ALGORITHMIC SOLUTIONS FOR EXTENSIVE GAMES

P. Battigalli
Istituto di Economia Politica,
Università Commerciale Luigi Bocconi
Via Sarfatti 25, 20136 Milano.

1. Introduction.

An *algorithmic solution* (AS) of a game is a step by step or iterative procedure yielding one outcome or a set of possible outcomes. I say that the game possesses a strong solution in the former case and a weak solution in the latter (the possibility that the algorithm produces no solution at all will also be considered). Such an algorithm is meant to represent a chain of deductions from the basic assumption that each player is rational and rationality is common knowledge or at least shared knowledge of a sufficiently high degree. The main difference between AS_s and equilibrium concepts such as Nash or correlated equilibria is that an AS shows *how* a player reaches the conclusion that she should not play "irrational" strategies. Moreover no condition of the sort "objective probability = subjective probability" is assumed to hold, though it may be the result of the algorithm (for this reason such solutions are sometimes called ex *ante equilibria*).

"Classic" game theory considers only one algorithm to solve extensive games: the so called *backward induction* algorithm (BI) originally proposed by Zermelo [1912] to prove the existence of a solution for the game of chess. Roughly speaking it works in the following way: whoever plays last can easily find out the optimal action, provided that he knows enough about previous moves. Assuming rationality, the second to last can therefore predict the actions of those who follow her and, with the same proviso as before, she can choose the best action accordingly. Thus, the third to last can predict the actions of those who follow him... Even this rough description should make clear that

at least two problems arise:

i) it may be that a player does not know enough about the previous play, as is normally the case in games of imperfect information;

ii) consider a position that can be reached only if someone plays irrationally: why should a player in that position assume that the followers will play rationally? But if this player's faith in rationality could be shaken, one of those who precede him might try to mislead him, by playing "irrationally", hoping to induce a more favourable outcome. This is the problem of *counterfactuals and strategic manipulation of beliefs.*

A number of algorithmic solutions of games has been recently proposed in the literature. While the theory seems quite settled as far as normal form games are concerned, it is less developed and more controversial when it deals with extensive games. To my knowledge, the only algorithm which can be applied to all finite extensive games, exploiting all the additional information that is not contained in the normal form, is still Pearce's *extensive form rationalizability* (EFR). But it turns out that EFR is a generalization of BI and therefore is subject to the objection that it does not deal adequately with the counterfactuals problem.

In this paper I propose a systematic approach to ASs for finite extensive games. I argue that the theory must explicitly state assumptions about how a player would update her beliefs if she faced an unexpected event. In section 2, I discuss this point through a number of examples and state the informal assumptions which are to be embedded in any AS or only in some ASs. In section 3, I formally define a generic AS and then present four specific algorithms, distinguished through two dichotomic criteria according to the different assumptions about updating processes embedded in the algorithms. When the updating process satisfies a particular hierarchy of beliefs, an *irreversible* algorithm is obtained; otherwise the algorithm is *reversible.* An irreversible algorithm yields always at least a weak solution, while reversible algorithms are able to represent the paradoxes generated by counterfactuals and therefore may yield no definite solution. The second demarcation criterion depends on whether or not beliefs about different opponents are treated independently when unexpected events occur. In the affirmative case, the AS is characterized by *independent updating.* A reversible algorithm may be used to check the robustness (with respect to counterfactuals) of the corresponding irreversible algorithm. In section 4 I show that the irreversible algorithm without independent updating is equivalent to EFR, while the reversible algorithm with independent updating is quite similar in its performance to a solution concept proposed by Bonanno [1989], being indeed equivalent for games of perfect information in which any player plays no more than once. The last section contains some concluding remarks.

2. Examples and informal assumptions.

Extensive games are worth studing only if strategic decision making is analyzed as an on-going process. Though players may *choose* a strategy at the beginning of the game, they actually *implement* it by choosing an action whenever they are called to play. Such choices might be different from those prescribed by the initially chosen strategy.

ASSUMPTION 1 (INTERIM RATIONALITY). *Whenever a player is called to play, she is endowed with a conjecture (a probability assessment, which trasforms the game in a one-person decision tree), consistent with her current information, and she chooses an action maximizing expected utility according to this conjecture and her plan for the future.*

Assumption 1 seems to be a minimal rationality requirement and in very simple games it eliminates non-creadible threats. However it allows the possibility of an individual playing a strictly dominated strategy. This may happen when a player changes her conjecture "without a good reason". In game Γ_1 player 1 has to guess the choice of 2. Assume that her conjecture is $P(l|R_1) < 1/2$ at the initial node and $P(l|R_1) > 3/4$ at information set h. Then 1 actually *plays* – though she does not *choose* at the beginning of the game – R_1L_2, a strictly dominated strategy.

