. By substituting any $3 < \Delta \leq 5$ and the optimal MF^* and TF^* in (72) we obtain the total welfare presented in Table 2. Figure D3 in Online Appendix D shows plots of the Social Planner's value function for the Taker-Maker case for the different support values ($\Delta \in \{3.5\tau, 4\tau, 4.5, \tau, 5\tau\}$) in Table 2.

Subcase $\boldsymbol{\Delta \in (3,5]}$: Under the Maker-Taker regime, the Social Planner maximizes total welfare from limit buy orders and market sell orders at P_{-1}, as well as limit buy order and market sell orders at P_1. As before, we present the welfare of the buy side of the market - the sell side being symmetric:

$$
\begin{aligned}
&\max_{\substack{MF,TF\\ MF\leq\tau\\ TF\leq\tau\\ MF+TF\geq 0\\ 3<\Delta\leq 5}} \sum_{t_z\in\{t_1,t_2\}} \left(W^{INV}_{t_z}(MF,TF)+\pi^{Ex}_{t_z}(MF,TF)\right) \qquad (72)\\
= \;& W^{INV}_{t_1}(x^{LB}_{-1,t_1}|\Xi,L_{t_0})+W^{INV}_{t_1}(x^{LB}_{1,t_1}|\Xi,L_{t_0})+W^{INV}_{t_2}(x^{MS}_{-1,t_2}|\Xi,L_{t_0})+W^{INV}_{t_2}(x^{MS}_{1,t_2}|\Xi,L_{t_0})\\
&+\left[Pr(x^{LB}_{-1,t_1},\theta_{t_1}^{x^{LB}_{-1}}|\Xi)+Pr(x^{LB}_{1,t_1},\theta_{t_1}^{x^{LB}_{1}}|\Xi)\right](MF+TF)
\end{aligned}
$$

Where the welfare from a limit buy at P_{-1} and from a limit buy at P_1 with $3<\Delta\leq 5$ are respectively:

$$
W^{INV}_{t_1}(x^{LB}_{-1,t_1}|\Xi,L_{t_0})=\int_{\beta_{t_1}\in B_{t_1}(x^{LB}_{-1,t_1},\Xi,L_{t_0})}[\beta_{t_1}-P(x^{LB}_{-1,t_1})-MF)]\frac{1}{\Delta}d\beta_{t_1}\times Pr(\theta_{t_1}^{x^{LB}_{-1}}|\Xi,L_{t_0}) \qquad (73)
$$

$$
=\begin{cases}
-\frac{(-\Delta+2TF+1)(-\Delta-2MF+2TF)(2(MF+TF-1)-\Delta)}{16\Delta^2} & (MF+\frac{3}{2}<TF\wedge MF\geq -1\wedge TF\leq 1\wedge((\Delta>3\wedge\Delta+2MF<2TF)\vee\\
& (\Delta+2MF>2TF\wedge\Delta\leq 5)))\vee(3<\Delta\leq 5\wedge((MF+\frac{3}{2}\geq TF\wedge\\
& TF\geq 0\wedge -1\leq MF\leq -\frac{1}{2})\vee(-\frac{1}{2}<MF\leq 0\wedge 0\leq TF\leq 1)))\\
0 & \text{Otherwise}
\end{cases}
$$

$$
W^{INV}_{t_1}(x^{LB}_{1,t_1}|\Xi,L_{t_0})=\int_{\beta_{t_1}\in B_{t_1}(x^{LB}_{1,t_1},\Xi,L_{t_0})}[\beta_{t_1}-P(x^{LB}_{1,t_1})-MF)]\frac{1}{\Delta}d\beta_{t_1}\times Pr(\theta_{t_1}^{x^{LB}_{1}}|\Xi,L_{t_0}) \qquad (74)
$$

$$
=\begin{cases}
-\frac{(MF-TF)(-\Delta+2TF-1)(-\Delta+MF+TF+1)}{4\Delta^2} & 0\leq TF\leq 1\wedge 3<\Delta\leq 5\wedge -1\leq MF<0\\
\frac{TF\left(\Delta^2+2TF^2-3\Delta TF+TF-1\right)}{4\Delta^2} & MF=0\wedge 0<TF\leq 1\wedge 3<\Delta\leq 5
\end{cases}
$$

Whereas the welfare from a market sell order at P_{-1} and from a market sell order at P_1 when $3<\Delta\leq 5$ are respectively:

$$
W^{INV}_{t_2}(x^{MS}_{-1,t_2}|\Xi,L_{t_0})=Pr(x^{LB}_{-1,t_1}|\Xi,L_{t_0})\times\int_{\beta_{t_2}\in B_{t_2}(x^{MS}_{-1,t_2},\Xi,L_{t_1})}[P(x^{LB}_{-1,t_1})-\beta_{t_2}-TF)]\frac{1}{\Delta}d\beta_{t_2} \qquad (75)
$$

$$
=\begin{cases}
\frac{(\Delta-2TF-1)^2(\Delta+2MF-2TF)}{16\Delta^2} & MF+\frac{3}{2}\geq TF\vee MF>-\frac{1}{2}\vee\Delta+2MF>2TF\\
0 & \text{Otherwise}
\end{cases}
$$

$$W^{INV}_{t_2}(x^{MS}_{1,t_2}|\Xi,L_{t_0}) = Pr(x^{LB}_{1,t_1}|\Xi,L_{t_0}) \times \int_{\beta_{t_2}\in B_{t_2}(x^{MS}_{1,t_2},\Xi,L_{t_1})} [P(x^{LB}_{1,t_1}) - \beta_{t_2} - TF)]\frac{1}{\Delta}d\beta_{t_2} \tag{76}$$

$$= \frac{(\Delta-2TF+1)^2}{8\Delta}\left(\begin{array}{cc} \frac{1}{2} & 2MF+\Delta \leq 2TF \wedge MF+\frac{3}{2} < TF \wedge MF \leq -\frac{1}{2} \\ \frac{TF}{\Delta} & MF=0 \wedge TF>0 \\ \frac{TF-MF}{\Delta} & MF<0 \wedge \left(2MF+1>0 \vee MF+\frac{3}{2} \geq TF \vee 2MF+\Delta > 2TF\right) \end{array}\right)$$

By substituting (73), (74), (75) and (76) into the welfare function of the Social Planner, (72), we obtain a functional form whose components are subject to different boundary conditions. The following component has the highest total welfare:

$$\max_{\substack{MF,TF \\ MF\leq\tau \\ TF\leq\tau \\ MF+TF\geq 0 \\ 3<\Delta\leq 5}} \sum_{t_z\in\{t_1,t_2\}} \left(W^{INV}_{t_z}(MF,TF) + \pi^{Ex}_{t_z}(MF,TF)\right)$$

$$= \frac{(\Delta-1)\Delta(2\Delta+1) + 8MF^2 - 4\Delta MF(\Delta-2TF+1) + 4(\Delta-2)TF^2 + 2(4-3\Delta)\Delta TF}{16\Delta^2} \tag{77}$$

The optimal fees presented in Table 2 are determined by the boundary conditions of the different parts of the total welfare functional form. By substituting any $3 < \Delta \leq 5$ and the optimal MF^* and TF^* in (77) we obtain the total welfare presented in Table 2. Figure D3 in Section **??** in the Online Appendix shows plots of the Social Planner's value function for the Maker-Taker case for the different values of the support ($\Delta \in \{3.5\tau, 4\tau, 4.5, \tau, 5\tau\}$) in Table 2.

B.3 Two-period market equilibrium with an arbitrary absolute tick size

Proof of Theorem 2: The proof follows from rescaling all of the variables in the model relative to an absolute tick size $\tau > 0$. In particular, we define scaled quantities $\hat{MF} = MF/\tau$, $\hat{TF} = TF/\tau$, $\hat{\beta}_{t_z} = \beta_{t_z}/\tau$, $\hat{\bar{\beta}} = \bar{\beta}/\tau$, $\underline{\hat{\beta}} = \underline{\beta}/\tau$, and prices $\hat{P}_j = P_j/\tau$ for all h. Given this rescaling, we next observe that all of the order-submission and order-execution probabilities are homogeneous of degree zero in the absolute tick size τ. In particular, the absolute tick size factors out of both the numerator and denominator and cancels. Similarly, the conditional payoffs on executed orders for the exchange and investors is homogeneous of order one. Thus, changing the tick size does not change relative comparisons for prices centered around a given fundamental valuation v. The rescaled optimization problems for the investors and exhange are, therefore, equivalent to the optimization problem with a tick of 1. This gives us the solutions to the exchange's optimal scaled fees. Multiplying the scaled fees by the absolute tick size τ gives the corresponding absolute fees $MF = \hat{MF}\tau$ and $MF = \hat{MF}\tau$. Q.E.D.

Equilibrium in trading subgame in two-period model: This rest of this section derives a property of the β thresholds that drives the solution of the 2-period models for the equilibrium order execution and order submission probabilities. We show that the distance between two consecutive thresholds is a linear function of the tick size and that it is equal to 2 ticks. This result has an important implication as it implies that by changing the tick size the distance between two adjacent thresholds changes proportionally and so do both the equilibrium probabilities of executions, and the equilibrium probabilities of submission.

Suppose that two limit buy orders posted at P_k and P_j, with $j < k$, are such that $P_k^{MS,cum} \in (\underline{\beta}, \overline{\beta})$ and $P_j^{MS,cum} \in (\underline{\beta}, \overline{\beta})$, so that there is a positive well-defined probability of an investor arriving at t_2 who would be willing to MS against these LBs. The expected profit on a limit buy at P_k at t_1 given $\beta_{t_1} > P_k$ is

$$\pi_{t_1}^{INV}(LBP_k) = Pr(\beta_{t_2} < P_k^{MS,cum})(\beta_{t_1} - P_k^{LB,cum}) = \frac{P_k^{MS,cum} - \underline{\beta}}{\overline{\beta} - \underline{\beta}}(\beta_{t_1} - P_k^{LB,cum}). \tag{78}$$

Similarly, the expected profit on a limit buy at P_j at t_1 given $\beta_{t_1} > P_k$ (and, thus, $\beta_{t_1} > P_j$) is

$$\pi_{t_1}^{INV}(LBP_j) = Pr(\beta_{t_2} < P_j^{MS,cum})(\beta_{t_1} - P_j^{LB,cum}) = \frac{P_j^{MS,cum} - \underline{\beta}}{\overline{\beta} - \underline{\beta}}(\beta_{t_1} - P_j^{LB,cum}). \tag{79}$$

Notation: Define $n_{j,k} = (P_k - \underline{\beta}) - (P_j - \underline{\beta}) > 0$, which is the integer number of ticks between P_j and P_k.

Consider the Taker-Maker market with a positive make fee $MF > 0$ and a negative take fee $TF < 0$.[32] Using the above notation, we have $P_k^{MS,cum} = P_k - TF = P_j + n_{j,k} - TF$ and $P_k^{LB,cum} = P_k + MF = P_j + n_{j,k} + MF$. In addition, we can write $P_k^{LB,cum} = P_j^{LB,cum} + n_{j,k}$ and $P_k^{MS,cum} = P_j^{MS,cum} + n_{j,k}$. Thus, we have

$$\begin{aligned} \pi_{t_1}^{INV}(LBP_k) &\geq \pi_{t_1}^{INV}(LBP_j) \\ \rightarrow \frac{P_k^{MS,cum} - \underline{\beta}}{\overline{\beta} - \underline{\beta}}(\beta_{t_1} - P_k^{LB,cum}) &\geq \frac{P_j^{MS,cum} - \underline{\beta}}{\overline{\beta} - \underline{\beta}}(\beta_{t_1} - P_j^{LB,cum}) \\ \rightarrow \beta_{t_1} &\geq P_j + P_k - \underline{\beta} - TF + MF \end{aligned} \tag{80}$$

so that the break-point $\beta_{t_1}^{j,k}$ where profits are equated for a LB at a fixed price P_j vs. LBs at higher prices $P_k > P_j$ is

$$\beta_{t_1}^{j,k} = P_j + P_k - \underline{\beta} - TF + MF \tag{81}$$

where $P_j + P_k - \underline{\beta} - TF + MF > 0$ in the Taker-Maker regime (where $-TF + MF > 0$) and $P_j - \underline{\beta} > 0$.

This has the following useful implication about the thresholds for a fixed limit price P_j and higher limit prices P_k: Consider two limit prices $P_k \in (P_j, \overline{\beta})$ and $P_{k'} \in (P_k, \overline{\beta}]$. The difference between the two respective breakpoints is $\beta_{t_1}^{j,k'} - \beta_{t_1}^{j,k} = k' - k$. Thus, the breakpoints for a given price P_j and higher limit prices are ordered in that $\beta_{t_1}^{j,k} < \beta_{t_1}^{j,k'}$ given $k < k'$, and, moreover, that breakpoints for consecutive higher prices with $(P_k' - \underline{\beta}) = (P_k - \underline{\beta}) + 1$ are equally spaced exactly one tick apart. Similarly, the breakpoints for a given price P_k and lower prices are ordered in that $\beta_{t_1}^{j',k} < \beta_{t_1}^{j',k}$ given $j' < j$, and, moreover, that breakpoints for consecutive lower prices with $(P_j' - \underline{\beta}) = (P_j - \underline{\beta}) - 1$ are equally spaced exactly one tick apart.

Another implication here is that if $k > j > 0$ and $k' > j' > 0$ where $j + k = j' + k'$, then $\beta_{t_1}^{j,k} = \beta_{t_1}^{j',k'}$. The same is true for $0 > k > j$ and $0 > k' > j'$ where $j + k = j' + k'$. The corresponding result when $k > 0 > j$

[32] Maker-Taker and positive fee pricing each lead to analogous results.

or $k' > 0 > j'$ is slightly different since there is no P_0.

