Strategies and Interactive Beliefs in Dynamic Games

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Introduction

- *Topic:* formal analysis of strategic thinking in dynamic games, backward and forward induction reasoning.
- *Tools:* epistemic (type) structures where states determine truth value of conditional statements.
- *Innovation:* we use **only epistemic conditionals** of the form: "if **i** learned **h** he would believe **E** with probability **p**"; state=(actual actions,epistemic conditionals).
- *Motivation:* strategies cannot be (irreversibly) chosen, nor observed, they are just beliefs on own contingent choices (objective *vs* subjective strategies).
- *Focus:* generic perfect information games.

Road map

1. Preview on conditionals and BI: semi-formal analysis of an example
2. Setup: PI games and epistemic structures
3. Strategies as epistemic constructs: some results
4. Conclusions

Preview: two kinds of type structures

Variants of Battigalli-Siniscalchi JET99: types are implicit representations of hierarchies of conditional beliefs. But, what are first-order beliefs about?

- $\mathbf{H}$=histories, $\mathbf{Z}$=paths (terminal nodes), $\mathbf{S_i}$,$\mathbf{S_{-i}}$,$\mathbf{S}$=strategies
- "Traditional" $\mathbf{S}$-*based structures*: $(\bar{\mathbf{T}}_\mathbf{i}, \bar{\beta}_\mathbf{i})_{\mathbf{i}\in\mathbf{I}}$, states in $\mathbf{\Omega = S \times T}$, $(\mathbf{s_i}, \mathbf{t_i})$=state of $\mathbf{i}$ ($\mathbf{s_i}$ is "objective"), $\mathcal{H}_\mathbf{i}$=conditioning events,

$$\begin{aligned} \mathcal{H}_\mathbf{i} &= \{\mathbf{S_{-i}(h)} \times \bar{\mathbf{T}}_{-\mathbf{i}} : \mathbf{h} \in \mathbf{H}\} \\ \bar{\beta}_\mathbf{i} &: \bar{\mathbf{T}}_\mathbf{i} \to \mathbf{\Delta}^{\mathcal{H}_\mathbf{i}}(\mathbf{S_{-i}} \times \bar{\mathbf{T}}_{-\mathbf{i}}) \subset [\mathbf{\Delta}(\mathbf{S_{-i}} \times \bar{\mathbf{T}}_{-\mathbf{i}})]^{\mathcal{H}_\mathbf{i}} \end{aligned}$$

- Our $\mathbf{Z}$-based structures: $(\mathbf{T_i}, \beta_\mathbf{i})_{\mathbf{i}\in\mathbf{I}}$, states in $\mathbf{\Omega = Z \times T}$, $\mathbf{t_i}$=epist.state of $\mathbf{i}$ (beliefs about others+plan),

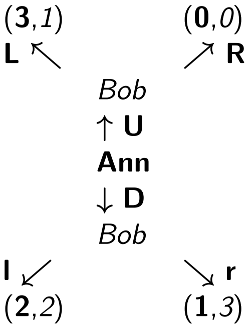
$$\begin{aligned} \mathbf{H} &\approx \{\mathbf{Z(h)} \times \mathbf{T_{-i}} : \mathbf{h} \in \mathbf{H}\} \\ \beta_\mathbf{i} &: \mathbf{T_i} \to \mathbf{\Delta}^\mathbf{H}(\mathbf{Z} \times \mathbf{T_{-i}}) \subset [\mathbf{\Delta}(\mathbf{Z} \times \mathbf{T_{-i}})]^\mathbf{H} \end{aligned}$$

Preview on conditionals and BI: an example

Stackelberg mini-game

outputs:	low (left)	high (right)
high (up)	**3**,*1*	**0**,*0*
low (down)	**2**,*2*	**1**,*3*

tree:



Preview on conditionals and BI: an example

Stackelberg mini-game

- *Ann believes "***Bob would go Right given Up***"*
  =belief about a behavioral conditional
- **Given Up**, *Ann would believe "***Bob goes Right***"*
  =conditional belief about behavior
- In **traditional analysis:** behavioral conditionals and beliefs about conditionals and opponents' beliefs
- In **our analysis:** actual actions (paths) and *conditional* beliefs about actions and beliefs
- Possible interpretation: static analysis of players' conjectures of what would be the case under different hypotheses (cf. Samet, GEB 1996)

