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Lectures on Macroeconomics

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is not particularly attractive as a description of preferences and is not recommended for general use. A nondegenerate steady state, when individual tastes differ, can also be achieved by assuming that agents have finite lives; this is a more plausible avenue which we develop in chapter 3.

Appendix A: Ruling Out Explosive Paths in the Ramsey Model

To show that the saddle point path DD in figure 2.2 is the optimal path, suppose that the initial capital stock is k_0, $0 < k_0 < k^*$. Consider any trajectory that starts above point D, at D', say. This path implies that the economy reaches zero capital in finite time. The proof turns on the fact that on such a path d^2k/dt^2 eventually becomes negative. Differentiating (2) gives

$$\frac{d^2k}{dt^2} = [f'(k) - n]\left(\frac{dk}{dt}\right) - \frac{dc}{dt} < 0, \qquad \text{as} \quad \frac{dc}{dt} > 0, \quad f'(k) - n > 0.$$

Thus $k_t = k_0 + \int_0^t (dk_v/dv)\, dv$ will reach zero in finite time.

Note that c is rising on the path starting at D' all the time until it hits the axis at point B. But when the path reaches B, k is zero, and the economy has to move to the origin. Thus c has to jump from a positive value to zero. But such a jump violates the necessary condition (7′), and it thus cannot have been optimal to start at D'.

Consider, alternatively, a trajectory starting below D, for example, at D''. This path converges asymptotically to A. But such a path violates the transversality condition. At points close to A, k is approximately constant, whereas from (7′) and $k > k_g$,

$$\frac{du'(c)/dt}{u'(c)} = \theta + n - f'(k) > \theta.$$

Thus as t tends to infinity and the trajectory approaches A, the transversality condition is violated.

Similar arguments apply if the initial capital stock is larger than k^*. It follows that the saddle point path DD is the unique path that satisfies conditions (2), (7′), and (8).

Appendix B: Local Behavior of Capital around the Steady State in the Ramsey Model

The characteristic equation associated with equation (15) is

$$x^2 - \theta x - \beta = 0.$$

It has two roots:

$$\lambda \equiv \frac{\theta - \sqrt{\theta^2 + 4\beta}}{2} < 0$$

and

$$\mu \equiv \frac{\theta + \sqrt{\theta^2 + 4\beta}}{2} > 0.$$

Thus paths that satisfy equation (15) are given by

$$k_t - k^* = c_0 \exp(\lambda t) + c_1 \exp(\mu t),$$

where c_0 and c_1 are arbitrary constants.

As k_0 is given from history, c_0 and c_1 must satisfy

$$k_0 - k^* = c_0 \exp(0) + c_1 \exp(0) = c_0 + c_1.$$

In addition, as μ is positive, c_1 must be equal to zero for k to converge to k^*. Thus $c_1 = 0$ and $c_0 = k_0 - k^*$. This implies in turn that

$$k_t = k^* + (k_0 - k^*) \exp(\lambda t).$$

Appendix C: Command Optimum and Decentralized Equilibrium in the Open Economy Model

We show here the equivalence of the command optimum and the decentralized competitive equilibrium in the open economy model of section 2.4. For notational simplicity, we assume that there are as many firms as families so that the same symbol denotes the ratio of a variable per capita or per firm.

The structure of the economy is the following: Firms rent labor services in the labor market but own the capital stock; they finance investment through retained earnings. Families supply labor services and own the firms, receiving profits net of investment expenses. They allocate their income between consumption and saving, where saving takes the form of lending to the rest of the world.[57]

Value Maximization by Firms

For simplicity, we do not explicitly model the labor market. Labor is supplied inelastically so that labor market equilibrium implies that each firm hires one worker, paying wages of $\{w_t\}$ $t = [0, \infty)$. The decision problem of a representative firm at time zero is then to choose the time path of investment that maximizes the present discounted value of cash flows:

$$\max V_0 = \int_0^\infty \left\{ f(k_t) - i_t\left[1 + T\left(\frac{i_t}{k_t}\right)\right] - w_t \right\} \exp(-\theta t)\, dt \tag{C1}$$

subject to $dk_t/dt = i_t$ and the same technology as the central planner.