Construction of upper envelope: To get started, we initially ignore the possibility of limit sells. That will be addressed below. Start at lowest ex ante feasible limit price on price grid. Call this price P_ℓ. No one at t_1 would be willing to LB at the next highest price $P_{\ell+1}$ until their β_{t_1} is at least above $P_{\ell+1}^{LB,cum}$. Thus, there is an interval β_{t_1} values above $P_{\ell}^{LB,cum}$ for which LBP_ℓ dominates LBs at any higher LB price. The interval for which LBP_ℓ dominates other LBs extends until the break-point $\beta_{t_1}^{\ell,\ell+1}$ at which point a LB at $P_{\ell+1}$ becomes optimal. In particular, we know that $P_{\ell+1}$ is the next optimal limit price rather than some higher price P_k because, holding price P_ℓ fixed, we have from above that $\beta_{t_1}^{\ell,\ell+1} < \beta_{t_1}^{\ell,k}$ for $k > \ell+1$. The interval for which $LBP_{\ell+1}$ is optimal extends to $\beta_{t_1}^{\ell+1,\ell+2}$ and so on.

Next, define ℓ^* as the index of the lowest price such that the interval $[\beta_{t_1}^{\ell^*-1,\ell^*}, \beta_{t_1}^{\ell^*,\ell^*+1}]$ includes the mid-valuation v. Since only potential buyers with $\beta_{t_1} > v$ submit limit buys, we have the following upper envelope for optimal limit buy orders when limit sell orders are also possible: The limit buy LBP_{ℓ^*} is used for $\beta_{t_1} \in [v, \beta_{t_1}^{\ell^*,\ell^*+1}]$, the limit buy LBP_{ℓ^*+1} is used for $\beta_{t_1} \in [\beta_{t_1}^{\ell^*,\ell^*+1}, \beta_{t_1}^{\ell^*+1,\ell^*+2}]$, the limit buy LBP_{ℓ^*+2} is used for $\beta_{t_1} \in [\beta_{t_1}^{\ell^*+1,\ell^*+2}, \beta_{t_1}^{\ell^*+2,\ell^*+3}]$, and so on. This continues up to the price P_m, which is defined as the lowest price such that $\beta_{t_1}^{m,m+1} > \overline{\beta}$. It follows, then, that investors at t_1 with $\beta_{t_1} \in [\beta_{t_1}^{m-1,m}, \overline{\beta}]$ optimally submit LBP_m limit orders.. Thus, the upper envelope continues with intervals up to $\ldots, [\beta_{t_1}^{m-2,m-1}, \beta_{t_1}^{m-1,m}], [\beta_{t_1}^{m-1,m}, \overline{\beta}]$. Putting everything together, the limit buys LBP_{ℓ^*}, $LB_{\ell^*+1}, \ldots LBP_m$ are optimal in the associated intervals $[v, \beta_{t_1}^{\ell^*,\ell^*+1}]$, $[\beta_{t_1}^{\ell^*,\ell^*+1}, \beta_{t_1}^{\ell^*+1,\ell^*+2}], \ldots [\beta_{t_1}^{m-1,m}, \overline{\beta}]$

Order-submission probabilities: Given our assumption that β is uniformly distributed, the order-submission probabilities for optimal LBs at t_1 are just the width of the interval for which they are optimal divided by the length Δ of the full valuation support.

Consider first an arbitrary "interior" interval $[\beta_{t_1}^{k-1,k}, \beta_{t_1}^{k,k+1}]$ for $1 < k < m$. Note here that the critical values, from above, are given by

$$\begin{aligned}\beta_{t_1}^{k-1,k} &= P_{k-1}+P_k-\underline{\beta}-TF+MF \\ \beta_{t_1}^{k,k+1} &= P_k+P_{k+1}-\underline{\beta}-TF+MF\end{aligned} \tag{82}$$

Thus, the width of any "interior" interval is

$$\beta_{t_1}^{k,k+1}-\beta_{t_1}^{k-1,k}=P_k+P_{k+1}-\underline{\beta}-TF+MF-(P_{k-1}+P_k-\underline{\beta}-TF+MF)=2 \tag{83}$$

The construction for the upper envelope of limit sells is symmetric.

C Exchange Competition in Access Pricing

Proof of Theorem 3: Part (i) The monopolistic equilibrium in (i) can be implemented by the exchange A setting its access pricing to be the monopolistic access pricing schedule and threatening to switch to the MF-undercutting/TF-matching schedule if exchange B competes at all via any positive-profit schedule that would attract orders given the two-sided nature of the market. If A has access to a commitment mechanism that makes this threat credible, then, in equilibrium, exchange B is indifferent between competing (in which case it earn profits of 0) and not competing (in which case exchange A earns monopolistic profits and exchange B again earns profits of 0) and, thus, is willing to not-compete.

Part (ii) The equilibrium in (ii) can be implemented by the exchange A setting its pricing to be the monopolistic pricing schedule and threatening to switch to the MF-undercutting/TF-matching schedule if exchange B competes using a positive-profit pricing schedule that differs from the monopolistic schedule. If A has access to a commitment mechanism that makes this threat credible, then, in equilibrium, exchange B strictly prefers matching exchange A's monopolistic pricing schedule since this lets exchange B earn positive expected profits if investors at t_1 randomize with positive probability on submitting limit orders to exchanges A and B, whereas any other pricing schedule results in zero-profits for exchange B. Q.E.D.

Proof of Theorem 4 : Part (i) The discriminating equilibrium can be implemented in part (i) by the same logic as for the equilibrium in part (i) of Theorem 3. Exchange operator A proposes the two discriminating price schedules for its two exchanges, $A1$ and $A2$ and then makes a grim trigger contingent undercutting strategy threat if operator B proposes any positive-profit pricing schedule that would attract trading volume.

Part (ii) The discriminating equilibrium can also be implemented in part (ii) by exchange operator A proposing to use one of the two discriminating pricing schedules in its exchange and making a grim trigger contingent threat to undercut operator B unless operator B uses the other discriminating price schedule in its exchange. Q.E.D.

Proof of Theorem 5 A Stackelberg equilibrium exists because this is a sequential finite game of full information with respect to the two exchanges. Thus, for each proposed price schedule for exchange A, there is an optimal response for exchange B. Thus, exchange A, as the Stackelbert leader, proposes using the pricing schedule that maximizes exchange A's expected profit conditional on exchange B's optimal response as the Stackelberg follower. Q.E.D.

Online Appendix

D 2-period trading game

This section examines the 2-period $\{t_1, t_2\}$ version of the general model. The model is solved by backwards induction in two steps. We first take market access pricing Ξ set by the Stackelberg leader as given and solve for the optimal investor trading strategies in the trading subgame. Given this characterization of optimal investor trading - the optimal response of the follower - we then solve for the optimal access pricing given the exchange's profit-maximization problem or given the Social Planner's total welfare-maximization problem.

D.1 Construction of optimal Maker-Taker fees

This section provides details for the construction of optimal Maker-Taker access pricing in the two-period LTM. The logic is similar to the construction of optimal Taker-Maker fees in Appendix B.

Maker-Taker: Now consider Maker-Taker pricing, $\Xi_{MT} = \{-1 \leq MF \leq 0, 0 \leq TF \leq 1\}$, with a make rebate and a positive take fee, as illustrated in Figure B2. Once again, we determine the optimal strategies for arriving investors at times t_1 and t_2 and the associated order-submission probabilities:

- First, given a positive take fee TF and an investor valuation support width $\Delta \leq 3$, the expected profit on limit buys at P_{-2} and limit sells at P_2 at t_1 is zero as there will be no sellers (buyers) at t_2 willing to sell (buy) at a cum-fee price smaller (higher) than P_{-2} (P_2). Thus, such limit orders are not used in this case.

- Second, the expected profit $(\beta_{t_1} - P_2^{cum,LB}) \times Pr(\theta_{t_1}^{x_2^{LB}} | \Xi, L_{t_0})$ on a limit buy at P_2 for a potential buyer with $\beta_{t_1} > P_2^{cum,LB}$ or $(P_{-2}^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-2}^{LS}} | \Xi, L_{t_0})$ on a limit sell at P_{-2} with $\beta_{t_1} < P_{-2}^{cum,LB}$ in non-negative.

$$\Big(|\beta_{t_1} - v| + \frac{3}{2} - MF\Big) \max\Big\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - TF]\Big\} > 0. \tag{84}$$

However with Maker-Taker pricing and $0 < \Delta \leq 5$ a limit buy at P_2 is always dominated by a limit buy at P_1

$$(\beta_{t_1} - P_2^{cum,LB}) \times Pr(\theta_{t_1}^{x_2^{LB}}|\Xi, L_{t_0}) > (\beta_{t_1} - P_1^{cum,LB}) \times Pr(\theta_{t_1}^{x_1^{LB}}|\Xi, L_{t_0}) \tag{85}$$

The conditions under which (85) holds are

$$\begin{cases} MF = 0 \,\&\, TF = 0 \,\&\, \beta > v + \dfrac{\Delta}{2} + 2 \\ \beta > v + 2 + MF - TF + \dfrac{1}{2}\Delta \,\&\, 0 < TF \leq 1 \,\&\, -TF \leq MF \leq 0 \end{cases} \tag{86}$$

It is straightforward to show that the first condition in (86) is never satisfied for $0 < \Delta \leq 5$ since $\beta > v + \frac{\Delta}{2} + 2$ is a β value that falls beyond $\overline{\beta}$ as $v + \frac{\Delta}{2} = \overline{\beta}$. The second condition is never satisfied as the minimum value of $MF - TF$ consistent with Maker-Taker is -2 and therefore for the second condition in (86) to be satisfied we need $\beta > v + \frac{\Delta}{2}$ which is impossible as $v + \frac{\Delta}{2} = \overline{\beta}$.

Third, the expected profit $(\beta_{t_1} - P_1^{cum,LB}) \times Pr(\theta_{t_1}^{x_1^{LB}}|\Xi, L_{t_0})$ from a limit buy at P_1 or $(P_{-1}^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}}|\Xi, L_{t_0})$ on limit sell at P_{-1} is non negative:

$$\big(|\beta_{t_1} - v| - \frac{1}{2} - MF\big) \max\Big\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - TF]\Big\}, \tag{87}$$

Lastly, the expected profit $(\beta_{t_1} - P_{-1}^{cum,LB}) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|\Xi, L_{t_0})$ from a limit buy at P_{-1} or $(P_1^{cum,LS} - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}}|\Xi, L_{t_0})$ from limit sell at P_1 is positive and equal to:

$$\big(|\beta_{t_1} - v| + \frac{1}{2} - MF\big) \max\Big\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - TF]\Big\}, \tag{88}$$

Hence we have to compute the threshold between a limit buy at P_{-1} and a limit buy at P_1

$$(\beta_{t_1} - P_{-1}^{cum,LB}) \times Pr(\theta_{t_1}^{x_{-1}^{LB}} | \Xi, L_{t_0}) - (\beta_{t_1} - P_{-1}^{cum,LB}) \times Pr(\theta_{t_1}^{x_1^{LB}} | \Xi, L_{t_0}) = 0 \tag{89}$$

The β_{t_1} values which satisfy (89) are:

$$\beta_{t_1}^{x_{P_{-1}}^{LB}, x_{P_1}^{LB}} = v + \frac{1}{2}\Delta + MF - TF \tag{90}$$

We can now compute the order-submission probabilities from (7) for a limit buy at P_{-1} and P_1

$$Pr(x_{-1,t_1}^{LB} | \Xi, L_{t_0}) = \begin{cases} \frac{1}{\Delta}\left[MF - TF + \frac{1}{\Delta}\right) & (MF \leq -\frac{1}{2} \wedge MF > TF - \frac{3}{2} \wedge \Delta + 2MF - 2TF > 0) \vee \\ & (MF > -\frac{1}{2} \wedge TF < 1 \wedge 1TF + \frac{1}{2} < \frac{1}{2}\Delta) \\ 0 & \text{Otherwise} \end{cases} \tag{91}$$

$$Pr(x_{1,t_1}^{LB} | \Xi, L_{t_0}) = \begin{cases} \begin{cases} \frac{1}{2} & MF \leq -\frac{1}{2} \\ \frac{\frac{1}{2}(\Delta - 2MF - 1)}{\Delta} & \text{Otherwise} \end{cases} & TF \geq 1 \vee TF + \frac{1}{2} \geq \frac{1}{2}\Delta \\ \begin{cases} \frac{1}{2} & MF < -\frac{1}{2} \wedge (MF + \frac{3}{2} \leq TF \\ & \vee (TF < MF + \frac{3}{2} \wedge \frac{1}{2}\Delta \leq TF - MF)) \\ \frac{TF}{\Delta} & MF = 0 \wedge TF > 0 \\ \frac{TF - MF}{\Delta} & (MF < -\frac{1}{2} \wedge TF < MF + \frac{3}{2} \wedge \\ & TF - MF < \frac{1}{2}\Delta) \vee -\frac{1}{2} < MF < 0 \\ \frac{TF + \frac{1}{2}}{\Delta} & MF = -\frac{1}{2} \wedge TF > 0 \\ 0 & \text{Otherwise} \end{cases} & \text{Otherwise} \end{cases} \tag{92}$$

We can now set the optimizing function for both the exchange and the Social Planner.

Exchange Problem: Maker-Taker $\Delta \in (0,3]$ Under the Maker-Taker regime the exchange will set the fees such that an investor arriving at t_1 will optimally choose either LBP_{-1,t_1} or LSP_{1,t_1}. As for the Taker-Maker regime, the exchange anticipates that the optimal order submission strategy for the buyer (seller) is to buy at P_{-1} (sell at P_1) or to buy at P_1 (sell at P_{-1}) (the case of the seller arriving at t_1 being symmetric):

$$\max_{\substack{MF,TF \\ 0\le TF\le\tau \\ MF\le TF \\ 0<\Delta\le 3\tau}} \pi^{Ex,LTM}(MF,TF) \tag{93}$$
$$= \left[Pr(x^{LB}_{-1,t_1}|\Xi,L_{t_0}) \times Pr(\theta^{x^{LB}_{-1}}_{t_1}|\Xi,L_{t_0}) + Pr(x^{LB}_{1,t_1}|\Xi,L_{t_0}) \times Pr(\theta^{x^{LB}_{1}}_{t_1}|\Xi,L_{t_0})\right] \times (MF+TF)$$

where (93) can be obtained by using (91) and (92) together with the probability of the limit buy order execution, $Pr(\theta^{x^{LB}_{-1}}_{t_1}|\Xi,L_{t_0})$ and $Pr(\theta^{x^{LB}_{1}}_{t_1}|\Xi,L_{t_0})$ from Table B1.