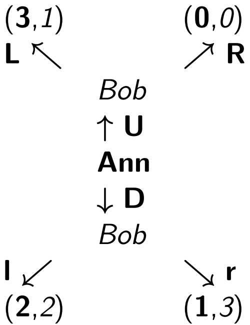
Preview on conditionals and BI: traditional analysis

States ω specify strategies $\mathbf{s_i} = \sigma_\mathbf{i}(\omega)$ **objectively**
'Bob would go Right given Up' **false** at ω iff $\sigma_{\mathbf{Bob}}(\omega) \in \{\mathbf{L.r}, \mathbf{L.l}\}$

$\mathbf{Rat_{Bob}} \subset [\mathbf{L.r}] = \{\omega : \sigma_\mathbf{i}(\omega) = \mathbf{L.r}\} \Rightarrow$
$\mathrm{B}_{\mathbf{Ann}}(\mathbf{Rat_{Bob}}) \subset \mathrm{B}_{\mathbf{Ann}}(\mathbf{L.r})$ (uncond. belief)

States also specify conditional beliefs
By built in independence:
$\mathrm{B}_{\mathbf{Ann}}(\mathbf{L.r}) \subset \mathrm{B}_{\mathbf{Ann}}(\mathbf{L}|\mathbf{U}) \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{r}|\mathbf{D}) =$
$= \mathrm{B}_{\mathbf{Ann}}(\mathbf{util} = \mathbf{3}|\mathbf{U}) \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{util} = \mathbf{1}|\mathbf{D})$

$\Rightarrow$ $\mathbf{Rat_{Ann}} \cap \mathbf{Rat_{Bob}} \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{Rat_{Bob}}) \subset$
$\subset [\mathbf{U}, \mathbf{L.r}] \subset [\mathbf{U}, \mathbf{L}]$



$\Rightarrow$ **BI strategies and path obtain**

Preview on conditionals and BI: our analysis

"strategy" is an epistemic concept:
i's cond. beliefs ⇒ contingent plan of **i**

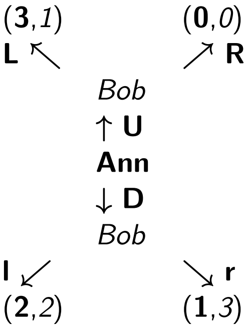
Rat. Planning: $\mathbf{RP_{Bob}}=\mathrm{B}_{\mathbf{Bob}}(\mathbf{L}|\mathbf{U})\cap\mathrm{B}_{\mathbf{Bob}}(\mathbf{r}|\mathbf{D})$

Material Consistency=path cons. with plan

Material Rationality=Material cons.+Rat.plan.
$\mathbf{MR_{Bob}}\subset [\mathbf{U},\mathbf{L}]\cup[\mathbf{D},\mathbf{r}]$

But no event $[\mathbf{L.r}]$. $\mathrm{B}_{\mathbf{Ann}}(\mathbf{L.r})$ not expressible!

$\mathrm{B}_{\mathbf{Ann}}(\mathbf{MR_{Bob}}) \subset\mathrm{B}_{\mathbf{Ann}}([\mathbf{U},\mathbf{L}]\cup[\mathbf{D},\mathbf{r}]) \not\subset$
$\not\subset\mathrm{B}_{\mathbf{Ann}}(\mathbf{L}|\mathbf{U})\cap\mathrm{B}_{\mathbf{Ann}}(\mathbf{r}|\mathbf{D})$ e.g. if $\mathrm{B}_{\mathbf{Ann}}(\mathbf{U})=\mathbf{0}$



Preview on conditionals and BI: an example

Our analysis

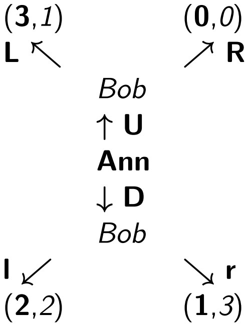
Recall: $\mathbf{MR_{Bob}} \subset \mathbf{[U, L]} \cup \mathbf{[D, r]}$, but
$\mathrm{B}_{\mathbf{Ann}}(\mathbf{MR_{Bob}}) \not\subset \mathrm{B}_{\mathbf{Ann}}(\mathbf{L}|\mathbf{U}) \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{r}|\mathbf{D})$ e.g. if $\mathrm{B}_{\mathbf{Ann}}(\mathbf{U}) = \mathbf{0}$

