Transparent Restrictions on Beliefs and Forward-Induction Reasoning in Games with Asymmetric Information

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Introduction

- Overall project: analysis of higher-order beliefs and *forward induction* (FI) *reasoning* in dynamic games (see Battigalli & Siniscalchi JET 2002, Battigalli & Friedenberg TE 2012)
- This paper:
  1. (partially) new epistemic justification of the Δ-rationalizability solution concept (version of Battigalli RE 2003)
  2. equivalence of two sets of epistemic assumptions representing FI reasoning and justifying Δ-rationalizability
  3. if restricted belief set Δ is "closed under compositions", Δ–rationalizability is shown to be equivalent to a simpler algorithm used by Battigalli & Siniscalchi BJTE 2003
  4. general analysis: incomplete information, imperfect asymmetric observation of past actions, chance moves

Example: Costly lies (Beer-Quiche in disguise)

- Chance chooses with prob. $\frac{1}{3} : \frac{2}{3}$ whether Adam (pl. 1) has low productivity (θ^L) or high productivity (θ^H).
- Adam observes this and then tells Bea (pl. 2) "I am high" (H) or "I am Low" (L). He incurs a small psychological cost c for lying ($c < \frac{1}{3}$).
- Bea can hire Adam (h) or leave him alone (ℓ).
- Adam would like to be hired. For Bea, h is a bet on the type of Adam.

The ugly picture

$$
\begin{array}{ccccccccc}
\binom{1-c}{1} & \nwarrow h & & L & \theta^H & H & & h \nearrow & \binom{1}{1} \\
 & & 2 & \longleftarrow - \quad - & 1 & - - - \quad \longrightarrow & 2 & & \\
\binom{-c}{0} & \swarrow \ell & \vdots & & \uparrow \frac{2}{3} & & \vdots & \ell \searrow & \binom{0}{0} \\
 & & \vdots & & \circ & & \vdots & & \\
\binom{1}{-1} & \nwarrow h & \vdots & & \downarrow \frac{1}{3} & & \vdots & h \nearrow & \binom{1-c}{-1} \\
 & & 2 & \longleftarrow - \quad - & 1 & - - - \quad \longrightarrow & 2 & & \\
\binom{0}{0} & \swarrow \ell & & L & \theta^L & H & & \ell \searrow & \binom{-c}{0}
\end{array}
$$

Figure 1. Costly lies.

FI epistemic assumptions (informal)

- Players have beliefs about everything, chance, each other's strategies, and each other's beliefs; they update according to the rules of conditional probabilities whenever they apply (denominator$\neq$ 0).
- First-order restr. (C): Initial beliefs about chance are $\frac{1}{3} : \frac{2}{3}$.
- First-order restr. (I): Adam regards θ and s_2 as *independent:* his conjecture on s_2 is independent of θ.
- (C) and (I) are "commonly believed at each node": **transparency of restrictions**.
- Given this, **best rationalization principle:** each player (Bea in particular) always ascribes to the other the "highest degree of strategic sophistication" consistent with what s/he observes (Battigalli GEB 1996).

Step-by-step FI analysis of "Costly Lies"

1. Adam would lie only to maximize the (subjective) probability of being hired, which is *independent* of θ [see (I)]: he sends the maximizing message $m \in \{H, L\}$ when $\theta = \theta^m$. Hence he either pools on one message, or is truthful.
2. If Bea hears H, *even if surprised*, she is infers that Adam is either pooling on H or is truthful (best rationalization of H). Hence *the ex ante more likely type,* θ^H*, becomes even more likely:* $\mu_B(\theta^H|H) \in [\frac{2}{3}, 1]$ [see (C)] and she *h*ires Adam.
3. Adam does not pool on L: he tells the truth if θ^H.
4. If Bea hears L, even if surprised, she infers that Adam is truthtelling (best rationalization of L) and ℓeaves him alone. The strategy of Bea is "h if H, ℓ if L".
5. Adam pools on H. END

Note: Bea would definitely be surprised by L. Still, she would reason and choose as in 4 (best rationalization).

Formal theory of FI: ingredients

- *Types structures for dynamic games*, focus on *canonical* structure (Battigalli & Siniscalchi JET 1999): information sets $h \in H_i$, belief maps from types to Cond. Prob. Systems about others
$$\beta_i = (\beta_{i,h})_{h \in H_i} : T_i \to CPS(\Omega_{-i}, H_i)$$
- *Conditional belief*: type t_i **believes** E **given** information set $\mathbf{h} \in H_i$ if $\beta_{i,h}(t_i)(E) = 1$; *monotone* operator $\mathrm{B}_{i,h}(E)$
- **Full belief**: t_i believes E conditional on each $h \in H_i$; *monotone* operator $\mathrm{B}_i(E) = \cap_h \mathrm{B}_{i,h}(H)$
- *Strong belief*: t_i **strongly believes** E if he believes E whenever possible: $E \cap [h] \neq \emptyset \Rightarrow \beta_{i,h}(E) = 1$; *non monotone* operator $\mathrm{SB}_i(E)$
- Non-monotonicity of $\mathrm{SB}_i(\cdot)$ makes the analysis more interesting and more difficult