From the first-order conditions of (93) — that we compute in the Online Appendix — we obtain the equilibrium optimal Make and Take fees for the exchange that are in (17). The determinant is:

$$\begin{aligned} Det(MF^*,TF^*) &= \delta_{MF,MF}(MF^*,TF^*) \times \delta_{TF,TF}(MF^*,TF^*) - (\delta_{MF,TF}(MF^*,TF^*))^2 \\ &= \begin{cases} 0 & \Delta \le 0 \vee \Delta > 3 \\ \dfrac{1}{3\Delta^2} & \text{Otherwise} \end{cases} \end{aligned} \tag{94}$$

where the three second partial derivatives are

$$\delta_{MF^*,MF^*}, \delta_{TF^*,TF^*}, \delta_{MF^*,TF^*} = \begin{cases} \left\{-\frac{2}{3\Delta},\ -\frac{2}{3\Delta},\ -\frac{1}{3\Delta}\right\} & 0 < \Delta \le 3 \\ \left\{0,\ 0,\ 0\right\} & \text{Otherwise} \end{cases} \tag{95}$$

Since the determinant is positive, the second-order condition is satisfied. Q.E.D.

Table 1 shows that when the exchange opts for a Taker-Maker (or Maker-Taker) pricing Proposition 1 holds in equilibrium:

$$Pr(x^{LB}_{-1,t_1}|\Xi,L_{t_0}) = Pr(\theta^{x^{LB}_{-1}}_{t_1}|\Xi,L_{t_0}) = \frac{(MF+TF)}{\Delta} = \frac{1}{3} \tag{96}$$

and

$$Pr(x^{LB}_{1,t_1}|\Xi,L_{t_0}) = Pr(\theta^{x^{LB}_{1}}_{t_1}|\Xi,L_{t_0}) = \frac{(MF+TF)}{\Delta} = \frac{1}{3} \tag{97}$$

As Figure B1 (and B2) shows, to maximize expected profits the exchange has to maximize the product of 3 components, $\bar{\beta} - P^{cum,LB}_{-1}$, (MF+TF), $P^{cum,MS}_{-1} - \underline{\beta}$ (and , $\bar{\beta} - P^{cum,LB}_{1}$,(MF+TF), $P^{cum,MS}_{1} - \underline{\beta}$), and the sum of these three components are constrained to be equal to Δ. Q.E.D.

Social Planner Problem: Maker-Taker $\Delta \in (0,3]$ Under Maker-Taker regime to determine the optimal MF and TF, the Social Planner maximizes the total welfare of market participants. Total welfare, welfare from both limit buy orders and market sell orders, and exchange profit are defined in (13), (14) and (15), as well as (22) and (12). We present the welfare of the buy side (the sell side being symmetric):

$$\begin{aligned} &\max_{\substack{MF,TF \\ 0\leq TF\leq\tau \\ MF+TF\geq 0 \\ 0<\Delta\leq 3\tau}} \sum_{t_z\in\{t_1,t_2\}} \left(W^{INV}_{t_z}(MF,TF) + \pi^{Ex}_{t_z}(MF,TF) \right) \\ = \;& W^{INV}_{t_1}(x^{LB}_{-1,t_1}|\Xi,L_{t_0}) + W^{INV}_{t_1}(x^{LB}_{1,t_1}|\Xi,L_{t_0}) + W^{INV}_{t_2}(x^{MS}_{-1,t_2}|\Xi,L_{t_0}) \\ & + W^{INV}_{t_2}(x^{MS}_{1,t_2}|\Xi,L_{t_0}) + \left[Pr(x^{LB}_{-1,t_1}, \theta^{x^{LB}_{-1}}_{t_1}|\Xi) + Pr(x^{LB}_{1,t_1}, \theta^{x^{LB}_{1}}_{t_1}|\Xi) \right] (MF+TF) \end{aligned} \tag{98}$$

$$W^{INV}_{t_1}(x^{LB}_{-1,t_1}|\Xi,L_{t_0}) = \int_{\beta_{t_1}\in B_{t_1}(x^{LB}_{-1,t_1},\Xi,L_{t_0})} [\beta_{t_1} - P(x^{LB}_{-1,t_1}) - MF] \frac{1}{\Delta} d\beta_{t_1} \times Pr(\theta^{x^{LB}_{-1}}_{t_1}|\Xi,L_{t_0}) \tag{99}$$

$$
= \begin{cases} -\frac{(-\Delta+2TF+1)(-\Delta-2MF+2TF)(2(MF+TF-1)-\Delta)}{16\Delta^2} & (2TF+1<\Delta\wedge((-1<MF<-\frac{1}{2}\wedge((MF+\frac{3}{2}=TF\wedge\Delta<3)\vee \\ & (MF+\frac{3}{2}<TF\wedge TF<1\wedge\Delta\leq 3)\vee \\ & (MF+\frac{3}{2}>TF\wedge TF\geq 0\wedge\Delta+2MF<2TF)))\vee \\ & (TF<1\wedge TF\geq 0\wedge MF\leq 0\wedge\Delta\leq 3\wedge(2MF+1>0\vee \\ & (MF+\frac{3}{2}>TF\wedge\Delta+2MF>2TF\wedge MF+1\geq 0)))))\vee \\ & (-1\leq MF\leq -\frac{1}{2}\wedge TF\geq 0\wedge\Delta\leq 3\wedge\Delta+2MF>2TF) \\ 0 & \text{Otherwise} \end{cases}
$$

where the region of integration is $B_{t_1}(x^{LB}_{-1,t_1},\Xi,L_{t_0})=[v,\beta_{t_1}^{x^{LB}_{P_{-1}},x^{LB}_{P_1}}]$, and $\beta_{t_1}^{x^{LB}_{P_{-1}},x^{LB}_{P_1}}$ is defined in (90).

$$
W^{INV}_{t_1}(x^{LB}_{1,t_1}|\Xi,L_{t_0})=\int_{\beta_{t_1}\in B_{t_1}(x^{LB}_{1,t_1},\Xi,L_{t_0})}[\beta_{t_1}-P(x^{LB}_{1,t_1})-MF)]\frac{1}{\Delta}d\beta_{t_1}\times Pr(\theta_{t_1}^{x^{LB}_1}|\Xi,L_{t_0}) \tag{100}
$$

$$
= \begin{cases} \frac{1}{18} & MF=0\wedge TF=1\wedge\Delta=3 \\ \frac{1}{18}(MF-1)^2 & -1\leq MF<0\wedge TF=1\wedge\Delta=3 \\ -\frac{(MF-TF)(-\Delta+2TF-1)(-\Delta+MF+TF+1)}{4\Delta^2} & -1\leq MF<0\wedge 0<TF<1\wedge 2TF+1\leq\Delta\wedge\Delta\leq 3 \\ -\frac{(\Delta-1)^2(-\Delta+2TF-1)}{16\Delta^2} & MF=0\wedge\Delta>1\wedge((TF=1\wedge\Delta<3)\vee(2TF+1>\Delta\wedge TF<1)) \\ \frac{(\Delta-2MF-1)^2(\Delta-2TF+1)}{16\Delta^2} & MF<0\wedge MF+1\geq 0\wedge\Delta>1\wedge((TF=1\wedge\Delta<3)\vee(2TF+1>\Delta\wedge TF<1)) \\ \frac{TF(\Delta^2+2TF^2-3\Delta TF+TF-1)}{4\Delta^2} & MF=0\wedge 0<TF<1\wedge 2TF+1\leq\Delta\wedge\Delta\leq 3 \end{cases}
$$

where the region of integration is $B_{t_1}(x^{LB}_{1,t_1},\Xi,L_{t_0})=[\beta_{t_1}^{x^{LB}_{P_{-1}},x^{LB}_{P_1}},\overline{\beta}]$.

$$
W^{INV}_{t_2}(x^{MS}_{-1,t_2}|\Xi,L_{t_0})=Pr(x^{LB}_{-1,t_1}|\Xi,L_{t_0})\times\int_{\beta_{t_2}\in B_{t_2}(x^{MS}_{-1,t_2},\Xi,L_{t_1})}[P(x^{LB}_{-1,t_1})-\beta_{t_2}-TF)]\frac{1}{\Delta}d\beta_{t_2} \tag{101}
$$

$$
= \begin{cases} \frac{(\Delta-2TF-1)^2(\Delta+2MF-2TF)}{16\Delta^2} & TF\geq 0\wedge 2TF+1<\Delta\wedge\Delta\leq 3\wedge(MF>-\frac{1}{2}\vee(MF+\frac{3}{2}>TF\wedge\Delta+2MF>2TF)) \\ 0 & \text{Otherwise} \end{cases}
$$

where the region of integration is $B_{t_2}(x^{MS}_{-1,t_2},\Xi,L_{t_1})=[\underline{\beta},P_{-1}-TF]$.

$$
W^{INV}_{t_2}(x^{MS}_{1,t_2}|\Xi,L_{t_0})=Pr(x^{LB}_{1,t_1}|\Xi,L_{t_0})\times\int_{\beta_{t_2}\in B_{t_2}(x^{MS}_{1,t_2},\Xi,L_{t_1})}[P(x^{LB}_{1,t_1})-\beta_{t_2}-TF)]\frac{1}{\Delta}d\beta_{t_2} \tag{102}
$$

$$= \frac{(\Delta-2TF+1)^2}{8\Delta}\left(\begin{array}{cc} \frac{1}{2} & (MF=-\frac{1}{2}\wedge(TF=1\vee(2TF+1\geq\Delta\wedge 0<TF\leq 1)))\vee \\ & (MF<-\frac{1}{2}\wedge(MF+\frac{3}{2}<TF\vee(MF+\frac{3}{2}\leq TF\wedge\Delta\leq 3)\vee(MF+\frac{3}{2}>TF\wedge 2MF+\Delta\leq 2TF))) \\ \frac{1}{3} & MF=0\wedge TF=1\wedge\Delta=3 \\ \frac{1-MF}{3} & -\frac{1}{2}<MF<0\wedge TF=1\wedge\Delta=3 \\ \frac{TF}{\Delta} & MF=0\wedge 0<TF<1\wedge 2TF+1\leq\Delta \\ \frac{TF-MF}{\Delta} & (-\frac{1}{2}<MF<0\wedge 0<TF<1\wedge 2TF+1\leq\Delta)\vee(MF<-\frac{1}{2}\wedge MF+\frac{3}{2}>TF\wedge 2MF+\Delta>2TF) \\ \frac{2TF+1}{2\Delta} & MF=-\frac{1}{2}\wedge 0<TF<1\wedge 2TF+1<\Delta \\ \frac{\Delta-1}{2\Delta} & MF=0\wedge((TF=1\wedge\Delta<3)\vee(2TF+1>\Delta\wedge 0<TF<1)) \\ \frac{-2MF+\Delta-1}{2\Delta} & -\frac{1}{2}<MF<0\wedge((TF=1\wedge\Delta<3)\vee(2TF+1>\Delta\wedge 0<TF<1)) \end{array}\right)$$

where the region of integration is $B_{t_2}(x^{MS}_{1,t_2},\Xi,L_{t_1})=[\underline{\beta},P_1-TF]$.

By substituting (99), (100), (101) and (102) into the welfare function of the Social Planner, (98), we obtain a functional form whose components are subject to different boundary conditions. The following component has the highest total welfare:

$$\max_{\substack{MF,TF \\ 0\leq TF\leq\tau \\ MF+TF\geq 0 \\ 0<\Delta\leq 3\tau}} \sum_{t_z\in\{t_1,t_2\}}\left(W^{INV}_{t_z}(MF,TF)+\pi^{Ex}_{t_z}(MF,TF)\right) \tag{103}$$

$$= \begin{cases} \frac{(\Delta-2MF-1)(\Delta-2TF+1)^2}{16\Delta^2}+\frac{1}{16}(\Delta-2TF+1) & \big(MF=-\frac{1}{2}\wedge 0<TF\leq\frac{1}{2}\wedge 0<\Delta\leq 1\wedge \\ & \Delta+2TF>1\big)\vee\big(MF=-\frac{1}{2}\wedge\frac{1}{2}<TF<1\wedge \\ & 2TF-\Delta<1\wedge 0<\Delta\leq 1\big) \\ \frac{\Delta(\Delta-2TF+1)^2-4(MF-TF)(-\Delta+2TF-1)(-\Delta+MF+TF+1)}{16\Delta^2} & 0<TF<1\wedge 1<\Delta\leq 3\wedge \\ & \Big((MF=-\frac{1}{2}\wedge 2TF+1=\Delta)\vee \\ & \big(-1\leq MF<-\frac{1}{2}\wedge 2TF+1\leq\Delta\wedge \\ & (MF+\frac{3}{2}\leq TF\vee\Delta+2MF\leq 2TF)\big)\Big) \end{cases}$$

The optimal fees MF^* and TF^* presented in Table 2 are those that maximize (103) for any value of $0<\Delta\leq 3$ that satisfies the boundary conditions of (103). For example, for $\Delta=2$ the second equation in (103) is maximized for $MF^* = -0.5$ and $TF^* = 0.5$.

The optimal fees presented in Table 2 are determined by the boundary conditions of the different parts of the total welfare functional form. By substituting any $0 < \Delta \leq 3$ and the optimal MF^* and TF^* in (103) we obtain the total welfare presented in Table 2. Figure D2 in Online Appendix D.2 shows plots of the Social Planner's value function for the Maker-Taker case for the different values of the support ($\Delta \in \{\tau, 2\tau, 2.5\tau, 3\tau\}$) in Table 2.

D.2 Additional numerical illustrations

STM equilibrium Tables D1 and D2 provides additional numerical results for the two-period STM equilibrium given a profit-maximizing exchange and the Social Planner.

Exchange's expected profit surface: Figure D1 illustrates the exchange's expected profit function surface for different combinations of fees and rebates given different investor valuation supports in the two-period market. The blue dots denote profit-maximizing combinations of make and take fees. The symmetric pairs of profit-maximizing MF and TF are clearly visible when the investor valuation supports are narrow. However, once again we see the profit-maximizing access pricing is unique once the valuation support is large enough.

Social Planner's objective function surface: Figure D2 illustrates the social planner expected total welfare function surface for different combinations of fees and rebates given different investor valuation supports.