In order to make *interim* rationality stronger than *ex ante* rationality a sort of intertemporal consistency assumption is necessary.

Γ_1

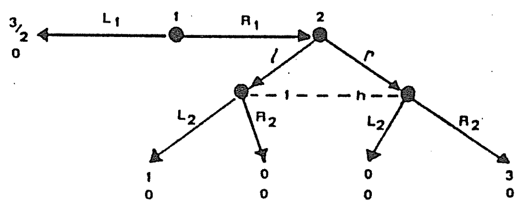

Figure 1.

ASSUMPTION 2 (INTERTEMPORAL CONSISTENCY). *No player changes her conjecture unless she observes a zero probability event.*

Assumption 2 is still not enough to take full advantage of the knowledge of the extensive game. Consider game Γ_2. It is usually claimed that $(L,\ l)$ is the unique solution of this game, though one may easily check that $(A,\ r)$ is a proper, perfect and sequential equilibrium. The argument runs as follows:

(i) R is not played because it is strictly dominated;

(ii) if 2 observes h, he deduces that L has been played and he chooses l (this is an instance of *forward induction*),

(iii) 1 predicts that 2 would play l, therefore she chooses L.

All the algorithms that I am going to define select $(L,\ l)$ as the unique solution of Γ_2, but it should

Γ_2

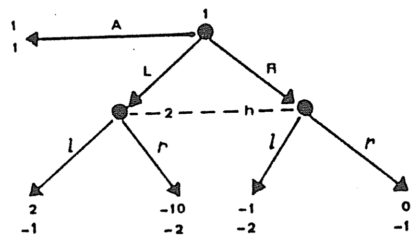

Figure 2.

be clear that more than common knowledge of rationality is at work. First of all the "trembling hand" story must be ruled out:

ASSUMPTION 3 (DELIBERATE CHOICES). *Any possibility of mistakes is fully represented by the extensive form of the game via chance moves. Hence choices are always deliberate.*

The most important point is that 2 must believe that 1 is rational even if 2 was sure of A and observes h, i.e. $L - or - R$. The reason is that $P(A) = 1$ is an *arbitrary conjecture*, because it can not be deduced from the hypotheses of the model. Therefore 2 is not allowed to reject the model when this initial conjecture is falsified.

ASSUMPTION 4 (UPDATING OF ARBITRARY CONJECTURES). *If a player knows the predictions of the model, she continues to believe them correct until they are falsified.*

Now modify Γ_2 interchanging player 1's payoffs corresponding to l with those corresponding to r. At first one may think that the only change is in step (iii): (iii') 1 predicts that 2 would play l, therefore she chooses A. Is this the end of the story? Perhaps not.

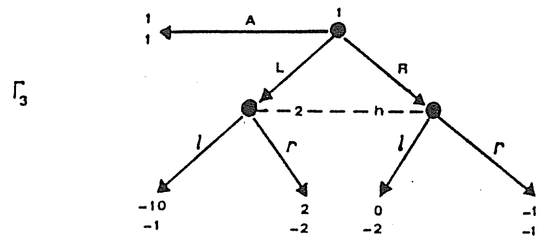

Figure 3.

According to this solution h is an impossible event and assumption 4 leaves player 2 free to reject the model if h is observed. h falsifies: "*everybody is rational*" & "*everybody believes that everybody is rational*" & "*everybody believes that everybody believes that everybody is rational*".

Moreover, if 2 observing h reached the conclusion that 1 is irrational, 1 could get his best payoff playing L. The solution is not common knowledge at every information set, so that the problem of counterfactuals and strategic manipulation of beliefs may arise. The most sensible way to rule out this possibility is to assume the following:

ASSUMPTION 5 (HIERARCHIES OF BELIEFS). *Each player has a sufficiently fine hierarchy of beliefs corresponding to implications of the model with increasing predictive power and she always holds the highest belief compatible with her current information.*

One may think that assumption 5 is quite ad hoc. In game Γ_3 player 2, observing h, should think that player 1 is rational, but she (i.e. 1) does not believe that 2 at h would believe that she is rational. More complex games need much more complex hierarchies of beliefs. But there are games in which a different and perhaps less ad hoc assumption does the same job. Consider game Γ_4.

