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A NOTE ON RATIONALIZABILITY AND REPUTATION WITH TWO LONG-RUN PLAYERS

by
PIERPAOLO BATTIGALLI*

1. *Introduction*

This note shows how a patient individual playing a repeated game against a relatively impatient opponent can build up a reputation that gives him a strategic advantage. The analysis relies on very weak assumptions about rationality and beliefs. In particular, it does not use the equilibrium hypothesis. To put this contribution in perspective, I start with a short review of some related results (for further references see Sorin, 1999, which provides an excellent and systematic analysis of reputation and learning).

In a seminal paper Kreps and Wilson (1982) proposed a model where incomplete information about the type of a long-run player facing a (finite) sequence of short-run players allows the long-run player to build a reputation for being "tough." An important point of their paper was that the backward-induction solution of a finite-horizon game (in their case, Selten's, 1978, chainstore game) is not robust to the introduction of "slightly incomplete" information, that is, a small prior probability that the long-run player's type is different from the "normal" one specified by the complete information game. To make their case stronger, they had to use the sequential equilibrium concept, a refinement of the Bayesian-Nash equilibrium concept. In one variant of their model, the "non-normal" type of the long-run player is a so called "commitment

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type" who is constrained to choose a particular "tough" action in every stage. In this variant of the game, the sequential equilibrium is (almost always) unique. In this equilibrium the "normal type" imitates the "non-normal" or "tough" type almost until the end of the game, thus building a reputation for being "tough" that gives him a strategic advantage. In the last few stages, long run-player and short-run opponents play rather complex randomized strategies.

From the methodological point of view, Kreps and Wilson's approach was the most appropriate to show the non-robustness of backward induction to the introduction of "slightly incomplete" information. But their sequential-equilibrium analysis relies heavily on the controversial notion of randomization, whereas the idea of building a reputation seems intuitively simple. Fudenberg and Levine (1989) approached the problem from a different perspective, which considerably simplified the analysis of reputation. They considered a repeated game where a long-run player faces a sequence of short-run opponents and there are (countably) many possible commitment types of the long-run player. Fix a commitment type θ who always plays the same action. By means of a simple result about statistical inference, Fudenberg and Levine showed that by imitating type θ long enough, the long-run player can convince the short-run opponents that he is going to play *as if* he were θ and therefore they should best respond to θ's behavior. Therefore, if the long-run player is very patient and there is a possible commitment type that would always play the Stackelberg action of the stage game, *in any Bayesian-Nash equilibrium* his average expected payoff must be approximately as large as the Stackelberg payoff of the stage game.

Note that to prove this *selection result*, that is, a result showing that the set of possible outcomes (or payoffs) is "small", Fudenberg and Levine (1989) used a rather weak solution concept. In particular, they did not have to rely on the full power of the sequential equilibrium concept. Ideally, one should prove a selection result with the weakest possible solution concept, that is, with the weakest assumptions on rationality and beliefs[1]. Watson (1993) made a further step in this direction,

[1]On the other hand, if one aims at showing a result of the "folk-theorem" kind, saying that the set of possible outcomes is large, he or she should use the strongest assumptions on rationality and beliefs. That is why folk-theorems are proved for the set of subgame perfect equilibria rather than the set of Nash equilibria.

showing that, to obtain the selection result, one only needs to assume that the long-run player is rational, that he assumes that his opponents are rational and that he knows that their beliefs are not too different from each other. The equilibrium assumption – forcing the short-run players to have the same (correct) beliefs about the type-contingent strategy of the long-run opponent – is not really needed.

Battigalli and Watson (1997) further clarified this point and also pointed out that only minimal assumptions concerning the prior beliefs about commitment types are needed. The latter remark is worth emphasizing. What these incomplete information games with commitment types really represent is a situation where there are "doubts" in the short-run players' mind about the true type of their long-run opponents. Assuming that such doubts can be expressed by common prior probabilities strains credulity. On the other hand, an explicit representation of the heterogeneous subjective beliefs of the short-run opponents about the long-run player type (and, maybe, subjective beliefs about subjective beliefs) would make the incomplete information model extremely complex. Fortunately, most differences in beliefs turn out to be immaterial for the reputation/selection result and there is no need to build explicitly a complex epistemic type space.

The presence of a sequence of short-run opponents complicates the analysis if one wants to account for heterogeneous beliefs. On the other hand, it ensures that these opponents (if rational) simply choose their stage-game best response. The situation is different when the long-run patient player faces another long-run opponent. Schmidt (1993) shows that in this case the analog of the selection result of Fudenberg and Levine (1989) does not hold in general, even if it is assumed that players' beliefs are consistent with the sequential equilibrium concept. However, he also shows that the selection result holds for any Bayesian-Nash equilibrium if the payoffs of the stage game satisfy a certain restriction called "conflicting interests". Battigalli and Watson (1997) mention that also this selection result does not need the full power of equilibrium analysis: in order to show that, under "conflicting interests", the long-run patient player can achieve (in terms of long-run average) almost his Stackelberg payoff by always playing the Stackelberg action, it is sufficient to assume that he correctly believes that the opponent is rational and assigns at least ϵ probability to the commitment type playing always the

Stackelberg action.