Table D1: 2-Period Small Tick Market (SMT): Equilibrium Fees and Trading Strategies. This table reports for different investor valuation support width, $\Delta = \overline{\beta} - \underline{\beta}$ still expressed in terms of the LTM tick size τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fees (mf* and tf*) (column 3 and 4), the sum and the absolute difference of the equilibrium MF^* and TF^* (column 5 and 6), the equilibrium trading strategies at t_1, x_{t_1} other than No Trade (column 7) and the associated probability of submission at t_1, $Pr(x_{t_1}|\Xi^*, L_{t_0})$ (column 8). The table also shows the cum-fee buy and sell prices ($P_j^{cum,LB}$ and $P_j^{cum,MS}$) (column 9 and 10), the probability of execution of the order posted at t_1, $Pr(\theta_{t_1}^{x_{t_1}}|\Xi^*, L_{t_0})$, which correspond to the unconditional probability of MS at t_2 (column 11), the equilibrium transaction probability $Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}}|S, \tau, \Xi)$ (column 12), and the exchange expected profit from both buyers and sellers, $\pi^{Ex}(\text{mf}^*, \text{tf}^*)$ (column 13). When the equilibrium pricing is rebate based, for each support we report first Taker-Maker set of fees and then the Maker-Taker set of equilibrium mf* and tf*.

Support $\Delta = \overline{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	mf*	tf*	mf*+tf*	|mf*-tf*|	Eq. Orders x_{t_1} at t_1	Pr. Submission $Pr(x_{t_1}\|\Xi^*, L_{t_0})$	$P_j^{cum,LB}$	$P_j^{cum,MS}$	Pr. Execution $Pr(\theta_{t_1}^{x_{t_1}}\|\Xi^*, L_{t_0})$	Pr. Trans	Exchange E[Profit] $\pi^{Ex}(mf^*, tf^*)$
$0.33\,\tau$	9.833, 10.167	0.222 -0.111	-0.111 0.222	0.111 0.111	0.333 0.333	LBp_{-1} LBp_1	0.333 0.333	10.056 10.056	9.944 9.944	0.333 0.333	0.222 0.222	0.025 0.025
τ	9.500, 10.500	0.333 0.000	0.000 0.333	0.333 0.333	0.333 0.333	LBp_{-1} LBp_1	0.333 0.333	10.167 10.167	9.833 9.833	0.333 0.333	0.222 0.222	0.074 0.074
$1.03\,\tau$	9.485, 10.515	0.333	0.017	0.350	0.316	LBp_{-1}	0.338	10.167	9.816	0.322	0.218	0.076
$1.1\,\tau$	9.450, 10.550	0.333	0.050	0.383	0.283	LBp_{-1}	0.348	10.167	9.783	0.303	0.211	0.081
$1.33\,\tau$	9.333, 10.667	0.333	0.167	0.500	0.167	LBp_{-1}, LBp_{-2}	0.375, 0.126	10.167, 9.833	9.667, 9.334	0.251, 0.001	0.188	0.085
$1.37\,\tau$	9.315, 10.685	0.333	0.171	0.504	0.162	LBp_{-1}, LBp_{-2}	0.369, 0.131	10.167, 9.833	9.662, 9.329	0.253, 0.009	0.189	0.095
$1.57\,\tau$	9.215, 10.785	0.333	0.204	0.537	0.129	LBp_{-1}, LBp_{-2}	0.353, 0.147	10.167, 9.833	9.629, 9.296	0.263, 0.051	0.197	0.106
$1.6\,\tau$	9.200, 10.800	0.167	0.333	0.500	0.166	LBp_1, LBp_{-1}	0.104, 0.396	10.334, 10.050	9.834, 9.500	0.396, 0.188	0.231	0.115
$1.67\,\tau$	9.165, 10.835	0.167	0.333	0.500	0.166	LBp_1, LBp_{-1}	0.100, 0.400	10.334, 10.050	9.834, 9.500	0.400, 0.200	0.240	0.120
$1.9\,\tau$	9.050, 10.950	0.222	0.333	0.555	0.111	LBp_1, LBp_{-1} LBp_{-2}	0.059, 0.351 0.091	10.389, 10.055 9.722	9.834, 9.500 9.167	0.412, 0.237 0.061	0.226	0.125
$2\,\tau$	9.000, 11.000	0.333	0.292	0.625	0.041	LBp_{-1}, LBP_{-2}	0.313, 0.187	10.167, 9.833	9.541, 9.208	0.271, 0.104	0.208	0.130
$3\,\tau$	8.500, 11.500	0.333	0.333	0.666	0.000	LBp_{-1}, LBp_{-2} LBp_{-3}	0.222, 0.222 0.056	10.167, 9.833 9.500	9.500, 9.167 8.834	0.333, 0.222 0.111	0.259	0.173
$4\,\tau$	8.000, 12.000	0.333	0.333	0.666	0.000	LBp_{-1}, LBp_{-2} LBp_{-3}	0.167, 0.167 0.167	10.166, 9.833 9.500	9.500, 9.167 8.834	0.375, 0.292 0.208	0.292	0.194
$5\,\tau$	7.500, 12.500	0.333	0.333	0.666	0.000	LBp_{-1}, LBp_{-2} LBp_{-3}, LBp_{-4}	0.133, 0.133 0.133, 0.100	10.166, 9.833 9.500, 9.167	9.500, 9.167 8.834, 8.500	0.400, 0.333 0.267, 0.200	0.307	0.204

Table D2: **2-Period Small Tick Market (STM): Social Planner Equilibrium Fees and Trading Strategies.** This table reports for different investor valuation support width, $\Delta = \overline{\beta} - \underline{\beta}$ expressed in terms of the LTM tick size, τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fees, mf and tf (column 3 and 4), the sum and the absolute difference of the equilibrium MF^* and TF^* (column 5 and 6), the buyer's equilibrium trading strategies at t_1, x_{t_1} other than No Trade (column 7) and the associated the cum-fee buy and sell prices ($P_k^{cum,LB}$ and $P_k^{cum,MS}$) (column 8 and 9). It also reports the equilibrium welfare of the limit order submitted at t_1, $W_{t_1}^{x_{t_1}}$, and the welfare associated with the market order posted at t_2, $W_{t_2}^{x_{t_2}}$, as well as the sum of limit and market orders $W_{t_1}^{x_{t_1}} + W_{t_2}^{x_{t_2}}$ (columns 10, 11 and 12). When the equilibrium pricing is rebate based, for each support we report first the Taker-Maker set of fees and then the Maker-Taker set of equilibrium MF and TF. Results are rounded to the third decimal.

Support width $\Delta = \overline{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	mf	tf	|mf-tf|	Eq.Strategy x_{t_1}	$P_k^{cum,LB}$	$P_k^{cum,MS}$	W. x_{t_1}	W. x_{t_2}	W. $x_{t_1} + x_{t_2}$
$0.33\,\tau$	9.833, 10.167	0.167	-0.167	0.334	LBp_{-1}	10.000	10.000	0.042	0.042	0.084
		-0.167	0.167	0.334	LBp_1	10.000	10.000	0.042	0.042	0.084
$0.67\,\tau$	9.665, 10.335	0.167	-0.167	0.334	LBp_{-1}	10.000	10.000	0.083	0.083	0.167
		-0.167	0.167	0.334	LBp_1	10.000	10.000	0.083	0.083	0.167
τ	9.500, 10.500	0.083	-0.083	0.166	LBp_{-1}	9.917	9.917	0.139	0.087	0.226
		-0.250	0.250	0.500	LBp_1	9.917	9.917	0.139	0.087	0.226
$1.33\,\tau$	9.333, 10.667	0.167	-0.167	0.334	LBp_{-1}, LBp_{-2}	10.000, 9.667	10.000, 9.667	0.188	0.104	0.292
		-0.167	0.167	0.334	LBp_1, LBp_{-1}	10.000, 9.667	10.000, 9.667	0.188	0.104	0.292
$1.67\,\tau$	9.165, 10.835	0.250	-0.250	0.500	LBp_{-1}, LBp_{-2}	10.083, 9.750	10.083, 9.750	0.237	0.132	0.369
		-0.083	0.083	0.166	LBp_1, LBp_{-1}	10.083, 9.750	10.083, 9.750	0.237	0.132	0.369
$2.33\,\tau$	8.833, 11.167	0.083	-0.083	0.166	LBp_{-1}, LBp_{-2}	9.917, 9.583	9.917, 9.583	0.337	0.176	0.513
		-0.250	0.250	0.500	LBp_1, LBp_{-1}	9.917, 9.583	9.917, 9.583	0.337	0.176	0.513
$3\,\tau$	8.500, 11.500	0.250	-0.250	0.500	LBp_{-1}, LBp_{-2} LBp_{-3}	10.083, 9.750 9.417	10.083, 9.750 9.417	0.434	0.224	0.658
		-0.083	0.083	0.166	LBp_1, LBp_{-1} LBp_{-2}	10.083, 9.750 9.417	10.083, 9.750 9.417	0.434	0.224	0.658
$3.67\,\tau$	8.165, 11.835	0.083	-0.083	0.166	LBp_{-1}, LBp_{-2} LBp_{-3}	9.917, 9.583 9.250	9.917, 9.583 9.250	0.532	0.272	0.804
		-0.250	0.250	0.500	LBp_1, LBp_{-1} LBp_{-2}	9.917, 9.583 9.250	9.917, 9.583 9.250	0.532	0.272	0.804
$4.33\,\tau$	7.833, 12.167	0.250	-0.250	0.500	LBp_{-1}, LBp_{-2} LBp_{-3}, LBp_{-4}	10.083, 9.750 9.417, 9.083	10.083, 9.750 9.417, 9.083	0.630	0.320	0.950
		-0.083	0.083	0.166	LBp_1, LBp_{-1} LBp_{-2}, LBp_{-3}	10.083, 9.750 9.417, 9.083	10.083, 9.750 9.417, 9.083	0.630	0.320	0.950
$5\,\tau$	7.500, 12.500	0.083	-0.083	0.166	LBp_{-1}, LBp_{-2} LBp_{-3}, LBp_{-4}	9.917, 9.583 9.250, 8.917	9.917, 9.583 9.250, 8.917	0.727	0.368	1.095
		-0.250	0.250	0.500	LBp_1, LBp_{-1} LBp_{-2}, LBp_{-3}	9.917, 9.583 9.250, 8.917	9.917, 9.583 9.250, 8.917	0.727	0.368	1.095

Figure D1: 2-Period Large Tick Market (LTM): Exchange Expected Profit Function and Access Pricing. This figure shows the exchange profit function and the equilibrium make fees and take fees for the LTM corresponding to different investor valuation supports with widths ranging from $0.33\,\tau$ to $5\,\tau$ (where $\tau = 1$ is the tick size in the LTM) as reported on the horizontal axis. The three-dimensional figures indicate (blue dots) the optimal make fee (MF), the optimal take fee (TF), and the associated equilibrium exchange expected profit for each support.

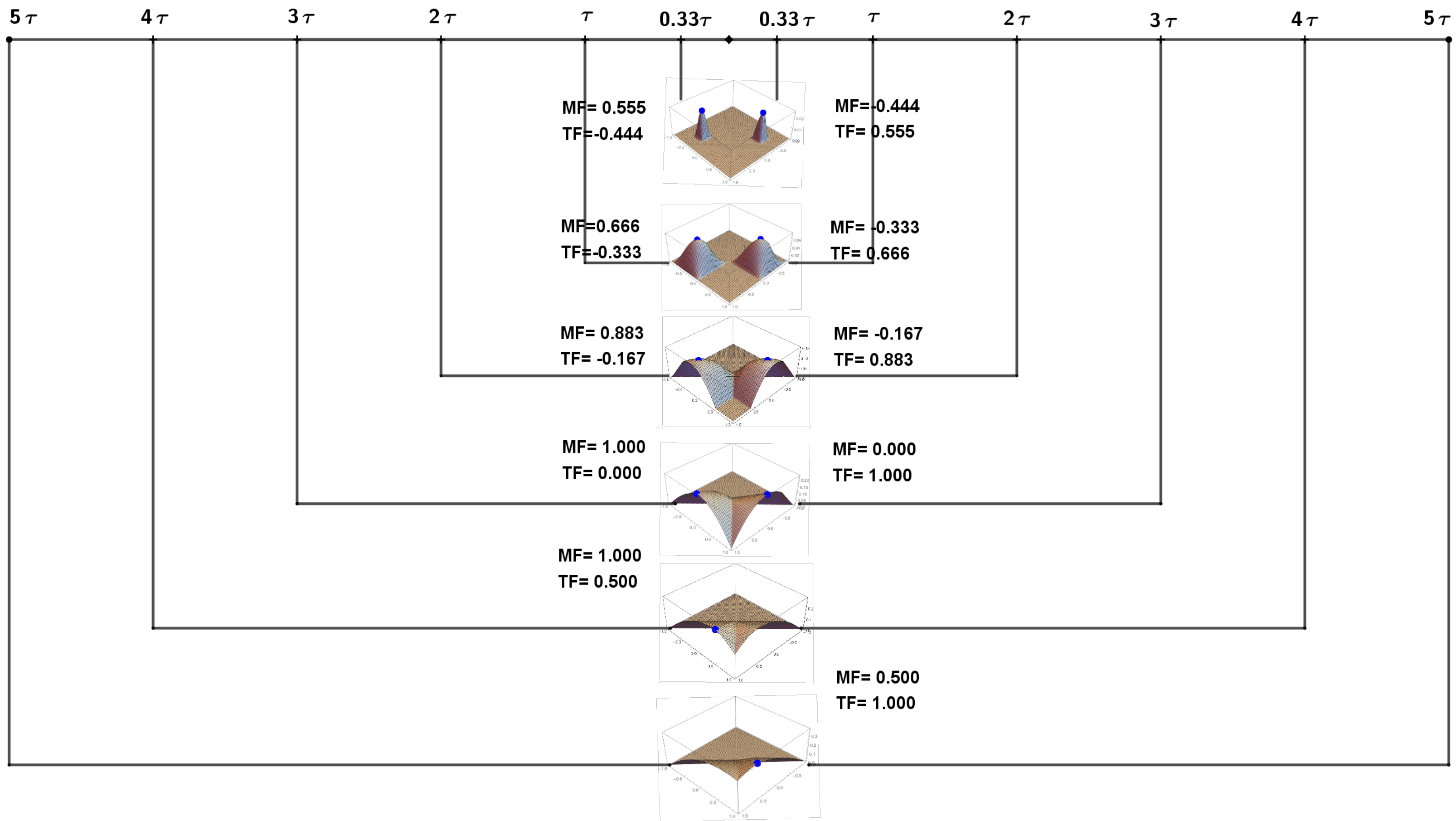
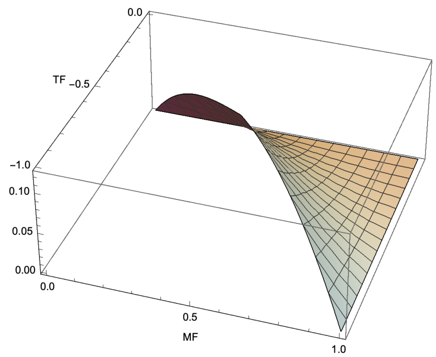
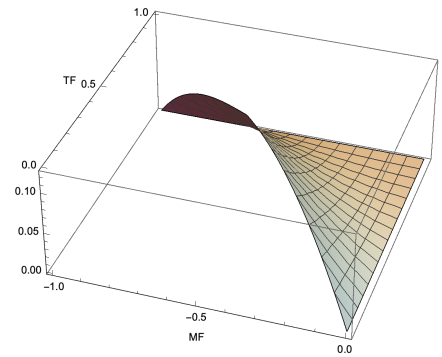