If it is assumed that $1a = 1b$, then the situation is similar to that of Γ_3 and assumption 5 is needed in order to solve the game. But if 1a and 1b are two different players (with the same payoffs) the following assumption is enough:

ASSUMPTION 6 (INDEPENDENT UPDATING). *Each player updates conjectures and beliefs concerning different opponents independently of one another, whenever it is clear which op-*

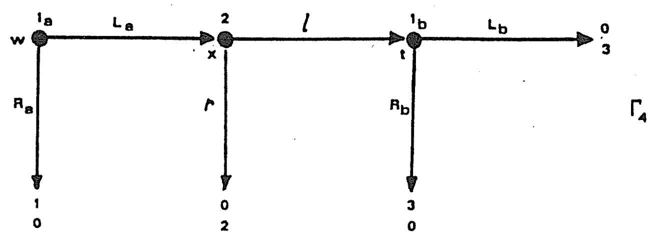

Figure 4.

ponent did not conform to the player's expectations.

Let us see how this applies to Γ_4. If node x were reached player 2 would have to modify his belief about $1a$ in a way that justified L_a, but his belief about $1b$ should be unaffected, therefore he should still predict that, since $1b$ is rational, if t were reached, L_b would be played. 1a anticipates this and therefore she has no hope to mislead 2. In all this discussion two hypotheses were implicit: complete information and common knowledge. They are stated now for completeness.

ASSUMPTION 7 (HARD CORE COMPLETE INFORMATION). *Each player knows the game at each of her information sets.*

This assumption is very strong. It states that the knowledge of the entire game is never touched by falsification, i.e. it is in the *hard core* of the structure of beliefs of any player. But one may object that such a role should be played by common knowledge of rationality, so that if current information falsifies the assumption that an opponent is rational, given the initial belief about the extensive game, it is this latter belief that should be modified. I think that this objection is reasonable when patently irrational choiches are observed (e.g. strictly dominated actions), but it loses a good deal of its strength in more interesting cases, e.g. when it is observed that player j chose an action which could be justified only if she does not know that her opponents are rational. In any case, to some extent, assumption 7 could also be weakened to become the assumption that each player knows the game before it begins and she knows at least the extensive form and her own payoffs at each information set. This is enough to justify the algorithm which I call "implementability" (see Battigalli [1988] and definition 3 of this paper), since this algorithm allows a player to hold a completely arbitrary conjecture, if her prior belief is inconsistent with her current information.

ASSUMPTION 8 (COMMON KNOWLEDGE). *Assumptions 1, 2, 3, 4 (and/or 5 and/or*

6) and 7 are common knowledge among the players at the beginning of the game.

Note that common knowledge at each information set is not required, otherwise a contradiction would arise for a large class of games, i.e. those games for which the theory predicts that at least one information set will not be reached. Reny [1987] expands on this point, showing how it constitutes a weakeness of such extensive form solution concepts as BI and Pearce's EFR. Then he develops a theory of conjectures formation, which is common knowledge at every information set, due to the fact that the theory does not assume anything about strategic choices. Here I take the alternative route of a theory of conjectures formation and strategic choice, which is *not* necessarily common knowledge at each information set. In my opinion the most important problem of extensive form solution concepts is the possibility of strategic manipulation of beliefs. Absence of common knowledge of the theory at some information set is a necessary but not sufficient condition for such a problem to arise. To see this more clearly consider game Γ_5. Common knowledge of assumptions 1 and 7 at the beginning of the play is sufficient to yield the only sensible solution of this game, i.e.

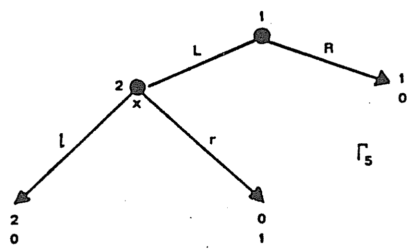

Figure 5.