The negative result in Schmidt (1993) is due to the fact that the set of commitment types considered in his paper (and in the other papers mentioned above) is not rich enough, as it encompasses only types that are constrained to play always a particular action. Evans and Thomas (1997) show that if the set also includes more complex commitment types playing history-dependent strategies the selection result holds in every Bayesian-Nash equilibrium.

In the same spirit of Watson (1993) and Battigalli and Watson (1997), in this note I show that also for the selection result of Evans and Thomas (1997) one does not need the full power of equilibrium analysis: in order to show that a patient long-run player can approximate his Stackelberg payoff, it is sufficient to assume that he correctly believes that the opponent is rational and assigns at least ϵ probability to a commitment type that "teaches" the opponent (by means of punishments) to play the best response to the Stackelberg action.

2. *The Reputation Result*[2]

Consider an infinitely repeated two-person game with discounting and one-sided incomplete information about feasible strategies. Player 2 does not know the set of feasible strategies of player 1. The finite or countable set of states of Nature Θ corresponds to the set of conceivable feasibility constraints for player 1. Let the (finite) stage game be $G = (A_1, A_2; v_1, v_2)$. The set of all (closed loop) strategies for player i in the repeated game is S_i. The set of feasible strategies for player 1's type θ is $\overline{S}_1(\theta) \subseteq S_1$. The graph of the feasibility correspondence $\overline{S}_1(\cdot)$ is denoted by $\Sigma_1 \subseteq \Theta \times S_1$. Let $\mathcal{H}$ be the set of partial histories, where h^0 denotes the empty initial history. The set of types and strategies for player 1 consistent with a given history $h \in \mathcal{H}$ is denoted by $\Sigma_1(h)$; in particular, $\Sigma_1(h^0) = \Sigma_1$.

For any feasible infinite history z, the long-run average payoff function for player i is $u_i(z) = (1-\delta_i)\sum_{t=1}^{\infty}\delta_i^{t-1}v_i(\alpha^t(z))$, where $\alpha^t(z)$ denotes the pair of actions chosen in period t along path z.

The stage game G satisfies the following assumptions:

[2]This result was first proposed in Battigalli (1999). I am presenting it now in a separate note because it is going to be deleted from the new version of that paper.

- Player 2 has a single-valued best response function $BR: A_1 \to A_2$ and

$$\min_{\alpha_1\in\Delta(A_1)}\max_{a_2\in A_2} v_2(\alpha_1, a_2) \geq 0$$

The domain of function v_2 is extended to $\Delta(A_1)\times A_2$ *via* expected payoff calculations.

- Player 1 has a pure maximin action, i.e. there is some "punishing" action a_1^P such that

$$v_2(a_1^P, BR(a_1^P)) = \min_{\alpha_1\in\Delta(A_1)}\max_{a_2\in A_2} v_2(\alpha_1, a_2)$$

The first assumption is made only for simplicity, the second is more substantial. Let

$$v_1^* = \max_{a_1\in A_1} v_1(a_1, BR(a_1))$$

denote player 1's static *Stackelberg payoff* and let a_1^* be a *Stackelberg action*, that is, an action attaining the maximum above. The best response to this action is $a_2^* = BR(a_1^*)$. Finally, $\underline{v}_i$ and $\overline{v}_i$ respectively denote the worst and best payoff for player i in G.

The feasibility correspondence $\overline{S}_1(\cdot)$ satisfies the following assumptions:

- There is a "*normal*" unconstrained type $\theta^0 \in \Theta$ such that $\overline{S}_1(\theta^0) = S_1$.

- There is a "*commitment*" type θ^* such that $\overline{S}_1(\theta^*) = \{s_1^*\}$ where s_1^* is a strategy "teaching" player 2 to play a_2^*. Strategy s_1^* plays a_1^* in normal phases and instigates punishment phases of increasing length when player 2 fails to play a_2^* in a normal phase (see Evans and Thomas, 1997). More precisely s_1^* is determined by an automaton with a countable set of states

$$Q = \{Norm(k), Punish(k,j); k = 0,1,...; j = 0,...,k\}$$

where $Norm(0)$ is the initial state; action and transition functions are given by the following table:

State	Action	Transition
$Norm(k)$	a_1^*	stay in $Norm(k)$ if $a_2 = a_2^*$, go to $Punish(k, k)$ otherwise
$Punish(k, 0)$	a_1^P	go to $Norm(k+1)$
$Punish(k, j), 1 \le j \le k$	a_1^P	go to $Punish(k, j-1)$

($Norm(k)$ is the normal phase after k defections and $Punish(k, j)$ is the punishment phase after k defections and with j punishments periods to come).