Figure D2: Social Planner - Total Welfare This figure shows the objective function of the Social Planner under both the Taker-Maker regime (Panels a,c,e and g), and the Maker-Taker regime (Panels b,d,f and h), for the 4 different supports reported in Table 2, $\Delta \in \{\tau, 2\tau, 2.5\tau, 3\tau\}$

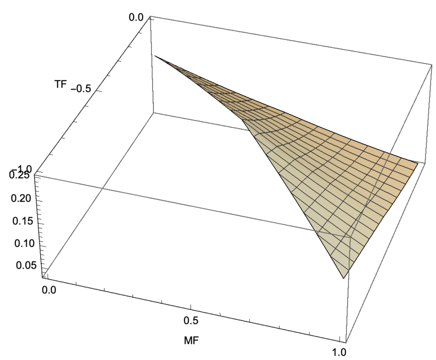
(a) Taker-Maker $\Delta = \tau$



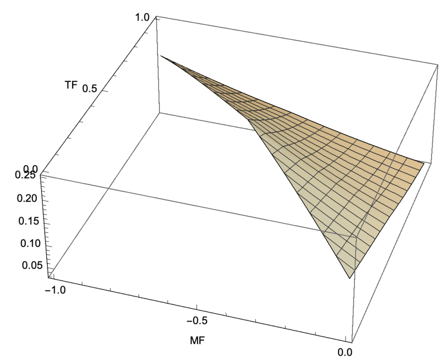
(b) Maker-Taker $\Delta = \tau$



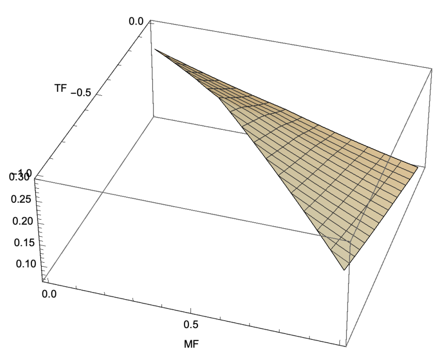
(c) Taker-Maker $\Delta = 2\,\tau$



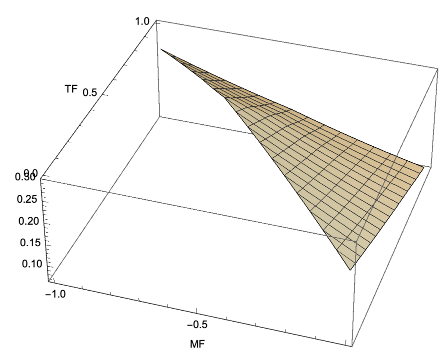
(d) Maker-Taker $\Delta = 2\,\tau$



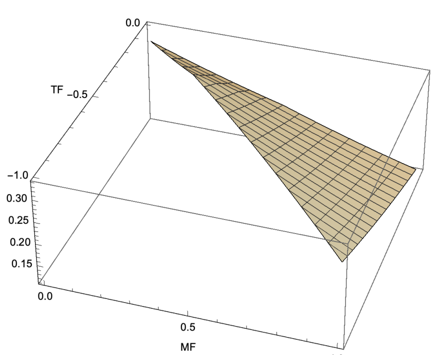
(e) Taker-Maker $\Delta = 2.5\,\tau$



(f) Maker-Taker $\Delta = 2.5\,\tau$



(g) Taker-Maker $\Delta = 3\,\tau$



(h) Maker-Taker $\Delta = 3\,\tau$

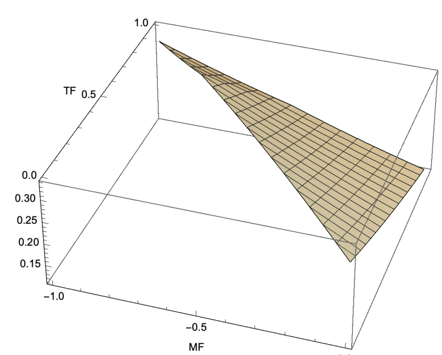
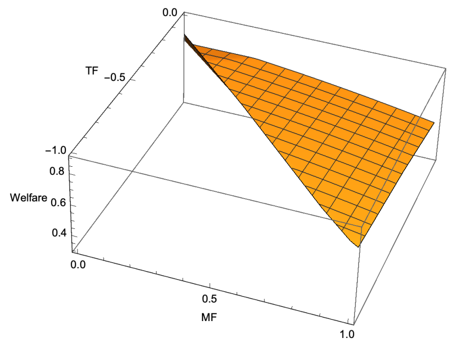
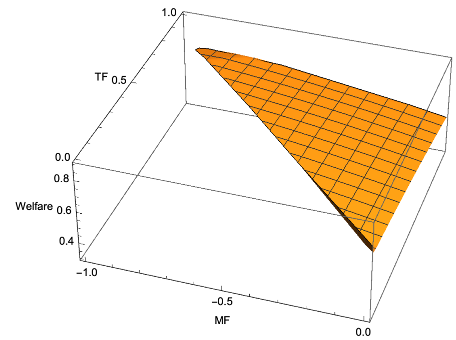


Figure D3: Social Planner - Total Welfare This figure shows the objective function of the Social Planner under both the Taker-Maker regime (Panels a,c,e and g), and the Maker-Taker regime (Panels b,d,f and h), for the 4 different supports reported in Table 2, $\Delta \in \{3.5\tau, 4\tau, 4.5\tau, 5\tau\}$

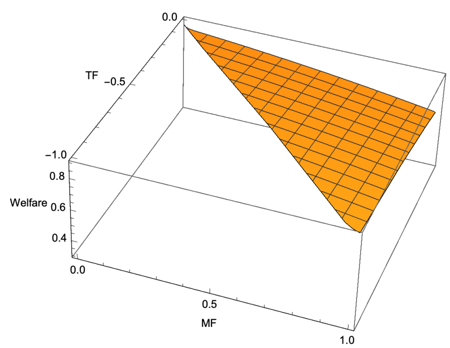
(a) Taker-Maker $\Delta = 3.5$



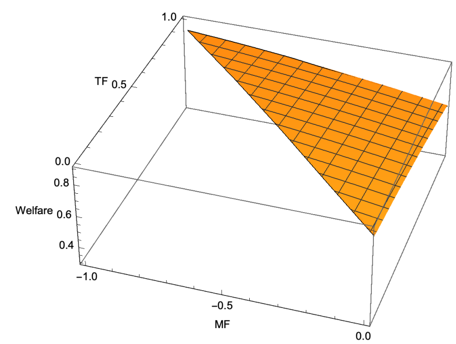
(b) Maker-Taker $\Delta = 3.5\tau$



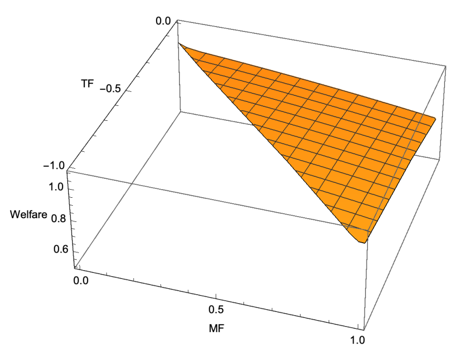
(c) Taker-Maker $\Delta = 4\tau$



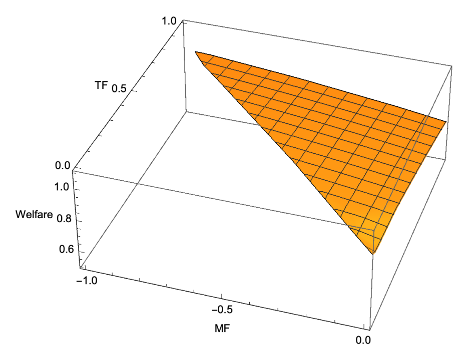
(d) Maker-Taker $\Delta = 4\,\tau$



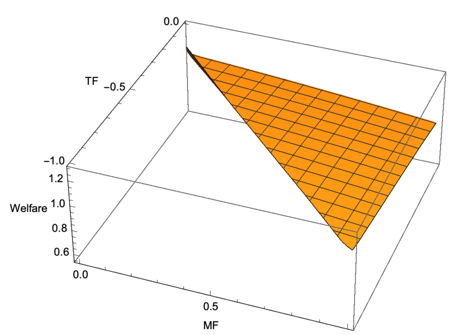
(e) Taker-Maker $\Delta = 4.5\,\tau$



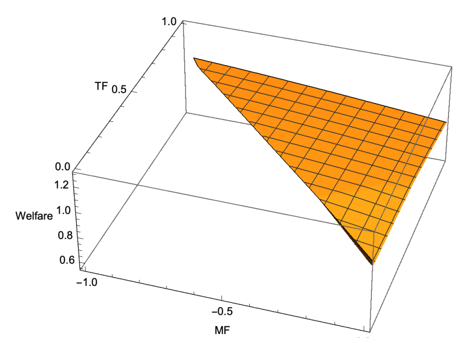
(f) Maker-Taker $\Delta = 4.5\,\tau$



(g) Taker-Maker $\Delta = 5\,\tau$



(h) Maker-Taker $\Delta = 5\,\tau$



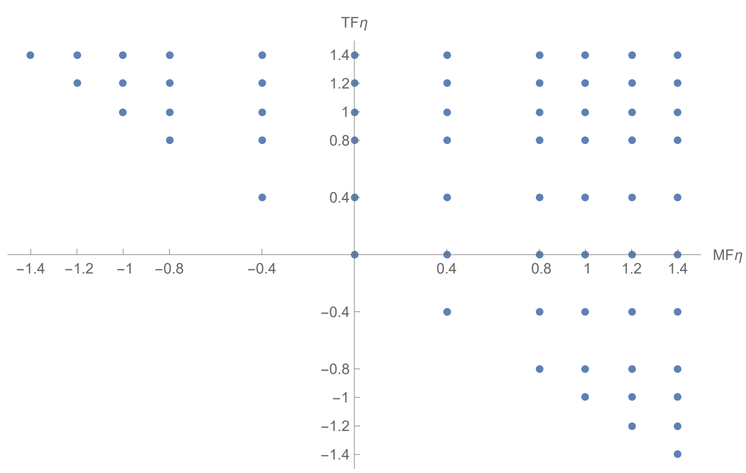
E Simulated Annealing (SA) and Grid Search (GS) Algorithms

This section explains how we implement the SA and GS algorithms to find the equilibrium profit-maximizing exchange fees $\Xi^* = \{MF^*, TF^*\}$. Given the valuation support and tick size, we choose a stratified collection of starting points that cover the set of regulatory-allowed fees ($TF \leq 1$ and $MF \leq 1$) with non-negative exchange profits (with $TF + MF \geq 0$). In particular, we use fee starting points from a larger set of fees with non-negative profits ($TF \leq 1.5$ and $MF \leq 1.5$ and $TM + MF \geq 0$). Figure E1 illustrates the collection of starting points for the large tick market. We included fee pairs outside of the set allowed by regulation to ensure that fees on the boundary of the regulatory set were not under-explored. To reduce computation time, we found that initial fees equal to ± 0.2 and ± 0.6 could be excluded. We then apply the SA algorithm to each of these starting points.

The simulated annealing (SA) algorithm is an iterative procedure. For each starting point Ξ_0 in our stratified collection of starting points of fees, the SA algorithm here searches for the fees that produce the largest profit for the exchange, π^{Ex}, conditional on the associated optimal behavior of investors given the exchange's fees. Our specific implementation of SA is based on Givens and Hoeting (2005).

Let η denote a generic round in the SA iteration associated with a given starting point Ξ_0, and let $\Xi_\eta = (MF_\eta, TF_\eta)$ denote the candidate pair of fees in round η. The next pair of candidate fees $\Xi_{\eta+1}$ is determined as follows: If Ξ_η achieves a greater expected exchange profit π_η^{Ex} than the expected exchange profit $\pi_{\eta-1}^{Ex}$ using the fees $\Xi_{\eta-1}$ from the prior round $\eta - 1$ (which is always true by definition for $\eta = 0$), then $MF_{\eta+1}$ is randomly selected from the interval $\{MF_\eta - \varepsilon, MF_\eta + \varepsilon\}$ and $TF_{\eta+1}$ is from the interval $\{TF_\eta - \varepsilon, TF_\eta + \varepsilon\}$, both with independent Uniform probability. Our SA implementation sets $\varepsilon = 0.25$ so that the sampling region has an amplitude of a half tick. If instead the exchange profit associated with Ξ_η is less than or equal to the exchange profit associated with $\Xi_{\eta-1}$, then $\Xi_{\eta+1}$ is randomly selected from a 2ε interval centered at Ξ_η with probability $\zeta_\eta = e^{\frac{\pi^{Ex}_\eta - \pi^{Ex}_{\eta-1}}{\chi_\eta}}$ or from a 2ε interval centered round $\Xi_{\eta-1}$ with probability $1-\zeta_\eta$, where χ_η is a parameter that starts at $\chi_0 = 0.8$ and reduces to $0.9\times\chi_\eta$ at each η iteration until it reaches its minimum that we set at 0.066667. This results in 162 iterative rounds for a given starting point.[33] The intuition for declining values of χ_η is the following: When $\pi^{Ex}_\eta > \pi^{Ex}_{\eta-1}$, our SA algorithm searches near the most recent best fees Ξ_η. However, when $\pi^{Ex}_\eta \le \pi^{Ex}_{\eta-1}$, the SA searches with some probability around the prior round's more profitable fees $\Xi_{\eta-1}$ with a probability that is increasing in how much bigger $\pi^{Ex}_{\eta-1}$ is than π^{Ex}_η. The fact that χ_η shrinks in later rounds causes the probability ζ_η with which the SA algorithm explores the neighborhood of the less profitable fees in later rounds to be smaller than in earlier rounds conditional on a given profitability difference $\pi^{Ex}_\eta - \pi^{Ex}_{\eta-1}$. This is because in earlier round η we want the algorithm to explore regions of the parameter space away from a prior better fee pair, but in later rounds we want the algorithm to explore more around the better for the two alternatives.