By letting $q_t \exp(-\theta t)$ be the Lagrange multiplier associated with the capital accumulation equation and setting up a present value Hamiltonian, the first-order conditions lead to equations identical to (43), (37′), and (44). Firms invest until the marginal cost of investment is equal to the shadow value of installed capital, q. This shadow value is itself equal to the present discounted value of future marginal products. Firms choose the same path of investment and capital accumulation as the central planner.

Given our assumption that firms finance investment through retained earnings, dividends paid by firms are therefore equal to net cash flows[58]

$$\pi_t = f(k_t) - i_t\left[1 + T\left(\frac{i_t}{k_t}\right)\right] - w_t. \tag{C2}$$

Utility Maximization by Families

Each family supplies one unit of labor inelastically, receiving wage w_t and dividends π_t. Its only decision problem is to choose a path of consumption that maximizes

$$U_0 = \int_0^\infty u(c_t)\exp(-\theta t)\,dt. \tag{C3}$$

It can borrow and lend on the world market at the rate θ. The dynamic budget constraint is therefore

$$\frac{db_t}{dt} = c_t + \theta b_t - \pi_t - w_t. \tag{C4}$$

To this we add the NPG condition:

$$\lim_{t\to\infty} b_t \exp(-\theta t) = 0. \tag{C5}$$

The solution to this maximization problem is given by

$$c_t = c_0 = \theta\int_0^\infty (\pi_t + w_t)\exp(-\delta t)\,dt. \tag{C6}$$

Replacing π_t by its value from (C2) in (C6) gives the same path of consumption as equation (42). Families will choose the same path of consumption as the central planner.

Appendix D: Saddle Point Equilibrium in the Linearized (k, q) System

Equation (47) linearizes the dynamic system that describes the behavior of q and k around the steady state values. The solution to such a linear system is given by

$$k_t - k^* = c_{11}\exp(y_1 t) + c_{12}\exp(y_2 t), \tag{D1}$$

$$q_t - 1 = c_{21}\exp(y_1 t) + c_{22}\exp(y_2 t),$$

where y_1 and y_2 are the roots of the characteristic equation associated with (47), namely,

$$\begin{vmatrix} 0 - y & k^*\varphi'(1) \\ -f''(k^*) & \theta - y \end{vmatrix} = 0,$$

and where $[c_{11}\ c_{21}]$ and $[c_{12}\ c_{22}]$ are eigenvectors associated with each of these two roots.

The roots are given by

$$y = \frac{\theta \pm \sqrt{\theta^2 - 4f''(k^*)k^*\varphi'(1)}}{2}.$$

Both roots are real, with one root negative and the other positive. The positive root exceeds θ.

Denote the negative root by y_1. The eigenvector associated with y_1 is given by

$$\begin{bmatrix} -y_1 & k^*\varphi'(1) \\ -f''(k^*) & \theta - y_1 \end{bmatrix}\begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = 0, \tag{D2}$$

so that $c_{21} = \{y_1[k^*\varphi'(1)]^{-1}\}c_{11}$. Examining (A7), we see that for the path that converges to $(k^*, 1)$, both c_{12} and c_{22} must be equal to zero. (Zero is always an eigenvector.)

To calculate the constants c_{11} and c_{21}, note that at time zero, the first row of (D1) is

$$k_0 - k^* = c_{11}. \tag{D3}$$

Replacing the c's by their values in (D1) gives the converging path for k and q.

On all paths other than the converging path, c_{12} and/or c_{22} are different from zero. Thus q and k eventually increase at rate y_2. This implies that qk eventually increases at rate no less than y_2, which is itself greater than θ. Thus they all violate the transversality condition (39).

Of course, the proof that the transversality condition is violated on all but the saddle point path in the linearized system does not establish the fact that the paths of the original system that are not saddle point paths explode at a rate greater than θ. A complete proof requires a characterization of the dynamics of the original nonlinear system along the lines of the proof presented in appendix A.

Problems

1. *The Solow growth model. (This follows Solow 1956.)*

(a) Consider an economy with a population growth rate equal to n, with constant returns to scale in production, and in which individuals save a constant fraction, s, of their income. Show that the differential equation describing the behavior of the capital stock per capita is given by

$$\frac{dk}{dt} = sf(k) - nk,$$

where $f(\cdot)$ is the production function per capita and s is the savings rate.
(b) Characterize the steady state capital stock per capita in this model.
(c) Examine the stability of the system, and characterize the adjustment of the capital stock toward its steady state.