FI reasoning: canonical structure

- With the canonical structure the analyst must state explicitly as an event assumptions about what is commonly believed
- Belief restrictions Δ, assumed *compact* (e.g. independence + prob. of chance moves are such and such), event $[\Delta]$
- **Transparency of** Δ: $\mathrm{B}^*([\Delta]) = \cap_{n\geq 0}\mathrm{B}^n([\Delta])$; note: $\mathrm{B}^*([\Delta]) = [\Delta] \cap \mathrm{B}(\mathrm{B}^*([\Delta])) = \times_{i\in I}\mathrm{B}^*_i([\Delta])$
- **Rationality:** conditional max. of EU, event R_i
- **Rat.**$\cap\mathrm{B}^*([\Delta])$ **and Common Strong Belief of Rat.**$\cap\mathrm{B}^*([\Delta])$

$$\begin{aligned} R^1_{i,\Delta} &= R_i \cap \mathrm{B}^*_i([\Delta]) \\ R^{m+1}_{i,\Delta} &= R^m_{i,\Delta} \cap \mathrm{SB}_i(R^m_{-i,\Delta}) \\ RCSBR_\Delta &= \times_{i\in I} \cap_{m\geq 1} R^m_{i,\Delta} \end{aligned}$$

Intermezzo: an equivalent approach

- $\mathrm{B}^*([\Delta])$ yields a type structure $\mathcal{T}^*_\Delta$ embedded in the canonical structure $\mathcal{T}^*$. Define events within any type structure $\mathcal{T}_\Delta$ hierarchy-equivalent to $\mathcal{T}^*_\Delta$.
- Transparency of Δ is captured implicitly within $\mathcal{T}_\Delta$.
- **Rationality** within $\mathcal{T}_\Delta$**:** conditional max. EU, event $R_{i,\Delta}$
- **Rationality and Common Strong Belief in Rationality** within $\mathcal{T}_\Delta$

$$
\begin{aligned}
R^1_{i,\Delta} &= R_{i,\Delta} \\
R^{m+1}_{i,\Delta} &= R^m_{i,\Delta} \cap \mathrm{SB}_i(R^m_{-i,\Delta}) \\
CSBR_\Delta &= \times_{i\in I} \cap_{m\geq 1} R^m_{i,\Delta}
\end{aligned}
$$

FI rationalizability solution: ingredients

- Players $i \in I$, information types (e.g. payoff types) $\theta_i \in \Theta_i$, information sets $h \in H_i$, strategies $s_i \in \times_{h \in H_i} A_i(h)$, chance is pl. $0 \notin I$, here $-i = (I \cup \{0\}) \setminus \{i\}$
- Iterated deletion of pairs (θ_i, s_i) for each $i \in I$
- First-order beliefs μ_i on $\Theta_{-i} \times S_{-i}$, restrictions may depend on θ_i:

$$\mu_i = (\mu_{i,h})_{h \in H_i} \in \Delta_{\theta_i} \subseteq CPS(\Theta_{-i} \times S_{-i}, H_i)$$

- BR correspondence

$$r_{\theta_i}(\mu_i) = \{s_i \in S_i : s_i \text{ is sequential BR for } \theta_i \text{ to } \mu_i\}$$

FI rationalizability: algorithm

- $\Delta^0_{\theta_i} = \Delta_{\theta_i}$
- given $\Delta^n = \left(\Delta^n_{\theta_j}\right)_{j \in I, \theta_j \in \Theta_j}$, let

$$\begin{aligned} \Sigma^{n+1}_{0,\Delta} &= \Theta_0 \times S_0 \text{ (no } (\theta_0, s_0) \text{ is ever deleted)} \\ \Sigma^{n+1}_{i,\Delta} &= \{(\theta_i, s_i) : s_i \in r_{\theta_i}(\Delta^n_{\theta_i})\} \quad (i \in I) \end{aligned}$$

$$\Delta^{n+1}_{\theta_i} = \left\{ \mu_i \in \Delta^n_{\theta_i} : \begin{array}{l} \forall h \in H_i, \\ \Sigma^{n+1}_{-i} \cap \Sigma_{-i}(h) \neq \emptyset \Rightarrow \mu_{i,h}(\Sigma^{n+1}_{-i}) = 1 \end{array} \right\}$$

- Note: $\Sigma^n_{i,\Delta}$ *may be empty* if Δ features some restrictions to endogenous beliefs, due to possible conflict between belief restrictions and strategic reasoning

A simpler algorithm

Battigalli & Siniscalchi BEJTE 2003 consider a simpler algorithm, closer to Pearce ECMA 1984 (who implicitly assumes as restrictions only the known probabilities of chance moves)