Figure E1: Simulated Annealing (SA) Algorithm: Large Tick Market (LTM) initial Sets of MF_η and TF_η. This Figure reports the initial combinations of MF_η and TF_η, from which the SA algorithm starts to numerically maximize the exchange profits π^{Ex}.



Our SA algorithm produces a total of 10692 pairs of candidate exchange fees given our initial collection of 66 starting pairs of exchange fees and iterative sequences of 162 rounds. At this point, we only inlude fee

[33]See Givens and Hoeting (2005).

pairs allowed by regulation. We identify 8 pairs of fees ($\Xi^{\dagger} = \{MF^{\dagger}, TF^{\dagger}\}$) to pass into our GS algorithm consisting of the 4 fee pairs with $MF^{\dagger} > TF^{\dagger}$ and the 4 fee pairs such that $MF^{\dagger} < TF^{\dagger}$ with the maximum exchange profit subject to a constraint that the differences in exchange profits are approximately 10^{-4} (i.e., subject to rounding). The GS algorithm further refines these eight best fees $\Xi^{\dagger}$ from the SA algorithm. For each input fee pair $\Xi^{\dagger}$, the GS algorithm sequentially generates a series of 6 grids of fees. The first grid (Grid #1) consists of fee pairs that differ from the input pair $(MF^{\dagger}, TF^{\dagger})$ by all combinations of 7 steps of 0.02 (i.e., $-0.06, -0.04, \ldots, 0, \ldots, 0.06$). Given the fee pair $\Xi^{\dagger,1} = \{MF^{\dagger,1}, TF^{\dagger,1}\}$ with the highest expected exchange profit on Grid #1, the GS algorithm then generates a second grid (Grid #2) of fees that differ from $\Xi^{\dagger,1}$ by 7 steps of $\Delta = 0.01$ $(-0.03, -0.02, \ldots, 0, \ldots, 0.03)$. The profit-maximizing fees $\Xi^{\dagger,2}$ are then used to construct a third grid (Grid #3) with a granularity of 0.005. This procedure continues with further grid granularities of 0.0025, 0.000125, and finally 0.000065 for Grid #6. The fees associated with the highest π^{Ex} from Grid #6 for each of the 8 best SA fee pairs $\Xi^{\dagger}$ are then compared. Our solution for the exchange's profit-maximizing equilibrium fees $\Xi^{*} = \{MF^{*}, TF^{*}\}$ is the pair of fees from this final comparison leading to the highest expected exchange profit.

Tables E1 and E2 illustrate how the GS algorithm works. Suppose one the 8 best SA fee pairs $\Xi^{\dagger}$ is $\{MF^{\dagger} = -0.270, TF^{\dagger} = 0.494\}$. Table E1 shows the 49 fee pairs associated with this $\Xi^{\dagger}$. In this example, the fee pair $\Xi^{\dagger,1} = \{MF^{\dagger,1} = -0.310, TF^{\dagger,1} = 0.514\}$ has the highest expected exchange profits on Grid #1. Given $\{MF^{\dagger,1} = -0.310, TF^{\dagger,1} = 0.514\}$, the GS algorithm then generates Grid #2. Table E2 shows the 49 variations of $\{MF^{\dagger,2} = -0.310, TF^{\dagger,2} = 0.514\}$ in this Grid #2. The best fees $\Xi^{\dagger,2}$ on this Grid #2 would then be used to generate a Grid #3, and so forth.

Table E1: **Example of Grid** #1 This Table shows combinations of MF and TF that differ by up to ± 3 steps of 0.02 from $\Xi^{\dagger}$. The two bold pairs are the input fees $\Xi^{\dagger}$ and the fees $\Xi^{\dagger,1}$ with the largest exchange profits in this illustration.

	-0.06	-0.04	-0.02	0	0.02	0.04	0.06
-0.06	-0.33,0.434	-0.31,0.434	-0.29,0.434	-0.27,0.434	-0.25,0.434	-0.23,0.434	-0.21,0.434
-0.04	-0.33,0.454	-0.31,0.454	-0.29,0.454	-0.27,0.454	-0.25,0.454	-0.23,0.454	-0.21,0.454
-0.02	-0.33,0.474	-0.31,0.474	-0.29,0.474	-0.27,0.474	-0.25,0.474	-0.23,0.474	-0.21,0.474
0	-0.33,0.494	-0.31,0.494	-0.29,0.494	**-0.27,0.494**	-0.25,0.494	-0.23,0.494	-0.21,0.494
0.02	-0.33,0.514	**-0.31,0.514**	-0.29,0.514	-0.27,0.514	-0.25,0.514	-0.23,0.514	-0.21,0.514
0.04	-0.33,0.534	-0.31,0.534	-0.29,0.534	-0.27,0.534	-0.25,0.534	-0.23,0.534	-0.21,0.534
0.06	-0.33,0.554	-0.31,0.554	-0.29,0.554	-0.27,0.554	-0.25,0.554	-0.23,0.554	-0.21,0.554

Table E2: **Example of Grid** #2 This Table shows combinations of MF and TF that differ by up to ± 3 steps of 0.01 from $\Xi^{\dagger,1}$. The two bold pairs are the input fees $\Xi^{\dagger,1}$ from Grid #1 in Table E1 and the fees $\Xi^{\dagger,2}$ with the largest exchange profits in this illustration.

	-0.03	-0.02	-0.01	0	0.01	0.02	0.03
-0.03	-0.34,0.484	-0.33,0.484	-0.32,0.484	-0.31,0.484	-0.30,0.484	-0.29,0.484	-0.28,0.484
-0.02	-0.34,0.494	-0.33,0.494	-0.32,0.494	-0.31,0.494	-0.30,0.494	-0.29,0.494	-0.28,0.494
-0.01	-0.34,0.504	-0.33,0.504	-0.32,0.504	-0.31,0.504	-0.30,0.504	-0.29,0.504	-0.28,0.504
0	-0.34,0.514	-0.33,0.514	-0.32,0.514	**-0.31,0.514**	-0.30,0.514	-0.29,0.514	-0.28,0.514
0.01	-0.34,0.524	-0.33,0.524	-0.32,0.524	-0.31,0.524	-0.30,0.524	-0.29,0.524	-0.28,0.524
0.02	-0.34,0.534	-0.33,0.534	-0.32,0.534	-0.31,0.534	-0.30,0.534	-0.29,0.534	**-0.28,0.534**
0.03	-0.34,0.544	-0.33,0.544	-0.32,0.544	-0.31,0.544	-0.30,0.544	-0.29,0.544	-0.28,0.544

F Three-Period Model

This Appendix reports our proofs for the three-period version of the model. In Section F.1 we consider the benchmark version of the model without fees ($MF = TF = 0$) and we show how to solve it in closed-form, given the investor valuation support and tick size. We present the solution for a valuation support width $\Delta = 2\tau$, i.e., $S = [\underline{\beta}, \overline{\beta}] = [v-\tau, v+\tau]$, and a tick size equal to τ (LTM). Solutions for other support widths and for the small tick size can be obtained in a similar way. In Section F.2 we show how to solve the 3-period benchmark model with HFTs in closed-form, and in Section E in the Online Appendix we show how the simulated annealing (SA) and the grid search (GS) algorithms work.

F.1 3-Period Benchmark Model

The model is solved by backward induction starting from period t_3 and Tables F1, F2 and F3 report the equilibrium thresholds, the equilibrium probabilities of execution and the equilibrium probabilities of submission respectively at

t_3, t_2 and t_1. We solve the model for the buy side, the sell side being symmetric.

Period t_3: Table F1 reports the equilibrium strategies (column 1), their payoffs (columns 2), the β thresholds and the associated order submission probabilities (columns 3 and 4) for the different states of the book at t_3, resulting from the equilibrium strategies at t_1 and t_2.

Equations (28) and (29) from Appendix B report a general expression for the market order submission probabilities at t_2 of the 2-period trading game; Table B1 reports the order submission probabilities associated with different order types. With $\Delta = 2\tau$, and $MF = TF = 0$, investors optimally choose orders at either P_{-1} or P_1. A buyer at t_1 or at t_2 would not choose to buy at P_{-2} as he anticipates that no seller would be willing to sell at a price $P_j \leq \underline{\beta}$. Likewise, a seller would not sell at a price $P_j \geq \overline{\beta}$. Hence both buyers and sellers would either buy at P_{-1}, or at P_1 or decide not to trade. The possible states of the book at t_3 depend on the equilibrium strategies at t_1 and t_2. For example, [00*B*0], [0*B*00] and [0000] are books that open with either a LBP_{-1}, or a LBP_1 and with no orders previously posted at any price level, respectively. The upper part of Table F1 indicates the equilibrium strategies at t_3 when at t_1 the investor submits an LBP_{-1}, and at t_2 investors post either an MSP_{-1} [0000], or an LSP_1 [0*SB*0], or an LBP_1 [0*BB*0].

If at t_3 the book opens empty, [0000], independently of his personal evaluation, the investor arriving at t_3 cannot trade: there are no limit orders to hit with a market order, and being t_3 the last period of the trading game, a limit order would never be executed. Hence, the investors payoff is equal to zero along all the support $[v-\tau, v+\tau]$. If instead the book opens [0*SB*0], an investor can either MSP_{-1}, or MBP_1, or decide not to trade (NT). If the investor opts for a MSP_{-1,t_3} his payoff will be $P_{-1} - \beta_{t_3} - TF = v - \frac{1}{2}\tau + \beta_{t_3}$, and he will market sell if $\beta_{t_3} < P_{-1}$. Given $P_{-1} = v - \frac{1}{2}\tau$ and $\Delta = 2\tau$ ($\underline{\beta} = v - \tau$), the MSP_{-1,t_3} probability of submission is:

$$Pr(MSP_{-1,t_3}|\Xi, L_{t_2}) = \frac{P_{-1} - \underline{\beta}}{\overline{\beta} - \underline{\beta}} = \frac{1}{4} = Pr(\theta_{t_2}^{LBP_{-1}}|\Xi, L_{t_2}). \tag{104}$$

Note that the probability of submission of MSP_{-1} is the probability that the investor's personal evaluation β_{t_3} falls between $\underline{\beta}$ and the threshold between MSP_{-1} and NT, $\beta_{t_3}^{MSP_{-1},NT}$. Such a threshold can be derived by equating the payoff of the two adjacent strategies, $v - \frac{1}{2}\tau + \beta_{t_3} = 0$. The order submission probabilities of the other possible orders at t_3 - both in the upper and in the lower panel of Table F1 - can be obtained in a similar way.

Period t_2**:** Having solved the model at t_3, it is now possible to determine the optimal trading strategies at t_2 as the probabilities of submission at t_3 are the execution probabilities of the limit orders standing in the book at t_2. Table F2 reports the equilibrium strategies at t_2 given a book that opens either $[00B0]$ or $[0B00]$. Note that the book at t_2 cannot open empty as at t_1 the investor either chooses LBP_{-1} or LBP_1. Table F1 indicates that if an investor posts a LBP_{-1} or a LBP_1, his order can be executed either at t_2 or at t_3, and therefore the probability of execution is non-zero. As a consequence, given any value of the investor's β_{t_1} within the 2τ support, the buyer (seller) will either choose a LBP_{-1} (LSP_1) or LBP_1 (LSP_{-1}).

For example, if the book opens $[00B0]$ at t_2, the incoming investor will either MSP_{-1}, or LSP_1, or LBP_1. The payoff and the equilibrium thresholds of the MSP_{-1} can be derived as we did for t_3. The payoff from a LSP_1 is instead $(P_1 - \beta_{t_2} - MF)Pr(\theta_{t_2}^{LSP_1}|\Xi, L_{t_1})$ where $Pr(\theta_{t_2}^{LSP_1}|\Xi, L_{t_1})$ is the probability of submission of a MBP_1 at t_3. Table F1 indicates that this probability is 0.250, and therefore the payoff of LBP_1 is equal to $(P_1 - \beta_{t_2}) \times 0.250 = 2.625 - 0.250\beta_{t_2}$ as reported in Table F2. The equilibrium thresholds between MSP_{-1} and LSP_1, $\beta_{t_2}^{MSP_{-1},LSP_1} = 9.167 = v - \frac{5}{6}\tau$, can be derived by equating the two payoffs $(9.500 - \beta_{t_2} = 2.625 - 0.250\beta_{t_2})$ and solving for β_{t_2}.

The payoffs, the equilibrium thresholds, and the order submission probabilities of the other orders posted at t_2 can be derived in a similar way.