(d) Can a constant saving rate along the path of adjustment be consistent with intertemporal utility maximization by infinitely long-lived individuals?
(e) Assume that factor markets are competitive. Show that the savings rate that leads to the golden rule capital stock is equal to the share of capital in production. Explain.

2. *Growth with exogenous technological progress.*

Suppose that, in a Ramsey economy, production is given (as in note 13) by the function

$$Y_t = F(K_t, \exp(\phi t)N_t),$$

where ϕ is the constant and exogenous rate of technical progress. Assume that the population grows at rate n and that the utility function is of constant relative risk aversion form, with a coefficient of relative risk aversion equal to γ.
(a) Derive and interpret the modified golden rule condition in this case.
(b) Characterize the dynamics of consumption and capital accumulation.
(c) Suppose that the economy is in steady state and that ϕ decreases permanently and unexpectedly. Describe the dynamic adjustment of the economy to this adverse supply shock.

3. *Optimal consumption with exponential utility.*

Consider a family, growing at rate n and with discount rate θ, that faces a given path of future wages and interest rates and has a constant absolute risk aversion utility function, with a coefficient of risk aversion α. Solve for the path of consumption, as is done in the text for the CRRA utility function.

4. *Government spending in the Ramsey model.*

(a) In the Ramsey model, suppose that the government unexpectedly increases government spending, raising it from a base level g_0 to the level g_1 (per capita in both cases), starting from steady state. Analyze the effects of this increase on the paths of consumption and capital accumulation.
Note: You may want to use the equivalence between the command and market solutions and treat the increase in g as a negative additive productivity shock.
(b) Do the same exercise, assuming that the economy is not initially in steady state. Characterize the dynamic effects when utility is of the CARA form. Explain.
(c) Suppose, instead, that the increase in government spending is announced at time t_0 to take place at time t_1, with $t_1 > t_0$. Characterize the dynamic effects on consumption and capital accumulation from t_0.
Note: Phase diagrams are convenient to use when characterizing the effects of such anticipated changes. Note that between t_0 and t_1 the equations of motion are given by the dynamic system with $g = g_0$, and that after t_1 the equations of motion are given by the dynamic system with $g = g_1$. Note further that c cannot jump anticipatedly at time t_1. Note finally that k at time t_0 is given and that the system must converge to the new equilibrium. Show that these conditions uniquely define the path of adjustment. (Abel 1981 characterizes the effects of anticipated or

temporary changes in taxation on investment within the q theory using such phase diagrams.)

5. *Savings and investment with costs of adjustment in a closed economy. (This follows Abel and Blanchard 1983.)*

Assume that there are costs of adjusting the capital stock, as in section 2.4, but that the economy is closed. Derive the optimal paths of consumption and capital accumulation in this case and provide an explanation of the difference between the Euler equation for this case and equation (7).

6. *Foreign debt and trade surpluses.*

(a) Using the relevant budget constraint, show that b_0, the initial value of external debt, is equal to the present value of net exports, provided an NPG condition is satisfied.
(b) Suppose that for some period of time a country's external debt is growing more rapidly than at the rate $r - n$. What can you conclude about the likelihood that the NPG condition will be violated in the long run? What then is the relevance of the NPG condition?

7. Suppose that in a closed economy there is an unexpected permanent reduction in the efficiency of production, represented in the symbols in the text as an increase in z_0. Assuming that the economy started in a steady state, derive and explain its optimal dynamic adjustment toward the new steady state.

8. *Growth with increasing returns, I.*

Consider an economy with the production function

$$Y = K^{a+b}N^{1-a}, \qquad b > 0, a + b < 1$$

so that there are increasing returns to scale but decreasing returns to capital given labor. Population is growing at the rate n, and there is no depreciation.
(a) Show that it is possible for capital, output, and consumption all to grow at the same rate g. This is known as *balanced growth*. Derive the balanced growth rate g, and explain its dependence on a, b, and n.
(b) Suppose that the felicity function for the representative family is

$$u(c_t) = \ln c_t$$

and that the family has a constant discount rate θ.

Assuming that the economy converges to a balanced growth path, characterize the steady state marginal product of capital. Compare it to the modified golden rule level that would obtain under constant returns (i.e., with $b = 0$). Explain the difference.