The beliefs of player 2 are represented by a probability measure on the set of feasible type-strategy pairs, $\mu^2 \in \Delta(\Sigma_1)$. I assume that the beliefs of player 2 satisfy the following restriction:

- For some fixed $\epsilon \in (0, 1)$, player 2 assigns at least ϵ prior probability to the commitment type θ^*, that is,

(1) $$\mu^2(\theta^*, s_1^*) \ge \epsilon$$

The set of beliefs satisfying (1) is denoted by Δ_ϵ^*.

A belief of player 1 is a probability measure over the set of player 2's strategies, $\mu^1 \in \Delta(S_2)$. Let $U_i(s_i, \mu^i)$ denote the expected utility of player i given strategy s_i and belief μ^i.

Definition 1. *A belief μ^1 of player 1 is Δ_ϵ^*-rationalizable if it assigns probability one to the set of player 2's strategies that maximize 2's expected utility for some belief in Δ_ϵ^*, i.e.*

$$\mu^1 \left\{ \bigcup_{\mu^2 \in \Delta_\epsilon^*} \arg \sup_{s_2 \in S_2} U_2(s_2, \mu^2) \right\} = 1$$

According to the following proposition, if player 1 is patient and rational and believes that player 2 is rational and assigns probability at least ϵ to the commitment type θ^*, then he expects to get a long-run average payoff approximately as large as the static Stackelberg payoff. Lemma 3 below implies that Player 1 can (approximately) achieve this lower bound if he builds up a reputation of behaving like the commitment type θ^*.

Proposition 2. *There is a positive integer $M = M(v_2, \delta_2, \epsilon)$ independent of δ_1 such that for every Δ_ϵ^*-rationalizable belief μ^1,*

$$\sup_{s_1 \in S_1} U_1(s_1, \mu^1) \ge U_1(s_1^*, \mu^1) \ge (1 - \delta_1^M)\underline{v}_1 + \delta_1^M v_1^*$$

Evans and Thomas (1997, Section 4) prove an analogous result about the lower bound to player 1's equilibrium payoffs in a Bayesian game where player 1 knows the beliefs of player 2 about his type θ and, in particular, knows the prior probability assigned by player 2 to the commitment type θ^*. The following proof is largely borrowed and adapted from their paper.

Proof of Proposition 2. We first look at how player 2's conditional beliefs evolve along a history generated by the "commitment" strategy s_1^* together with some strategy s_2. Then we will be able to find an upper bound on the number of times player 2 does not play a_2^* in any such history. If player 1 is patient enough his long-run average payoff is affected only slightly by what happens in a finite number of periods. Therefore by playing s_1^* player 1 would get almost the Stackelberg payoff.

Let $h^t(s)$ denote the history of length t induced by strategy pair s. Thus the actions selected in period $t+1$ by the strategy pair s are $s_i(h^t(s))$, $i = 1, 2$, and for any $\mu \in \Delta_\epsilon^*$ (the player superscript is suppressed for notational simplicity)

$$\mu\left(\Sigma_1(h^{t+k}(s_1^*, s_2)) | \Sigma_1(h^t(s_1^*, s_2))\right)$$

is the conditional probability that player 1 will behave as the Stackelberg commitment type from period $t+1$ to period $t+k$ given that he did so through period t and provided that player 2 is playing s_2 (note that this conditional probability is well defined because (1) implies that $\mu(\Sigma_1(h^t(s_1^*, s_2))) > 0$).

Lemma 3. (cf. Evans and Thomas, 1997). *If $\mu \in \Delta_\epsilon^*$, for any strategy s_2, for any real number $\eta \in (0, 1)$ and any positive integer k, there exists a finite integer $N(k, \eta, \epsilon)$ such that there are at most $N(k, \eta, \epsilon)$ periods $n_1, n_2, \ldots, n_\ell, \ldots$ in which:*
(i) s_2 triggers a punishment phase in period n_ℓ and
(ii) $\mu\left(\Sigma_1(h^{n_\ell+k-1}(s_1^*, s_2)) | \Sigma_1(h^{n_\ell-1}(s_1^*, s_2))\right) \le 1 - \eta$.