Period t_1**:** The equilibrium strategies, the payoffs, the thresholds and the order submissions at t_1 can be derived by considering the order submission probabilities obtained both at t_3, and at t_2. For example, Table F3 reports the payoff from LBP_{-1} as

$$(\beta_{t_1} - P_{-1} - MF)Pr(\theta_{t_1}^{LBP_{-1}}|\Xi, L_{t_0}) = -2.375 + 0.250\beta_{t_1} \tag{105}$$

where

$$Pr(\theta_{t_1}^{LBP_{-1}}|\Xi,L_{t_0}) = Pr(MBP_{-1,t2}|\Xi,L_{t_1}) + \left(1 - Pr(LBP_{1,t_2}|\Xi,L_{t_1})\right) \times Pr(MBP_{-1,t_3}|\Xi,L_{t_2}) = 0.250 \tag{106}$$

and the probabilities in (106) have been derived by backward induction at t_3 and t_2. The probability of execution of LBP_{-1} at t_1 is equal to the probability that the buy order at P_{-1} is executed at t_2 $(Pr(MBP_{-1,t2}|\Xi,L_{t_1}))$, plus the probability that it can be executed at t_3. The latter is equal to the probability that at t_2 the order is not undercut by a trader buying aggressively at P_1 $(1 - Pr(LBP_{1,t_2}|\Xi,L_{t_1}))$, times the probability that it is executed at t_3 $(Pr(MBP_{-1,t_3}|\Xi,L_{t_2}))$. Note that $Pr(\theta_{t_1}^{LBP_{-1}}|\Xi,L_{t_0}) > 0$ and therefore the payoff from LBP_{-1} is equal to zero for $\beta_{t_3} = P_{-1} < v$, which confirms that at t_1 NT cannot be an equilibrium strategy.

Lemma 3 shows that investors with $\beta > v$ $(\beta < v)$ are potential buyers (sellers), therefore the equilibrium probability of submission of LBP_{-1} at t_1 can be obtained by computing the probability that the buyer's personal evaluation falls between v and the threshold $\beta_{t_1}^{LBP_{-1},LBP_1}$. This threshold can be derived by equating the payoffs from the two equilibrium strategies:

$$\begin{aligned}(\beta_{t_1} - P_{-1} - MF)Pr(\theta_{t_1}^{LBP_{-1}}|\Xi,L_{t_0}) &= (\beta_{t_1} - P_1 - MF)Pr(\theta_{t_1}^{LBP_1}|\Xi,L_{t_0}) \\ -2.375 + 0.250\beta_{t_1} &= -9.844 + 0.938\beta_{t_1} \\ \beta_{t_1}^{LBP_{-1},LBP_1} &= v + \frac{259}{300}\tau\end{aligned} \tag{107}$$

It follows that the probability of LBP_{-1} submission at t_1 is equal to $\frac{\beta_{t_1}^{LBP_{-1},LBP_1} - v}{\overline{\beta} - \underline{\beta}} = 0.432$.

The other order submission probabilities presented in Table F3 can be derived in a similar way. We have shown that the 3-period benchmark model can be solved in closed-form. Along the same lines the 3-period model can be solved for any possible set of trading fees, $\Xi = [MF, TF]$.

F.2 3-Period Benchmark Model with HFT

As for the 3-period model, we now show how to obtain a closed-form solution for the benchmark model (this time with HFTs) without fees, a support equal to 2τ and the large tick size, τ. Tables F4, F5 and F6 report the equilibrium strategies, the payoffs, the β thresholds and the order submission probabilities obtained solving the model by backward induction from t_3 to t_2 and then t_1.

The solution of this model with HFTs is similar to the previous one (Section F.1) a part from the interaction of regular investors with HFT firms which act as opportunistic liquidity providers. As before, given a $\Delta = 2\tau$ support and a tick size equal to τ, investors are not willing to post orders at prices below P_{-1} or above P_1, and therefore the $INVs$' set of profitable strategies only include P_{-1} and P_1.

HFTs have a personal evaluation equal to v ($\beta = v$) and therefore they are willing to take the other side of a trade only if - in absence of fees - they buy (sell) at prices below (above) v. Precisely, an HFT is always willing to use a market sell (buy) MSP_1 (MBP_{-1}) to immediately execute an INV's limit buy (sell) LBP_1 (LSP_{-1}). Because HFT firms are faster than $INVs$, they can send flash orders that take the other side of a profitable aggressive limit order and execute immediately. Following Budish et al. (2015), we assume that HFT firms are unwilling to provide ex ante liquidity via limit orders.

The striking difference between this model with HFTs and the 3-period game without HFTs is that when the book opens empty on the ask, on the bid or on both sides, there is now an investor willing to post an aggressive limit order and exploit the latent liquidity demand offered by the HFT firm. We observe this behaviour in all the three periods of the trading game. When there are no standing limit orders in the limit order book at any price, which may be the case at time $t = t_1$ and also at $t = t_2, t_3$ if there are no standing LOs after time t_1, t_2 (Tables F4, F5 and F6), an INV natural buyer with $\beta_t > v$ not only may choose NT as in Section F.1, but he can now also choose a LBP_1 and trade with the HFT firm. In particular, with no limit sell orders at prices $P \leq P_1$ (against which a LBP_1 would be automatically crossed), a posted LBP_1 will be immediately executed by a MSP_1 from an HFT seller with probability 1. A natural seller has the same option of either posting an aggressive LSP_{-1} or decide not to trade.

For example, when the book opens empty at t_3 following a LBP_{-1} at t_1 and a MSP_{-1} at t_2 (Table F4), a natural seller will post an aggressive LSP_{-1} that will immediately execute against an HFT's MBP_{-1}. As the payoff is equal to $P_{-1} - \beta_{t_3} - MF$, the seller will limit sell up to $\beta_{t_3} < P_{-1}$, i.e., $\beta_{t_3} < v - \frac{1}{2}\tau$ which is the β threshold between LSP_{-1} and NT, $\beta_{t_3}^{LSP_{-1},NT}$. Therefore the β range consistent with LSP_{-1} is between $\underline{\beta}$ and $v - \frac{1}{2}\tau$, and therefore given that β is uniformly distributed, the probability of LSP_{-1} submission is 0.250. The other equilibrium thresholds, order execution and order submission probabilities at t_3 can be obtained in a similar way.

Table F5 reports the equilibrium strategies at t_2. Compared to the previous case without HFTs, now the book can open either [00B0] or [0000]. If the book opens [00B0] the INV can only attract HFTs by posting an aggressive limit buy at P_1, whereas if the book opens empty, the INV can attract HFTs' flash orders both by aggressively selling at P_{-1} or by aggressively buying at P_1. The INV can alternatively limit sell at P_1 or limit buy at P_{-1} and wait for his order to be executed at t_3 with probability $Pr(\theta_{t_2}^{LBP_{-1}}|\Xi, L_{t_1})$ and $Pr(\theta_{t_2}^{LSP_1}|\Xi, L_{t_1})$, where as before the execution probability of limit orders at t_2 is equal to the INV equilibrium order submission probabilities at t_3 that are equal to 0.250.

The model is finally solved by backward induction at t_1 (Table F6) where the equilibrium order strategies are the same as in Table F3 with the difference that now aggressive limit orders are immediately executed against the HFTs flash orders with the usual payoff of $(P_{-1} - \beta_{t_1} - MF)$ for a LSP_{-1} and $(P_1 - \beta_{t_1} - MF)Pr(\theta_{t_1}^{LSP_1}|\Xi, L_{t_0})$ for a LSP_1. The β threshold between LSP_{-1} and LSP_1 being determined by equating the two payoffs and solving for $\beta_t^{LSP_{-1},LSP_1}3 = v - \frac{5}{6}\tau$.

The other order submission probabilities presented in Table F6 can be derived in a similar way. We have shown that the 3-period benchmark model with HFTs can be solved in closed-form. Along the same lines the 3-period model with HFTs can be solved for any possible set of trading fees, $\Xi = [MF, TF]$.

The next step is to show how we obtain the optimal trading fees set either by a profit-maximizing exchange, or by a Social Planner. To find the fees Ξ^* that maximize either an exchange's profit π^{Ex}, or the Social Planner total welfare W, we implement a two-stage process using first the Simulated Annealing (SA) algorithm and then a grid-search (GS) algorithm to further refine the SA solutions. Results are shown in Table 4. Section E in the Online Appendix discusses how the SA and the GS work.

Table F1: 3-Period Large Tick Market (LTM). Equilibrium Strategies at t_3. This table shows how to derive the equilibrium order submission strategies at t_3 for the benchmark model which has no trading fees ($MF = TF = 0.00$) and for an investors' support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D_{t_z}^{P_i}$, corresponds to the depth of the book at each price level at time t_z, $[L_{t_z}^{P_2}, L_{t_z}^{P_1}, L_{t_z}^{P_{-1}}, L_{t_z}^{P_{-2}}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1, LBP_1 and LBP_{-1} on the buy side and LSP_1 and LSP_{-1} on the sell side. Table F3 presents both the buy and the sell equilibrium strategies at t_1. However, as the equilibrium strategies consistent with the states of the book derived from the buy side are symmetric to those derived from the sell side, to economize space at t_2 we only present the equilibrium strategies that are consistent with the states of the book derived from the buy equilibrium strategies at t_1. Given the equilibrium limit buy orders at t_1, the possible states of the books at the beginning of t_2 are: [00B0] following a LBP_{-1} and [0B00] following a LBP_1. Given the equilibrium strategies at t_2 and therefore the possible states of the books at the beginning of t_3, this table shows the equilibrium strategies at t_3 (column 1), their payoffs (column 2), the β thresholds (column 3) and the order submission probabilities (column 4).

Equilibrium Strategy	Payoff	β Threshold	Order Submission Probability
at t_1 the book opens empty [0000]: equilibrium strategy LBP_{-1}			
at t_2 the book opens [00B0]			
	t_2 equilibrium strategy: MSP_{-1,t_2}		
at t_3 the book opens empty [0000]			
NT_{t_3}	0	$\{v-\tau, v+\tau\}$	1
	t_2 equilibrium strategy: LSP_1		
at t_3 the book opens [0SB0]			
MSP_{-1,t_3}	$P_{-1} - \beta_{t_3} - TF = v - \frac{1}{2}\tau + \beta_{t_3}$	$\{v-\tau, v-\frac{1}{2}\tau\}$	0.250
NT_{t_3}	0	$\{v-\frac{1}{2}\tau, v+\frac{1}{2}\tau\}$	0.500
MBP_{1,t_3}	$\beta_{t_3} - P_1 - TF = -10.500 + \beta_{t_3}$	$\{v+\frac{1}{2}\tau, v+\tau\}$	0.250
	t_2 equilibrium strategy: LBP_1		
at t_3 the book opens [0BB0]			
MSP_{1,t_3}	$P_1 - \beta_{t_3} - TF = 10.500 - \beta_{t_3}$	$\{v-\tau, v+\frac{1}{2}\tau\}$	0.750
NT_{t_3}	0	$\{v+\frac{1}{2}\tau, v+\tau\}$	0.250
at t_1 the book opens empty [0000]: equilibrium strategy LBP_1			
at t_2 the book opens [0B00]			
	t_2 equilibrium strategy: MSP_{1,t_2}		
at t_3 the book opens empty [0000]			
NT_{t_3}	0	$\{v-\tau, v+\tau\}$	1
	t_2 equilibrium strategy: NT_{t_2}		
at t_3 the book opens [0B00]			
MSP_{1,t_3}	$P_1 - \beta_{t_3} - TF = 10.500 - \beta_{t_3}$	$\{v-\tau, v+\tau\}$	0.750
NT_{t_3}	0	$\{v+\frac{1}{2}\tau, v+\tau\}$	0.250

Table F2: **3-Period Large Tick Market (LTM). Equilibrium Strategies at t_2.** This table shows how to derive the equilibrium order submission strategies at t_2 for the benchmark model which has no trading fees ($MF = TF = 0.00$) and for a support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D_{t_z}^{P_i}$, corresponds to the depth of the book at each price level at time t_z, $[L_{t_z}^{P_2}, L_{t_z}^{P_1}, L_{t_z}^{P_{-1}}, L_{t_z}^{P_{-2}}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1: LBP_1 and LBP_{-1} on the buy side and LSP_1 and LSP_{-1} on the sell side. At t_1 we only present buy equilibrium strategies, and to economize space, at t_2 we present the equilibrium strategies that are consistent with the states of the book derived from the sell equilibrium strategies at t_1, as the equilibrium strategies consistent with the states of the book derived from the buy side are perfectly symmetric. Given the equilibrium limit buy orders at t_1, the possible states of the books at the beginning of t_2 are: [00B0] following a LBP_{-1} and [0B00] following a LBP_1. Column 1 shows the Equilibrium Strategies at t_2, column 2 shows the corresponding payoffs, and columns 3 and 4 show the β thresholds and the order submission probabilities respectively.

Equilibrium Strategy	Payoff	β Threshold	Order Submission Probability
at t_1 the book opens empty [0000]: equilibrium strategy LBP_{-1}			
at t_2 the book opens [00B0]			
MSP_{-1,t_2}	$P_{-1} - \beta_{t_2} - TF = 9.500 - \beta_{t_2}$	$\{v - \tau, v - \frac{5}{6}\tau\}$	0.083
LSP_1	$(P_1 - \beta_{t_2} - MF)Pr(\theta_{t_2}^{LSP_1} \vert \Xi, L_{t_1}) = 2.625 - 0.250\beta_{t_2}$	$\{v - \frac{5}{6}\tau, v + \frac{1}{2}\tau\}$	0.667
LBP_1	$(\beta_{t_2} - P_1 - MF)Pr(\theta_{t_2}^{LBP_1} \vert \Xi, L_{t_1}) = -7.875 + 0.750\beta_{t_2}$	$\{v + \frac{1}{2}\tau, v + \tau\}$	0.250
at t_1 the book opens empty [0000]: equilibrium strategy LBP_1			
at t_2 the book opens [0B00]			
MSP_{1,t_2}	$P_1 - \beta_{t_2} - TF = 10.500 - \beta_{t_2}$	$\{v - \tau, v + \frac{1}{2}\tau\}$	0.750
NT_{t_2}	0	$\{v + \frac{1}{2}\tau, v + \tau\}$	0.250

Table F3: **3-Period Large Tick Market (LTM). Equilibrium Strategies at t_1.** This table shows how to derive the equilibrium order submission strategies at t_1 for the benchmark model which has no trading fees ($MF = TF = 0.00$) and for a support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D_{t_z}^{P_i}$, corresponds to the depth of the book at each price level at time t_z, $[L_{t_z}^{P_2}, L_{t_z}^{P_1}, L_{t_z}^{P_{-1}}, L_{t_z}^{P_{-2}}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1: LBP_1 and LBP_{-1} on the buy side and LSP_1 and LSP_{-1} on the sell side (column 1). Column 2 shows their payoffs, and columns 3 and 4 shows the β thresholds and the order submission probabilities respectively.