9. *Growth with increasing returns, II. (This follows Rebelo 1987.)*

Consider the following economy: Population is constant and normalized to unity, and the representative individual maximizes

$$\int_0^\infty U(C) \exp(-\theta t)\, dt.$$

K is the capital stock in the economy and can be used either to produce consumption goods or new capital goods. Let x, $0 \leqslant x \leqslant 1$, be the proportion of capital used in the production of consumption goods. The two production functions for consumption and investment goods are given by

$$C = F(xK),$$

$$F(0) = 0, \qquad F'(\cdot) > 0, F''(\cdot) < 0.$$

$dK/dt = I = B(1 - x)K$; B is a positive constant. Capital does not depreciate.
(a) What is the maximum growth rate of capital in this economy? What is the associated level of consumption?
(b) Derive the first-order conditions associated with this maximization problem. Interpret them. Give, in particular, an interpretation of the Lagrange multipliers and costate variables as shadow prices.
(c) Assume that $F(xK) = A(xK)^a$, where $0 < a < 1$, and that $U(C) = \ln(C)$. Show that if the economy converges to a balanced growth path, the rate of growth of consumption is given by $a(B - \theta)$. Explain in words.
What happens to the relative price of capital goods in terms of consumption goods along the balanced growth path?
(d) Contrast your results with those obtained in the conventional Ramsey model. Explain why they differ.
(e) How does this model do in terms of explaining the basic facts of growth as laid out by Kaldor and Solow, and summarized in chapter 1? What is the relation of consumption to income along the balanced growth path? What is the relation of output to capital? (Be careful about how you define capital—value or volume—here.)

Notes

1. In chapter 3 we show that people who have finite lives may still act as if they in effect had infinite lives.

2. Frank Ramsey was a Cambridge, England, mathematician and logician who died at the age of 26. His genius is evidenced by the fact that he had written three classic articles in economics by the age at which many economists are contemplating leaving graduate school. J. M. Keynes (1930) eulogizes Ramsey.

3. If depreciation is exponential at the rate λ, then gross output is $Y + \lambda K = F(K, N) + \lambda K = G(K, N)$. If $F(K, N)$ is degree one homogeneous, so is $G(K, N)$.

4. An alternative plausible formulation is the so-called Benthamite welfare function in which the felicity function becomes $N_t u(c_t)$ so that the number of family members receiving the given utility level is taken into account. Recognizing that $N_t = N_0 e^{nt}$, we see that the Benthamite formulation is equivalent to reducing the rate of time preference to $(\theta - n)$ because the larger size of the family at later dates in effect

increases the weight given to the utility of the representative individual in a later generation.

In assuming that $\theta > 0$, we depart from Ramsey who, interpreting the maximization problem as the problem solved by a central planner, argued that there was no ethical case for discounting the future.

5. Ordinary calculus optimization methods have to be augmented to handle the presence of a time derivative in constraint (2). Intriligator (1971) provides an introduction to intertemporal optimization methods.

6. A warning is in order here. First, under weaker assumptions than those made in the text, for example, a linear production function or no discounting, an optimum may not exist. Even if an optimum does exist, the transversality condition, equation (8), may not be necessary. But if one is ready to set sufficiently strong conditions for the maximization problem, these problems can usually safely be ignored. For a more careful statement and further discussion, see Shell (1969) and Benveniste and Scheinkman (1982).

7. Note from the formulation of the central planner's problem that it is implicitly assumed that capital can be consumed.

8. We emphasize again that as intuitive as this argument for the transversality condition is, there are infinite horizon problems in which the transversality condition is not necessary for the optimal path. See Shell (1969) and Michel (1982).

9. To show that the utility function converges to the logarithmic function as γ tends to unity, use L'Hospital's rule.

10. On the basic measures of risk aversion, see J. Pratt in Diamond and Rothschild (1978); see also the following articles in Diamond and Rothschild by Yaari and by Rothschild and Stiglitz.

Behavior toward risk and the degree of substitution between consumption at different times are conceptually two different issues. Under the assumption that the von Neumann-Morgenstern utility integral is additively separable over time, however, the two depend only on the curvature of the instantaneous utility function and are thus directly related. See chapter 6 for further discussion.

11. In steady state, with $dk/dt = 0$, we have from (2),

$$c^* = f(k^*) - nk^*.$$

Maximization of c^* with respect to k^* gives the golden rule, that the marginal product of capital (or interest rate) is equal to the growth rate of population.

12. We freely interchange the marginal product and interest rates. We show later that in the decentralized Ramsey economy, the two are indeed equal.