Proof of Lemma 3. Essentially we translate into our notation the proof of the analogous lemma in Evans and Thomas (1997). Since $\mu((\theta^*, s_1^*)|\Sigma_1) \geq \epsilon$, then for any $t = 0, 1, \ldots$ $\mu\left(\Sigma_1(h^{t+k}(s_1^*, s_2))|\Sigma_1(h^t(s_1^*, s_2))\right) \geq \epsilon$ because the latter probability is at least as large as the conditional probability of the commitment type, which cannot decrease along any history consistent with the commitment type. Consider a sequence of normal phase periods $n_1, n_2, \ldots$ such that (i) and (ii) both hold. By Bayes rule

$$\mu\left((\theta^*, s_1^*)|\Sigma_1(h^{n_\ell+k-1}(s_1^*, s_2))\right) =$$

$$= \frac{\mu\left((\theta^*, s_1^*)|\Sigma_1(h^{n_\ell-1}(s_1^*, s_2))\right)}{\mu\left(\Sigma_1(h^{n_\ell+k-1}(s_1^*, s_2))|\Sigma_1(h^{n_\ell-1}(s_1^*, s_2))\right)}$$

For brevity we write μ^t for the conditional probability of the commitment type given history $h^t(s_1^*, s_2)$. Using (ii) and the equation above we obtain $\mu^{n_\ell+k-1} \geq (1-\eta)^{-1}\mu^{n_\ell-1}$. If $\ell \geq k$, then period n_ℓ is followed by at least k periods of punishment and thus $n_{\ell+1}$ (another normal period) must come after $n_\ell + k$ (the last punishment period after the "deviation" in period n_ℓ). Taking into account this and the fact that μ^t is non-decreasing in t, for all $\ell \geq k$,

$$\mu^{n_{\ell+1}-1} \geq \frac{\mu^{n_\ell-1}}{1-\eta}$$

Since $\mu^{n_k-1} \geq \epsilon$, the equation above yields for all $\ell \geq k$

$$\mu^{n_\ell-1} \geq \frac{\epsilon}{(1-\eta)^{\ell-k}}$$

Since $1 > \eta > 0$, $\epsilon > 0$, $\mu^{n_1-1} \leq 1$, taking logs we obtain $0 \geq \log\epsilon - (\ell - k)\log(1-\eta)$ or

$$\ell \leq k + \frac{\log\epsilon}{\log(1-\eta)} =: N(k, \eta, \epsilon)$$

Q.E.D.

To prove Proposition 2, suppose s_2 is a best response to player 2' beliefs. Then, for any given payoff function v_2 satisfying our assumptions and any discount factor δ_2, we can choose η small enough and k large enough that s_2 fails to play a_2^* at most $K = k + N(k, \eta, \epsilon)$ times. This implies that the path induced by (s_1^*, s_2) contains at most $M = (1/2)K(K+3)$ periods when either a_2^* is not played or player 2 is punished (clearly M depends on ϵ, v_2 and δ_2). Therefore player 1's long-run payoff is at least as large as if he got the worst payoff on the first M periods and the Stackelberg payoff afterward. Q.E.D.

As noted in Battigalli and Watson (1997), the result stated in Proposition 2 holds also when θ^* is a commitment type always playing the Stackelberg action and the stage game G has "conflicting interests" in the sense of Schmidt (1993).

3. *Concluding Remarks*

This note strengthens the point made by Watson (1993) and Battigalli and Watson (1997), that some of the most important results about reputation can be obtained without using equilibrium analsysis and without assuming that there is a common prior on the set of payoff-relevant types (in this case, commitment types). Simple and transparent assumptions about rationality of the players and their first and second-order beliefs[3] suffice to obtain outcome selection results under the same hypotheses on payoffs and commitment types that were used to obtain such results for Bayes-Nash equilibria.

This illustrates a general methodological approach to the analysis of incomplete information games proposed by Battigalli (1999). Economists who apply Harsanyi's theory of incomplete information games often assume that there is a one-to-one relationship between payoff-types and epistemic types and/or that there is a common prior on the set of types. Both assumptions are typically made for tractability reasons, but they are unwarranted when we analyze situations with *genuine* incomplete information, where the type structure simply describes all the possible payoff functions and constraints as well as the possible hierarchies of beliefs. As an alternative to this approach, one can state simple and transparent assumptions about first-order beliefs and rationality and then explore the consequences of iterated mutual belief that

[3]First-order beliefs concern the payoff types and strategies of the other players. Second-order beliefs also concern the first-order beliefs of other players.

such assumptions hold. Such assumptions about rationality and beliefs are captured by iterative solution procedures akin to rationalizability (see Battigalli, 1999, and Battigalli-Siniscalchi, 1999). In this note, I have applied the first two steps of one such procedure.

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ABSTRACT

This note shows how a patient player in a two-person repeated game can build a reputation and achieve his stage-game Stackelberg (average) payoff, provided that his opponent is rational and assigns positive prior probability to the commitment type the patient player wants to imitate. This result is proposed as an illustration of a general approach to the analysis of incomplete information games, whereby simple and transparent assumptions about players' rationality and beliefs replace the traditional Bayesian-Nash equilibrium cum common prior hypothesis.

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