Since there is no event $\mathbf{[L.r]}$,
$\mathrm{B}_{\mathbf{Ann}}(\mathbf{L.r})$ not expressible

If $\mathrm{B}_{\mathbf{Ann}}(\mathbf{D})$ (Ann plans **D**) the following is possible

$\mathrm{B}_{\mathbf{Ann}}(\mathbf{[U, L]} \cup \mathbf{[D, r]}) \cap \neg \mathrm{B}_{\mathbf{Ann}}(\mathbf{L}|\mathbf{U})$

$\exists \omega \in \mathbf{[D, r]} \cap \mathbf{MR_{Ann}} \cap \mathbf{MR_{Bob}} \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{MR_{Bob}})$



⇒ **Imperfect Nash** equilibrium **may obtain**

Preview on conditionals and BI: an example

How to recover elementary BI, route 1: Strong Belief in rationality

- Ann *strongly believes* **E** if she believes **E** *given* each **C** with $\mathbf{C} \cap \mathbf{E} \neq \emptyset$
- $[\mathbf{U}] \cap \mathbf{MR_{Bob}} \neq \emptyset$, $[\mathbf{D}] \cap \mathbf{MR_{Bob}} \neq \emptyset$
- hence $\mathrm{SB}_{\mathbf{Ann}}(\mathbf{MR_{Bob}}) \subset \mathrm{B}_{\mathbf{Ann}}(\mathbf{MR_{Bob}}|\mathbf{U}) \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{MR_{Bob}}|\mathbf{D})$
- but $\mathbf{MR_{Bob}} \subset [\mathbf{U}, \mathbf{L}] \cup [\mathbf{D}, \mathbf{r}]$
- hence $\mathrm{SB}_{\mathbf{Ann}}(\mathbf{MR_{Bob}}) \subset \mathrm{B}_{\mathbf{Ann}}(\mathbf{L}|\mathbf{U}) \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{r}|\mathbf{D})$ (BI cond. beliefs)
- thus $\mathbf{MR_{Ann}} \cap \mathbf{MR_{Bob}} \cap \mathrm{SB}_{\mathbf{Ann}}(\mathbf{MR_{Bob}}) \subset [\mathbf{U}, \mathbf{L}]$ (BI path)

Preview on conditionals and BI: an example

How to recover elementary BI, route 2: Own-Action Independence

- Suppose
  (i) Ann's conditional beliefs about Bob's beliefs (hence his plan) are *independent* of her actions (**Ind**),
  (ii) Ann strongly believes the Bob is materially consistent:
- $\mathbf{Ind} \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{RP}_{\mathbf{Bob}}) \cap \mathrm{SB}_{\mathbf{Ann}}(\mathbf{MC}_{\mathbf{Bob}}) \subset \mathrm{B}_{\mathbf{Ann}}(\mathbf{L}|\mathbf{U}) \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{r}|\mathbf{D})$
- Thus
  $\mathbf{MR}_{\mathbf{Ann}} \cap \mathbf{MR}_{\mathbf{Bob}} \cap \mathbf{Ind} \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{RP}_{\mathbf{Bob}}) \cap \mathrm{SB}_{\mathbf{Ann}}(\mathbf{MC}_{\mathbf{Bob}}) \subset [\mathbf{U}, \mathbf{L}]$
- (Note: such independence and belief in consistency are implicit in traditional analysis)

- Route 1 (SB, with higher-level epistemic assumptions) leads to the BI-path in generic PI games
- Route 2 (OAI) leads to the BI-path in generic two-stage PI games, not in longer games such as the Centipede (even with higher-level epistemic assumptions)

"Centipede" example: strong belief analysis

Forward induction reasoning yields the BI outcome

$$
\begin{array}{ccccccc}
 & \mathbf{C} & & c & & \mathbf{C'} & \\
\mathbf{Ann} & \longrightarrow & Bob & \longrightarrow & \mathbf{Ann} & \longrightarrow & (\mathbf{0}, \mathit{3}) \\
\downarrow \mathbf{D} & & \downarrow d & & \downarrow \mathbf{D'} & & \\
(\mathbf{1}, \mathit{0}) & & (\mathbf{0}, \mathit{2}) & & (\mathbf{3}, \mathit{0}) & &
\end{array}
$$