13. The result that the steady state interest rate does not depend on the utility function can, however, be easily overturned. If labor-augmenting (Harrod-neutral) technical progress is taking place at the rate μ, so that

$$Y_t = F[K_t, \exp(\mu t)N_t]$$

and if the utility function is of the CRRA class, then the modified golden rule condition becomes $f'(k^*) = \theta + \sigma\mu + n$. [In this case k^* is the ratio of capital to effective labor, i.e., $K_t/\exp(\mu t)N_t$, and the steady state is one in which consumption per capita is growing at the rate μ.]

14. The analysis can also be undertaken in (k, λ) space, using the first-order condition (6).

15. The behavior of consumption on the horizontal axis, where $c = 0$, depends on the value of the instantaneous elasticity of substitution $\sigma(c)$ for $c = 0$. Equation (7″) implies that

$$\frac{dc}{dt} = \sigma(c)[f'(k) - \theta - n]c.$$

If $\sigma^{-1}(0)$ is not zero, then $dc/dt = 0$ when $c = 0$. We assume this to be the case. If the condition is not met, one must examine the behavior of $c\sigma(c)$ at $c = 0$.

16. Throughout the book we will encounter phase diagrams in which there is only one convergent path. Although we will often simply assume that the economy proceeds on this converging path, an argument must be made in each case that the converging path is the only one that satisfies the conditions of the problem. As we will see in chapter 5, there are cases in which we cannot rule out some of the diverging paths.

17. Changes in f'' and θ affect both the rate of convergence to the steady state and the steady state capital stock itself.

18. The condition that the rental rate on capital is equal to the interest rate is special to this one-good model. If the relative price of capital, p_k, could vary, asset market equilibrium would ensure that the expected rate of return from holding capital would be equal to the interest rate. The rate of return from holding capital is the rental rate, r_k plus any capital gains on capital minus depreciation, all expressed relative to the price of the capital:

$$\text{rate of return} = \frac{r_k + (dp_k/dt) - \delta p_k}{p_k} = \text{real interest rate},$$

where δ is the rate of depreciation. In the single-good model, p_k is identically one, so there are no changes in the relative price of capital, and we are assuming that δ is zero; accordingly, the rate of return on capital is r_k, which is equal to the interest rate. (We are implicitly assuming that the economy never specializes completely; if it did not save at all, the relative price of capital goods could be less than one; if it did not consume at all, the relative price of capital could exceed one.)

19. For notational convenience we shall assume that there is just one family and one firm, both acting competitively.

20. There are many alternative ways of describing the decentralized economy. For example, firms can own the capital and finance investment by either borrowing or issuing equity. Or, instead of operating with spot factor markets, the economy may

operate in the Arrow-Debreu complete market framework in which markets for current and all future commodities, including services, are open at the beginning of time; all contracts are made then, and the rest of history merely executes these contracts. Under perfect foresight, all these economies will have the same allocation of resources.

21. We limit ourselves in what follows to paths of wages and rental rates such that the following condition is satisfied:

$$\lim_{t\to\infty} \exp\left[-\int_0^t (r_v - n)\, dv\right] = 0.$$

This condition says, roughly, that asymptotically the interest rate must exceed the rate of population growth. We will show that the equilibrium path indeed satisfies this condition. A complete argument would show that if this condition is not satisfied, there is no equilibrium. See note 25 below for further elaboration.

22. In the present model, in which all families are the same, they will in equilibrium have the same wealth position and hold the same fraction of the capital stock. Since the aggregate capital stock must be positive, each family will, in equilibrium, have positive wealth. This is, however, a characteristic of equilibrium, not a constraint that should be imposed a priori on the maximization problem of each family. In an economy with heterogeneous families, or families with different paths of labor income, positive aggregate capital may coexist with temporary borrowing by some families.

23. Charles Ponzi, one of Boston's sons, made a quick fortune in the 1920s using chain letters. He was sent to prison and died poor.

24. This raises the question of how the no-Ponzi-game condition is actually enforced. The fact that parents cannot, for the most part, leave negative bequests to their children implies that family debt cannot increase exponentially. It may in fact impose a stronger restriction on borrowing than the no-Ponzi-game condition used here.