- $\hat{\Sigma}^0_{i,\Delta} = \Theta_i \times S_i$
- given $(\hat{\Sigma}^n_{i,\Delta})_{i \in I \cup \{0\}}$

$$\hat{\Sigma}^{n+1}_{0,\Delta} = \Theta_0 \times S_0 \text{ (no } (\theta_0, s_0) \text{ is ever deleted)}$$

$$\hat{\Sigma}^{n+1}_{i,\Delta} = \left\{ (\theta_i, s_i) \in \hat{\Sigma}^n_{i,\Delta} : \begin{array}{l} \exists \mu_i \in \Delta_{\theta_i}, s_i \in r_{\theta_i}(\mu_i), \forall h \in H_i, \\ \hat{\Sigma}^n_{-i,\Delta} \cap \Sigma_{-i}(h) \neq \emptyset \Rightarrow \mu_{i,h}(\hat{\Sigma}^n_{-i,\Delta}) = 1 \end{array} \right\}$$

Comparison

- Note subtle difference: the def. of $\hat{\Sigma}^n_{i,\Delta}$ does *not* require μ_i to satisfy the best rationalization principle w.r.t. $\left(\hat{\Sigma}^m_{-i,\Delta}\right)^{m=n}_{m=1}$, endogenous restrictions on beliefs from previous steps can be ignored (making the algorithm simpler)
- In general, $\Sigma^2_{i,\Delta} = \hat{\Sigma}^2_{i,\Delta}$, $\Sigma^3_{i,\Delta} \subseteq \hat{\Sigma}^3_{i,\Delta}$, possibly $\Sigma^n_{i,\Delta} \not\subseteq \hat{\Sigma}^n_{i,\Delta}$ for $n > 3$ due to non-monotonicity of strong belief
- Differences my follow from the failure of Δ_i to contain "compositions" of CPS's in Δ_i
- The price of greater simplicity is that the algorithm is not "conceptually correct", i.e. there is some Δ such that $\left(\Sigma^n_\Delta\right)_{n\in\mathbb{N}}$ does not capture what it is meant to.

Main result

Behavioral predictions of epistemic assumptions $EA = \times_{i \in I} EA_i$ given by $\mathrm{proj}_{\Theta_i \times S_i} EA_i$

Theorem

For every $i \in I$ and $m \in \mathbb{N}$

$$\mathrm{proj}_{\Theta_i \times S_i} R^m_{i,\Delta} = \Sigma^m_{i,\Delta}$$

Hence

$$\mathrm{proj}_{\Theta \times S} RCSBR_\Delta = \Sigma^\infty_\Delta$$

Alternative epistemic justification of FI rationalizability

- See Battigalli & Siniscalchi RE 2007, "the sausage" for friends and family :-)

$$\hat{R}^1_{i,\Delta} = R_i \cap [\Delta_i]$$

(note: $\mathrm{B}^*_i([\Delta]) \subseteq [\Delta_i]$, hence $R^1_{i,\Delta} = R_i \cap \mathrm{B}^*_i([\Delta]) \subseteq \hat{R}^1_{i,\Delta}$)

$$\hat{R}^{m+1}_{i,\Delta} = \hat{R}^m_i \cap \mathrm{SB}_i(\hat{R}^m_{-i,\Delta})$$

- Although $R^1_{i,\Delta} \subseteq \hat{R}^1_{i,\Delta}$, $R^m_{i,\Delta} \subseteq \hat{R}^m_{i,\Delta}$ cannot be proved using standard monotonicity arguments because SB_i is not monotone. But ...
- B&S 2007 shows (essentially) $\mathrm{proj}_{\Theta_i \times S_i} \hat{R}^m_{i,\Delta} = \Sigma^{m+1}_{i,\Delta}$ for every i and m. Hence

Theorem

For every $i \in I$ and $m \in \mathbb{N}$

$$\begin{aligned} \mathrm{proj}_{\Theta_i \times S_i} R^m_{i,\Delta} &= \Sigma^m_{i,\Delta} = \mathrm{proj}_{\Theta_i \times S_i} \hat{R}^m_{i,\Delta} \\ R^m_{i,\Delta} &\subseteq \hat{R}^m_{i,\Delta} \end{aligned}$$

Results about algorithms

- It can be easily be shown by example that $(\hat{\Sigma}^n_\Delta)_{n\in\mathbb{N}}$ *may be different from the conceptually correct algorithm* $(\Sigma^n_\Delta)_{n\in\mathbb{N}}$.
- Furthermore, *for some game and restrictions* Δ *there is no type structure* T *such that* $\mathrm{proj}_{\Theta\times S} RCSBR_{\mathcal{T}} = \Sigma^\infty_\Delta$, i.e. Σ^∞_Δ is not an "extensive form best response set". Thank you Amanda Friedenberg :-)
- We define the property "Δ_i is closed under compositions" so that:

  Proposition *If* Δ_i *is closed under compositions for every* $i \in I$, *then* $\hat{\Sigma}^n_\Delta = \Sigma^n_\Delta$ *for every* $n \in \mathbb{N}$.
- The property is satisfied in many applications of interest:
  - Δ_i only captures restrictions on exogenous beliefs (beliefs about θ_{-i} and chance moves)
  - Δ_i only captures restrictions on initial beliefs (e.g. consistency with a distribution of paths) and/or independence restrictions (connection with FI refinements of Nash eq. based on strategic stability)

I never have time for conclusions

Thank you for your patience

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