- $\mathbf{MR_{Ann}} \subset \neg[\mathbf{C}, \mathbf{c}, \mathbf{C'}]$
- $\mathbf{MR_{Bob}} \cap \mathrm{SB}_{\mathbf{Bob}}(\mathbf{MR_{Ann}}) \subset [\mathbf{D}] \cup [\mathbf{C}, \mathbf{d}]$
- $\mathbf{MR_{Ann}} \cap \mathrm{SB}_{\mathbf{Ann}}(\mathbf{MR_{Bob}} \cap \mathrm{SB}_{\mathbf{Bob}}(\mathbf{MR_{Ann}})) \subset [\mathbf{D}]$

"Centipede" example: initial common belief analysis

Not enough to get the BI outcome

$$
\begin{array}{ccccccc}
 & \mathbf{C} & & c & & \mathbf{C}' & \\
\mathbf{Ann} & \longrightarrow & \mathit{Bob} & \longrightarrow & \mathbf{Ann} & \longrightarrow & (\mathbf{0}, \mathit{3}) \\
\downarrow \mathbf{D} & & \downarrow d & & \downarrow \mathbf{D}' & & \\
(\mathbf{1}, \mathit{0}) & & (\mathbf{0}, \mathit{2}) & & (\mathbf{3}, \mathit{0}) & &
\end{array}
$$

- $\mathbf{MR_{Ann}} \subset \neg[\mathbf{C}, \mathbf{c}, \mathbf{C}']$
- But, if $\mathrm{SB}_{\mathbf{Bob}}(\mathbf{MR_{Ann}})$ is **not** assumed, $\mathbf{MR_{Bob}}$ has no behavioral implication! Thus
- $\mathbf{MR_{Ann}} \cap \mathbf{MR_{Bob}} \cap \mathbf{Ind} \cap \mathrm{B}_{\mathbf{Ann}}(\mathbf{RP_{Bob}}) \cap \mathrm{SB}_{\mathbf{Ann}}(\mathbf{MC_{Bob}}) \subset \neg[\mathbf{C}, \mathbf{c}, \mathbf{C}']$
- $\Rightarrow$ initial common belief in $\mathbf{MR} \cap \mathbf{Ind} \cap \mathrm{SB}(\mathbf{MC})$ only buys $\neg[\mathbf{C}, \mathbf{c}, \mathbf{C}']$

(We can show this by example and as corollary of a general theorem)

Setup

Perfect Information (PI) games

- $\mathbf{i} \in \mathbf{I}$, *players*
- $\mathbf{h} \in \mathbf{H}$, *histories/nodes* ($\mathbf{H_i}$, owned by $\mathbf{i}$), **H finite**
- $\mathbf{z} \in \mathbf{Z} \subset \mathbf{H}$, *terminal* histories/*paths*; $\mathbf{Z(h) = \{z : h \preceq z\}}$
- $\mathbf{a} \in \mathbf{A(h)}$, *actions* at $\mathbf{h} \in \mathbf{H \backslash Z}$
- $\mathbf{u_i : Z} \to \mathbb{R}$, *utility/payoff* s.t. **no relevant ties**

Setup

Epistemic structures for PI games: general

- $\omega \in \mathbf{\Omega}$, *states of the world*
- $\mathbf{x} \in \mathbf{X}$, *eXternal* (non-epistemic) *states*, **finite** (e.g. $\mathbf{X} = \mathbf{S}$ or $\mathbf{X} = \mathbf{Z}$)
- (implicitly understood *path function* $\pi : \mathbf{X} \to \mathbf{Z}$)
- $\xi : \mathbf{\Omega} \to \mathbf{X}$, $\xi(\omega)$ external state at ω, $[\mathbf{Y}] = \xi^{-1}(\mathbf{Y})$ external events
- $\hat{\beta}_i : \mathbf{\Omega} \to \mathbf{\Delta}_i$, $\hat{\beta}_i(\omega)$ epistemic state at ω ($\mathbf{\Delta}_i = \mathbf{Beliefs}_i$ to be specified)

- Conditioning events (hypotheses) correspond to histories:

$$\mathbf{[h] = \{\omega : \pi(\xi(\omega)) \in Z(h)\}}$$

- Probability measures concentrated on conditioning events $\mathbf{C = [h]}$, $\mathbf{D = [h']}$ ... related by chain rule:

$$\mathbf{E \subset C \subset D \Rightarrow \mu(E|D) = \mu(E|C)\mu(C|D)} \qquad \text{(ch.r)}$$