25. Following up on note 21, there is one loose end in our proof of equivalence, which we now tie up. We have restricted ourselves to paths where the interest rate exceeds asymptotically the population growth rate. Given this restriction, we showed that there is an equilibrium path, which is the same as the central planning one, so that r converges asymptotically to $n + \theta$. We now need to show that paths on which the interest rate is asymptotically less than n, cannot be equilibria. To see why, rewrite the budget constraint facing the family as

$$\frac{da_t}{dt} = (r_t - n)a_t + (c_t - w_t).$$

Consider then two paths of consumption, which have the same level of consumption after some time T, so that $c_t - w_t$ is the same on both paths after T. Then, if $r_t - n$ is asymptotically negative, both paths will lead to the same asymptotic value of a (the same level of net indebtedness if a is negative). If one path satisfies the no-Ponzi-game condition, so will the other. But this implies that the family will always want to have very high (possibly infinite) consumption until time T. This cannot be an equilibrium.

26. We consider endogenous government spending in chapter 11.

27. Government spending, for instance, on education, might substitute for private spending, in which case the utility function would have to be amended appropriately. Similarly, government spending on defense and public safety might contribute to the economy's productive capacity, but we do not model any such effects.

28. The dynamics of investment and savings in a closed economy with adjustment costs are studied in Abel and Blanchard (1983).

29. Blanchard (1983), Fischer and Frenkel (1972), and Svensson (1984) have used similar models to examine the dynamics of foreign debt and the current account.

30. Investment decisions based on adjustment costs have been modeled by Abel (1981), Eisner and Strotz (1963), Lucas (1967), and Tobin (1969). Our specification is that of Hayashi (1982).

31. The conditions specified after equation (31) ensure the properties of the installation cost function $iT(i/k)$. Note that, in practice, when capital depreciates, the costs of small rates of disinvestment, which can take place through depreciation, are likely to be very small or zero.

Instead of defining both a production and an installation cost function, we could have defined a 'net' production function that gives output available for consumption or export, $H(K, N, I)$. This is the approach taken, for example, by Lucas (1967). In our case $H(K, N, I) = F(K, N) - I[1 + T(I/K)]$, where uppercase letters are total amounts of corresponding per capita variables. The function $H(\cdot)$ has constant returns to scale if $F(\cdot)$ does.

32. If the world interest rate had differed from the rate of time preference, the country would either accumulate or decumulate forever. This follows from the Euler equation in the absence of population growth, which from section 2.2 will give $[du'(c_t)/dt]/u'(c_t) = \theta - r$, where r is the interest rate. If the country accumulates forever because $\theta < r$, then it eventually becomes a large economy and begins to affect the world interest rate; if $\theta > r$, then the country runs its wealth down as far as it can. To avoid these difficulties, we set $\theta = r$. We could also obtain convergence to a steady state if we specified a time path for the world interest rate that converges to θ, rather than always being equal to θ. We assume $r = \theta$ for simplicity.

33. We state the NPG condition as an equality. We could again state it as an inequality, requiring the present discounted value of debt to be nonnegative. But if marginal utility is positive, the central planner will not want to accumulate increasing claims on the rest of the world forever. Thus the NPG condition will hold with equality.

34. Defining the costate variable on (31) as $\mu_t q_t \exp(-\theta t)$ rather than as a single variable is a matter of convenience, as will become clear later when we show that q plays a key role in determining investment.

35. Note that because of the equality of the interest rate and the subjective discount rate, the marginal propensity to consume out of wealth is equal to θ independently of the form of the felicity function.

36. Note that given constant μ_t, equation (39) implies that $\lim q_t k_t \exp(-\theta t) = 0$ as t goes to ∞. This is, however, not the same as $\lim q_t \exp(-\theta t) = 0$ as t goes to ∞, which is the condition needed to derive (44). To derive (44), one must characterize the phase diagram associated with equations (37′) and (43) and show that the only path that satifies these equations and the transversality condition (39) is a path where both k and q tend to k^* and q^*, respectively, so that $\lim q_t \exp(-\theta t) = 0$ as t goes to ∞.

37. This way of thinking about the investment decision was developed by Tobin. For that reason, q is often called Tobin's q. See Hayashi (1982) for a discussion of the relation of q to its empirical counterparts; in particular, Hayashi discusses the conditions under which average q, as reflected, say, in the stock market valuation of a firm, is equal to marginal q, the shadow value of an additional unit of installed capital. Marginal and average q are equal, leaving aside tax issues, if the firm's production function and the adjustment cost function $iT(\cdot)$ are each first-degree homogeneous and firms operate in competitive markets. Under those assumptions one would expect a tight relation between the market valuation of firms and their investment decisions. Empirically, although average q and investment rates are indeed correlated, the relation is far from tight (see Hayashi 1982).