Equilibrium Strategy	Payoff	β Threshold	Order Submission Probability
at t_1 the book opens empty [0000]			
LSP_{-1}	$(P_{-1} - \beta_{t_1} - MF)Pr(\theta_{t_1}^{LSP_{-1}} \vert \Xi, L_{t_0}) = 8.906 - 0.938\beta_{t_1}$	$\{v - \tau, v - \frac{41}{300}\tau\}$	0.068
LSP_1	$(P_1 - \beta_{t_1} - MF)Pr(\theta_{t_1}^{LSP_1} \vert \Xi, L_{t_0}) = 2.625 - 0.250\beta_{t_1}$	$\{v - \frac{41}{300}\tau, v\}$	0.432
LBP_{-1}	$(\beta_{t_1} - P_{-1} - MF)Pr(\theta_{t_1}^{LBP_{-1}} \vert \Xi, L_{t_0}) = -2.375 + 0.250\beta_{t_1}$	$\{v, v + \frac{259}{300}\tau\}$	0.432
LBP_1	$(\beta_{t_1} - P_1 - MF)Pr(\theta_{t_1}^{LBP_1} \vert \Xi, L_{t_0}) = -9.844 + 0.938\beta_{t_1}$	$\{v + \frac{259}{300}\tau, v + \tau\}$	0.068

Table F4: 3-Period Large Tick Market (LTM) with HFTs. Equilibrium Strategies at t_3. This table shows how to derive the equilibrium order submission strategies at t_3 for the benchmark model with HFTs which has no trading fees ($MF = TF = 0.00$) and for the Investors' support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D_{t_z}^{P_i}$, corresponds to the depth of the book at each price level at time t_z, $[L_{t_z}^{P_2}, L_{t_z}^{P_1}, L_{t_z}^{P_{-1}}, L_{t_z}^{P_{-2}}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1: LBP_1 (followed by a MSP_1 from an HFT firm) and LBP_{-1} on the buy side, and LSP_1 and LSP_{-1} (followed by a MBP_{-1} from an HFT firm) on the sell side. Table F6 presents both the buy and the sell equilibrium strategies at t_1. However, as the equilibrium strategies consistent with the states of the book derived from the buy side are symmetric to those derived from the sell side, to economize space at t_2 we only present the buy equilibrium strategies at t_1. Given the equilibrium limit buy orders at t_1, the possible states of the books at the beginning of t_2 are: [00B0] following a LBP_{-1} and [0000] following a LBP_1 and a MSP_1 from an HFT firm. Given the equilibrium strategies at t_2 and therefore the possible states of the books at the beginning of t_3, this table shows the equilibrium strategies at t_3 (column 1), their payoffs (column 2), the β thresholds (column 3) and the order submission probabilities (column 4).

Equilibrium Strategy	**Payoff**	**β Threshold**	**Order Submission Probability**
at t_1 the book opens empty [0000]: equilibrium strategy LBP_{-1}			
at t_2 the book opens [00B0]			
t_2 equilibrium strategy: MSP_{-1,t_2}			
at t_3 the book opens empty [0000]			
$LSP_{-1} \longrightarrow$ HFT: MBP_{-1,t_3}	$P_{-1} - \beta_{t_3} - MF = 9.500 - \beta_{t_3}$	$\{v-\tau, v-\frac{1}{2}\tau\}$	0.250
NT_{t_3}	0	$\{v-\frac{1}{2}\tau, v+\frac{1}{2}\tau\}$	0.500
$LBP_1 \longrightarrow$ HFT: MSP_{1,t_3}	$\beta_{t_3} - P_1 - MF = -10.500 + \beta_{t_3}$	$\{v+\frac{1}{2}\tau, v+\tau\}$	0.250
t_2 equilibrium strategy: LSP_1			
at t_3 the book opens [0SB0]			
MSP_{-1,t_3}	$P_{-1} - \beta_{t_3} - TF = 9.500 - \beta_{t_3}$	$\{v-\tau, v-\frac{1}{2}\tau\}$	0.250
NT_{t_3}	0	$\{v-\frac{1}{2}\tau, v+\frac{1}{2}\tau\}$	0.500
MBP_{1,t_3}	$\beta_{t_3} - P_1 - TF = -10.500 + \beta_{t_3}$	$\{v+\frac{1}{2}\tau, v+\tau\}$	0.250
t_2 equilibrium strategy: $LBP_1 \longrightarrow$ HFT: $MSP_{1_{t_2}}$			
at t_3 the book opens [00B0]			
MSP_{-1,t_3}	$P_{-1} - \beta_{t_3} - TF = 9.500 - \beta_{t_3}$	$\{v-\tau, v-\frac{1}{2}\tau\}$	0.250
NT_{t_3}	0	$\{v-\frac{1}{2}\tau, v+\frac{1}{2}\tau\}$	0.500
$LBP_1 \longrightarrow$ HFT: MSP_{1,t_3}	$\beta_{t_3} - P_1 - MF = -10.500 + \beta_{t_3}$	$\{v+\frac{1}{2}\tau, v+\tau\}$	0.250
at t_1 the book opens empty [0000]: equilibrium strategy $LSP_{-1} \longrightarrow$ HFT: $MBP_{-1_{t_2}}$			
at t_2 the book opens empty [0000]			
t_2 equilibrium strategy: $LBP_1 \longrightarrow$ HFT: $MSP_{1_{t_2}}$			
at t_3 the book opens empty [0000]			
$LSP_{-1} \longrightarrow$ HFT: MBP_{-1,t_3}	$P_{-1} - \beta_{t_3} - MF = 9.500 - \beta_{t_3}$	$\{v-\tau, v-\frac{1}{2}\tau\}$	0.250
NT_{t_3}	0	$\{v-\frac{1}{2}\tau, v+\frac{1}{2}\tau\}$	0.500
$LBP_1 \longrightarrow$ HFT: MSP_{1,t_3}	$\beta_{t_3} - P_1 - MF = -10.500 + \beta_{t_3}$	$\{v+\frac{1}{2}\tau, v+\tau\}$	0.250
t_2 equilibrium strategy: LBP_{-1}			
at t_3 the book opens [00B0]			
MSP_{-1,t_3}	$P_{-1} - \beta_{t_3} - TF = 9.500 - \beta_{t_3}$	$\{v-\tau, v-\frac{1}{2}\tau\}$	0.250
NT_{t_3}	0	$\{v-\frac{1}{2}\tau, v+\frac{1}{2}\tau\}$	0.500
$LBP_1 \longrightarrow$ HFT: MSP_{1,t_3}	$\beta_{t_3} - P_1 - MF = -10.500 + \beta_{t_3}$	$\{v+\frac{1}{2}\tau, v+\tau\}$	0.250
t_2 equilibrium strategy: LSP_1			
at t_3 the book opens [0S00]			
$LSP_{-1} \longrightarrow$ HFT: MBP_{-1,t_3}	$\beta_{t_3} - P_{-1} - MF = 9.500 - \beta_{t_3}$	$\{v-\tau, v-\frac{1}{2}\tau\}$	0.250
NT_{t_3}	0	$\{v-\frac{1}{2}\tau,\ v+\frac{1}{2}\tau\}$	0.500
MBP_{1,t_3}	$\beta_{t_3} - P_1 - TF = -10.500 + \beta_{t_3}$	$\{v+\frac{1}{2}\tau, v+\tau\}$	0.250
t_2 equilibrium strategy: $LSP_{-1} \longrightarrow$ HFT: $MBP_{-1_{t_2}}$			
at t_3 the book opens empty [0000]			
$LSP_{-1} \longrightarrow$ HFT: MBP_{-1,t_3}	$P_{-1} - \beta_{t_3} - MF = 9.500 - \beta_{t_3}$	$\{v-\tau, v-\frac{1}{2}\tau\}$	0.250
NT_{t_3}	0	$\{v-\frac{1}{2}\tau, v+\frac{1}{2}\tau\}$	0.500
$LBP_1 \longrightarrow$ HFT: MSP_{1,t_3}	$\beta_{t_3} - P_1 - MF = -10.500 + \beta_{t_3}$	$\{v+\frac{1}{2}\tau, v+\tau\}$	0.250

Table F5: 3-Period Large Tick Market (LTM) with HFTs. Equilibrium Strategies at t_2. This table shows how to derive the equilibrium order submission strategies at t_2 for the benchmark model with HFT which has no trading fees ($MF = TF = 0.00$) and for an investors' support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D_{t_z}^{P_i}$, corresponds to the depth of the book at each price level at time t_z, $[L_{t_z}^{P_2}, L_{t_z}^{P_1}, L_{t_z}^{P_{-1}}, L_{t_z}^{P_{-2}}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1: LBP_1 (followed by a MSP_1 from an HFT firm) and LBP_{-1} on the buy side, and LSP_1 and LSP_{-1} (followed by a MBP_{-1} from an HFT firm) on the sell side. At t_1 Table F6 presents both the buy and the sell equilibrium strategies. However, as the equilibrium strategies consistent with the states of the book derived from the buy side are symmetric to those derived from the sell side, to economize space at t_2 we only present the equilibrium strategies that are consistent with the states of the book derived from the sell equilibrium strategies at t_1. Given the equilibrium limit buy orders at t_1, the possible states of the books at the beginning of t_2 are: [00B0] following a LBP_{-1} and [0000] following a LBP_1 and a MSP_1 from an HFT firm. Column 1 shows the Equilibrium Strategies at t_2, column 2 shows the corresponding payoffs, and columns 3 and 4 show the β thresholds and the order submission probabilities respectively. We present the β Thresholds and the Order Submission Probabilities only for the regular investors; HFT firms have $\beta = 1$ and take profitable liquidity offered by aggressive orders with probability 1.

Equilibrium Strategy	**Payoff**	**β Threshold**	**Order Submission Probability**
at t_1 the book opens empty [0000]: equilibrium strategy LBP_{-1}			
at t_2 the book opens [00B0]			
MSP_{-1,t_2}	$P_{-1} - \beta_{t_2} - MF = 9.500 - \beta_{t_2}$	$\{v - \frac{1}{2}\tau, v - \frac{5}{6}\tau\}$	0.083
LSP_1	$(P_1 - \beta_{t_2} - MF)Pr(\theta_{t_2}^{LSP_1} \vert \Xi, L_{t_1}) = 2.625 - 0.250\beta_{t_2}$	$\{v - \frac{5}{6}\tau, v + \frac{1}{2}\tau\}$	0.667
$LBP_1 \longrightarrow HFT: MSP_{1,t_2}$	$(\beta_{t_2} - P_1 - TF) = -10.500 + \beta_{t_2}$	$\{v + \frac{1}{2}\tau, v + \tau\}$	0.250
at t_1 the book opens empty [0000]: equilibrium strategy $LBP_1 \longrightarrow$ HFT: MSP_{1,t_2}			
at t_2 the book opens empty [0000]			
$LSP_{-1} \longrightarrow HFT: MBP_{-1,t_2}$	$(P_{-1} - \beta_{t_1} - MF) = 9.500 - \beta_{t_1}$	$\{v - \tau, v - \frac{5}{6}\tau\}$	0.083
LSP_1	$(P_1 - \beta_{t_1} - MF)Pr(\theta_{t_1}^{LSP_1} \vert \Xi, L_{t_1}) = 2.625 - 0.250\beta_{t_1}$	$\{v - \frac{5}{6}\tau, v\}$	0.417
LBP_{-1}	$(\beta_{t_1} - P_{-1} - MF)Pr(\theta_{t_1}^{LBP_{-1}} \vert \Xi, L_{t_1}) = -2.375 + 0.250\beta_{t_1}$	$\{v, v + \frac{5}{6}\tau\}$	0.417
$LBP_1 \longrightarrow HFT: MSP_{1,t_2}$	$(\beta_{t_1} - P_1 - MF) = -10.500 + \beta_{t_1}$	$\{v + \frac{5}{6}\tau, v + \tau\}$	0.083

Table F6: 3-Period Large Tick Market (LTM) with HFTs. Equilibrium Strategies at t_1. This table shows how to derive the equilibrium order submission strategies at t_1 for the benchmark model with HFTs which has no trading fees ($MF = TF = 0.00$) and for an investors' support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D_{t_z}^{P_i}$, corresponds to the depth of the book at each price level at time t_z, $[L_{t_z}^{P_2}, L_{t_z}^{P_1}, L_{t_z}^{P_{-1}}, L_{t_z}^{P_{-2}}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1: LBP_1 (followed by a MSP_1 from an HFT firm) and LBP_{-1} on the buy side, and LSP_1 and LSP_{-1} (followed by a MBP_{-1} from an HFT firm) on the sell side (column 1). Column 2 shows their payoffs, and columns 3 and 4 shows the β thresholds and the order submission probabilities respectively. We present the β Thresholds and the Order Submission Probabilities only for the regular investors; HFT firms have $\beta = 1$ and take profitable liquidity offered by aggressive orders with probability 1.

Equilibrium Strategy	**Payoff**	**β Threshold**	**Order Submission Probability**
at t_1 the book opens empty [0000]			
$LSP_{-1} \longrightarrow HFT: MBP_{-1,t_1}$	$(P_{-1} - \beta_{t_1} - MF) = 9.500 - \beta_{t_1}$	$\{v - \tau, v - \frac{5}{6}\tau\}$	0.083
LSP_1	$(P_1 - \beta_{t_1} - MF)Pr(\theta_{t_1}^{LSP_1} \vert \Xi, L_{t_0}) = 2.625 - 0.250\beta_{t_1}$	$\{v - \frac{5}{6}\tau, v\}$	0.417
LBP_{-1}	$(\beta_{t_1} - P_{-1} - MF)Pr(\theta_{t_1}^{LBP_{-1}} \vert \Xi, L_{t_0}) = -2.375 + 0.250\beta_{t_1}$	$\{v, v + \frac{5}{6}\tau\}$	0.417
$LBP_1 \longrightarrow HFT: MSP_{1,t_1}$	$(\beta_{t_1} - P_1 - MF) = -10.500 + \beta_{t_1}$	$\{v + \frac{5}{6}\tau, v + \tau\}$	0.083