- Axiomatization: decision-theoretic (Myerson, 1991, Siniscalchi 2011), epistemic (Di Tillio-Halpern-Samet, 2010)⇒ Fairly well understood epistemic analysis of conditionals.
- $\mathbf{\Delta_i = \Delta^H(\Omega) \subset [\Delta(\Omega)]^H}$, CPS's on $\mathbf{(\Omega, H)}$: set of $\mathbf{(p_i(\cdot|[h]))_{h \in H} \in [\Delta(\Omega)]^H}$ s.t. $\mathbf{p_i([h]|[h]) = 1}$ and (ch.r) holds

Setup

We focus on epistemic **type structures** with ext.states $\mathbf{X = Z}$:

$$\Omega = Z \times \prod_i T_i, \ \beta_i : T_i \to \Delta^H(Z \times T_{-i}), \ \xi = \zeta = \mathrm{proj}_Z,$$

$$\hat{\beta}_i(z, t_i, t_{-i})(E|[h]) = \beta_i(t_i)(E_{t_i}|Z(h) \times T_{-i})$$

meaning:

- **no knowledge** about **z at the outset**
- **information** about moves **cannot disclose** anything about types/**beliefs**

Technical (in a sense, w.l.o.g.)

- $\forall i$, T_i **compact metrizable**, $\Rightarrow \Delta^H(Z \times T_{-i})$ compact metrizable
- β_i **continuous**

Definition

A structure is *complete* if β_i is **onto** for every $i \in I$ (e.g., the *canonical* structure)

Setup

Belief operators

Standard (monotonic) belief operators

- conditional belief $\mathrm{B}_{\mathbf{i}}(\mathbf{E}|\mathbf{h}) = \{\omega : \hat{\beta}_{\mathbf{i}}(\mathbf{E}|[\mathbf{h}]) = \mathbf{1}\}$,
- initial belief $\mathrm{B}_{\mathbf{i}}(\mathbf{E}) = \mathrm{B}_{\mathbf{i}}(\mathbf{E}|\boldsymbol{\Omega})$ ($\boldsymbol{\Omega} = [\phi]$, ϕ=empty hist., root)

Nonmonotonic belief operator

- strong belief $\mathrm{SB}_{\mathbf{i}}(\mathbf{E}) = \bigcap_{\mathbf{h}:\mathbf{E}\cap[\mathbf{h}]\neq\emptyset} \mathrm{B}_{\mathbf{i}}(\mathbf{E}|\mathbf{h})$

Given $\mathbf{p} \in \mathbf{\Delta^H(\Omega)}$ derive

$$\mathbf{p(a|h) = p([h, a]|[h])}$$

for all $\mathbf{h} \in \mathbf{H \backslash Z}$, $\mathbf{a} \in \mathbf{A(h)}$.
Consider only the conditional prob. of **i**'s opponents' actions:

$$\mathbf{p(a|h)}, \mathbf{h} \in \mathbf{H_{-i}}, \mathbf{a} \in \mathbf{A(h)}$$

$\Rightarrow$ subjective decision tree $\mathbf{\Gamma_i(p)}$ for **i**
p is *consistent with dynamic programming* on $\mathbf{\Gamma_i(p)}$ iff (OSD)

$$\forall \mathbf{h} \in \mathbf{H_i}, \mathbf{p}\left(\arg\max_{\mathbf{a} \in \mathbf{A(h)}} \mathbf{V_i((h, a), p)|h}\right) = \mathbf{1},$$

$$\text{with } \mathbf{V_i((h, a), p) = \sum_{z \in Z(h,a)} u_i(z) p(z|h, a)}$$

Strategies as beliefs

Rational planning

The plan of $\mathbf{i}$ at $(\mathbf{z}, \mathbf{t_i}, \mathbf{t_{-i}})$ (plan of $\mathbf{t_i}$) is derived from $\beta_\mathbf{i}(\mathbf{t_i})$

Definition

Pl. $\mathbf{i}$ *plans rationally* at $(\mathbf{z}, \mathbf{t_i}, \mathbf{t_{-i}})$ if

$$\forall \mathbf{h} \in \mathbf{H_i},\ \beta_\mathbf{i}(\mathbf{t_i})\left(\arg\max_{\mathbf{a}\in\mathbf{A(h)}} \mathbf{V_i}((\mathbf{h},\mathbf{a}),\mathbf{p})|\mathbf{h}\right) = 1.$$

Event: $\mathbf{RP_i}$

- *$RP_\mathbf{i}$ is just a property of* $\mathbf{i}$*'s beliefs/types:* $\mathbf{i}$ expects to take locally maximizing actions conditional on each $\mathbf{h} \in \mathbf{H_i}$.
- **Interpretation:** $\mathbf{i}$ has beliefs about others and computes his plan (beliefs about himself) by dynamic programming on the corresponding subjective decision tree.