38. If there is population growth at the rate n, then q^* is given by $n = \phi(q^*)$ so that $q^* > 1$, and k^* is given by

$$\theta q^* = f'(k^*) - n^2 T'(n).$$

39. The restriction to local dynamics ensures that $dq/dt = 0$ is negatively sloped; away from the steady state there is no assurance that the slope of $dq/dt = 0$ is negative without imposing more conditions on the $T(\cdot)$ function. However, the restrictions imposed on $T(\cdot)$ are sufficient to ensure that there is a unique steady state in the neighborhood of which the $dq/dt = 0$ locus is negatively sloped.

40. In appendix D we show that the transversality condition suffices in the linearized system to rule out any divergent paths that satisfy the necessary conditions (47).

41. The current account always has present discounted value equal to zero when condition (32) is satisfied; it is only when the initial debt is zero that the same applies to the trade account.

42. Given the equivalence between the command optimum and the decentralized economy, the shocks can also be interpreted as taxes, where the government is using the proceeds of the taxes to finance government spending that does not affect the utility function, as in section 2.3.

43. This experiment raises the methodological issue of how unexpected changes can occur in a model in which there is perfect foresight. The correct way to analyze such changes would be to set up the maximizing problems of the central planner or economic agents explicitly as decision problems under uncertainty. This substantially complicates the analysis, and we defer this to chapter 6; we can think of the approach taken here as a shortcut in which the surprise is an event that was regarded as so unlikely as not to be taken into account up to the time it occurs.

44. If individuals dislike changes in the rate of consumption so that the felicity function is, for instance, $u(c, dc/dt)$, the reduction in z_1 would cause a smaller decline in consumption than in output initially; the country would in that case initially borrow abroad temporarily to cushion the shock of the reduction in the standard of living, and end up with permanently higher debt and lower consumption.

45. We briefly return in chapter 7 to the issue of nonseparability in the context of a discussion of labor supply.

As we shall see, however, the distinction between the felicity function and the discount factor becomes somewhat blurred when we allow for more general formulations of this discount factor.

46. The argument to this point does not eliminate the possibility that there is no steady state. The argument of this paragraph can be seen, however, to imply the existence of a steady state with $r = \theta_m$.

47. The no-Ponzi-game condition prevents the shortsighted from going further and further into debt.

48. Ramsey (1928) conjectured this result; it was proved by Becker (1980).

49. Note the similarity between the discussion here and that of the relationship between the world interest rate and rate of time preference of a small country in section 2.4.

50. Because the point we are about to make about the optimal program does not depend on the presence of $x(t)$ in the discount function, we omit that argument henceforth.

51. This result is due to Strotz (1956).

52. An example is $D(\cdot) = \max[0, A - \theta(t - s)]$.

53. See Elster (1979) and Schelling (1984) for more extensive discussion of how people do and should deal with inconsistencies. Issues of time consistency also arise in the context of games between agents or between agents and the government. We will study these in chapter 11.

54. There is no "correct" way to behave when tastes are dynamically inconsistent, for there is no way of knowing which is the right set of tastes: the title "Ulysses and the Sirens" (Elster 1979) refers to Ulysses's strategy of having himself tied to the mast to avoid succumbing to the Sirens' cry—but maybe the real Ulysses was the one who would have succumbed if the other Ulysses hadn't tied him to the mast.

55. Epstein and Hynes (1983) suggest an alternative specification, namely,

$$\int_0^\infty \exp\left[-\int_0^s u(c_v)\,dv\right] ds.$$

This specification has the same qualitative implication as Uzawa's but is more tractable analytically. Note that in this form there is no longer any distinction between the discount rate and the instantaneous felicity function.

56. Lucas and Stokey (1984) work with a model of this type.

57. Once again, there are many alternative ways of describing the decentralized economy. Firms could, instead, finance investment by issuing shares or by borrowing either abroad or domestically. The real allocation would be the same in all cases.

58. If investment is so high that net cash flows are negative, the firm is, in effect, issuing equity by paying a negative dividend, that is, making a call on stockholders for cash.

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