Connect beliefs to behavior:

Definition

Pl. $\mathbf{i}$ is *materially consistent at* $(\mathbf{z}, \mathbf{t_i}, \mathbf{t_{-i}})$ if he does not violate his plan on path $\mathbf{z}$

$$\begin{aligned} \forall \mathbf{h} &\in \mathbf{H_i}, \forall \mathbf{a} \in \mathbf{A}(\mathbf{h}), \\ (\mathbf{h}, \mathbf{a}) &\preceq \mathbf{z} \Rightarrow \beta_\mathbf{i}(\mathbf{t_i})(\mathbf{a}|\mathbf{h}) > \mathbf{0}. \end{aligned}$$

Event: $\mathbf{MC_i}$

Definition

Pl. $\mathbf{i}$ is *materially rational at* $(\mathbf{z}, \mathbf{t_i}, \mathbf{t_{-i}})$ if he plans rationally and does not violate his plan at $(\mathbf{z}, \mathbf{t_i}, \mathbf{t_{-i}})$.Event: $\mathbf{MR_i} = \mathbf{MC_i} \cap \mathbf{RP_i}$.

Strategies as beliefs

Common belief in MR and Nash equilibrium

- Recall: results apply to **finite PI games with NRT**, in such games the **BI strategy** is **unique** and (mixed) **Nash** equilibria **yield a unique path** with prob. 1.
- Similar to traditional analysis: initial common belief in material rationality does not yield BI or Nash paths in games of depth $\mathbf{d} > \mathbf{2}$ (e.g. Centipede).

- *Unlike traditional analysis:* in games of **depth 2**, correct belief in rationality does **not** yield **BI**, only a **Nash path** (cf. initial example).

Proposition

In a game $\mathbf{\Gamma}$ *of depth 2,* $\forall \mathbf{z} \in \mathbf{Z}$, $\mathbf{z}$ *is a Nash path if and only if*

$$\mathbf{(z,t)} \in \bigcap_{\mathbf{i}} \mathbf{MR_i} \cap \mathrm{B}_\mathbf{i}(\mathbf{MR_{-i}})$$

for some $\mathbf{t}$*, in some type structure* $(\mathbf{T_i}, \beta_\mathbf{i})_{\mathbf{i} \in \mathbf{I}}$ *for* $\mathbf{\Gamma}$*.*

Strategies as beliefs

Common Strong Belief in Material Rationality

- One way to obtain elementary BI (games of depth 2) is to assume that the first mover strongly believes in the material rationality of the co-player.
- We can go further and replicate "traditional" results on common strong belief in rationality and Nash eq. (Battigalli-Friedenberg, 2010), or EFR and BI in complete structures (Battigalli-Siniscalchi, 2002).

- $\mathbf{MR_i^0{=}MR_i}$, $\mathbf{MR_i^{k+1}{=}MR_i^k} \cap \mathrm{SB}_\mathbf{i}(\mathbf{MR^k_{-i}})$
- $\mathbf{CSBMR{=}\bigcap_{k,i} MR_i^k}$, correct Common Strong Belief in MR
- $\pi : \mathbf{S} \to \mathbf{Z}$ strategy-path function

Proposition

(i) In any type structure

$$proj_{\mathbf{Z}}\mathbf{CSBMR} \subset \{\mathbf{Nash}\text{-}paths\}$$

(ii) In a **complete** *(or otherwise "sufficiently rich") structure*

$$proj_{\mathbf{Z}}\mathbf{CSBMR} = \pi(\mathbf{EFR}) = \{\mathbf{BI}\text{-}path\}.$$

Strategies as beliefs

Own Action Independence

- Reason for non-BI result: Ann may initially believe in $\mathbf{MR_{Bob}}$ but she need not believe in $\mathbf{MR_{Bob}}$ conditional on taking an unplanned action.
- This cannot happen if Ann's beliefs about Bob's type *conditional on her own actions* do not depend on the conditioning action, and she strongly (hence always) believes in $\mathbf{MC_{Bob}}$.
- *Own-action* **independence**=**i**'s conditional beliefs about $\mathbf{t_{-i}}$ do not depend on **i**'s actions (event $\mathbf{Ind_i}$)

Proposition

In every "rich" (e.g., complete) type structure for a game of depth 2

$$proj_{\mathbf{Z}} \left(\bigcap_{\mathbf{i}} \mathbf{MR_i} \cap \mathbf{Ind_i} \cap \mathrm{B}_{\mathbf{i}} \left(\mathbf{MR_{-i}} \right) \cap \mathrm{SB}_{\mathbf{i}}(\mathbf{MC_{-i}}) \right) = \{ \mathbf{BI}\text{-}path \}.$$

What about longer games (e.g. Centipede)?

- $\mathbf{MRInd_i^0 = MR_i \cap Ind_i \cap \mathrm{SB}_i(MC_{-i})}$,
- $\mathbf{MRInd_i^{k+1} = MRInd_i^k \cap \mathrm{B}_i(MRInd_{-i}^k)}$
- $\mathbf{CBMRInd = \bigcap_{i,k} MRInd_i^k}$

All the paths consistent with the "Dekel-Fudenberg procedure", $\boldsymbol{\pi(S^\infty W)}$, are also consistent with **CBMRInd** (cf. Ben Porath, 1997):

Proposition

For every $\mathbf{s} \in \mathbf{S^\infty W}$ *there is a type structure with a state* $\mathbf{(z,t)} \in$ **CBMRInd** *such that* $\mathbf{z} = \boldsymbol{\pi}(\mathbf{s})$.

Corollary

There are non-BI paths consistent with **CBMRInd** *in Centipede (of depth* $\mathbf{d} > \mathbf{2}$*).*

Conclusions

- Many results of the "traditional analysis" with behavioral conditionals make a lot of sense. They are built on often implicit assumptions of plan/behavior consistency, strong belief in consistency (perceived intentionality) and self/opponents independence.
- We rule out behavioral conditionals, but allow for epistemic ones. This forces an interpretation of strategies as epistemic constructs.
- Consistency and independence have to be assumed explicitly. This seems fitting for a formal analysis of strategic reasoning.

Conclusions

Imperfect information (with perfect recall)

- $\mathbf{h_i} \in \mathbf{H_i}$ information sets (personal histories)
- Conditions/hypotheses for $\mathbf{i}$: $\mathbf{[h_i]}$ and $\mathbf{[h_i, a_i]}$ ($\mathbf{a_i} \in \mathbf{A_i(h_i)}$)
- Value of action $\mathbf{a_i}$ at $\mathbf{h_i}$: $\mathbb{E}_{\beta_{\mathbf{i}}(\omega)}[\mathbf{u_i}|\mathbf{h_i}, \mathbf{a_i}]$
- "Newcomb paradoxes" $\Rightarrow$ potential conflict between our analysis and standard decision theory
- Own-action independence and strong belief in material consistency allow to reconcile our analysis with traditional decision theory.

Under consistency and independence we obtain analogs of the "traditional results".

For generic PI games:

any type structure	complete structure
CBMRInd $\approx$ $\mathbf{S^\infty W}$ (?)	$\mathrm{proj}_{\mathbf{Z}}$**CBMRInd** $\supset \pi(\mathbf{S^\infty W})$ (=?)
CBMRInd $\approx$ **BI** if depth=2	**CSBMR** $\approx$ **EFR** ($\approx$ **BI** if $\mathbf{d}=\mathbf{2}$)
CSBMR $\approx$ **Nash** path	$\mathrm{proj}_{\mathbf{Z}}$**CSBMR** $= \pi(\mathbf{EFR}) = \{\mathbf{z}^{\mathbf{BI}}\}$
$\mathrm{proj}_{\mathbf{Z}}$**CSBMR** $= \{\mathbf{z}^{\mathbf{BI}}\}$ in Centip.	**CSBMR** $\approx$ **BI** in Centipede

($\approx$ means "characterization w/ strategies as beliefs", (?)=conj., $\mathbf{z}^{\mathbf{BI}}$ = **BI**-path)

We conjecture that the results on $\mathbf{S^\infty W}$ and **EFR** can be extended to all games with perfect recall (assuming OAI).

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