$(R,\ r)$. The theory is not common knowledge at node x, but this is immaterial, since player 2 does not need to know the theory in order to choose the optimal action. A similar route is also taken by Basu [1988]. Essentially he assumes the following: Let information set $\{x\}$ (in a game of perfect information) be reached only if all the players $j \in J$ deviate form the rationalizable path. Then it is common knowledge that all the players i not in J are rational, while those in J are unpredictable, i.e. they could play any of their strategies. Unpredictability of players in J could induce someone to deviate form rationalizability in the subgame starting from x; this in turn could give to some $j \in J$ the incentive to deviate form EFR so that x is reached. This is, again, strategic manipulation of beliefs. Though outcomes of this kind could hardly be regaded as part of a weak solution set, one can reasonably argue that they are possible. One may object, however, that if such outcomes are

possible, each player could think that they do not violate the rationality principle and it is not clear why, if x occurred, players in J should be regarded as unpredictable. As I will argue later, when the possibility of strategic manuipulation makes the set of possible outcomes larger than the set of EFR-outcomes, it is more sensible to say that the game has no solution.

3. Four algorithms to solve extensive games.

I assume that the reader is familiar with the formalism of extensive games. Here I use the notation of Kreps and Wilson [1982], which is quite well-known. The only difference is that I do not assume that chance moves occur only at the beginning of the game.

An extensive form is a structure $(\mathrm{T}, <; \iota; \mathrm{H}; A; \rho)$ where:

- $(\mathrm{T}, <)$ is a finite topological tree with a set of terminal nodes Z, a set of decision nodes $\mathrm{X} = \mathrm{T}\backslash\mathrm{Z}$ and a distinguished root w; S(x) denotes the set of immediate successors of x and Z(x) the set of terminal successor of x; $\mathrm{S(h)} = \cup_{\mathrm{x}\in\mathrm{h}}\mathrm{S(x)}$ and $\mathrm{Z(h)} = \cup_{\mathrm{x}\in\mathrm{h}}\mathrm{Z(x)}$.
- $\iota : X \to \mathcal{N}$ is the players' function inducing the players' partition (ι(x) is the player who moves at x); let $|\iota(\mathrm{X})| = \mathrm{n} + 1$, then without loss of generality $\iota(\mathrm{X}) = \{0, 1, \ldots \mathrm{n}\}$, where 0 is the chance player; $\mathrm{I} = \{1, 2, \ldots, \mathrm{n}\}$ is the personal players set.
- $\mathrm{H} \subset 2^{\mathrm{X}}$ is the information partition, a refinement of the players' partition over $\mathrm{X}\backslash\iota^{-1}(0)$. I write ι(h) for the individual who is called to play at information set $\mathrm{h} \in \mathrm{H}$ and $-\iota(\mathrm{h})$ to denote the set of her opponents. H is such that $\mathrm{x}, \mathrm{t} \in \mathrm{h} \in \mathrm{H} \Rightarrow \mathrm{S(x)} = \mathrm{S(t)}$.
- $A : \mathrm{H} \to 2^{\mathrm{A}}$ is a correspondence, which assigns to each information set H the set of available actions $A(\mathrm{h}) \subset A$. By convention, $\mathrm{h}' \neq \mathrm{h} \Rightarrow A(\mathrm{h}') \cap A(\mathrm{h}) = \emptyset$.
- $\rho : S(\iota^{-1}(0)) \to (0, 1]$ is the probability assignement for chance moves: $\forall \mathrm{x} \in \iota^{-1}(0)\ \sum_{t\in \mathrm{S(x)}} \rho(t) = 1$.

We obtain an n-person *extensive game*, denoted by Γ, by adding to an extensive form an n-*tuple* of Von Neumann-Morgenstern utility indexes $\mathrm{u} = (\mathrm{u}^1, \mathrm{u}^2, \ldots \mathrm{u}^n), \mathrm{u}^i : \mathrm{Z} \to \Re$.

I assume, as usual, that each player is able to recall all her past actions and information:

ASSUMPTION G1. Γ *is a game with perfect recall.*

G1 implies that the partial order relation < induces a corresponding order over each player i's collection of information sets H^i; moreover for h, $h' \in H^i$ $h < h' \Leftrightarrow Z(h') \subseteq Z(h)$.

I also make a technical assumption: each player i has a unique "first" information set h^i_o:

ASSUMPTION G2. $\forall i \in I \; \exists \, h^i_o \in H^i : \forall h \in H^i \backslash \{h^i_o\} \; h^i_o < h$.

G2 makes the assumptions of "intertemporal consistency" and "interim rationality" sufficient for "ex ante rationality": an individual has to play a strategy which is a best response to her "first" conjecture. I claim that G2 is completely innocuous.

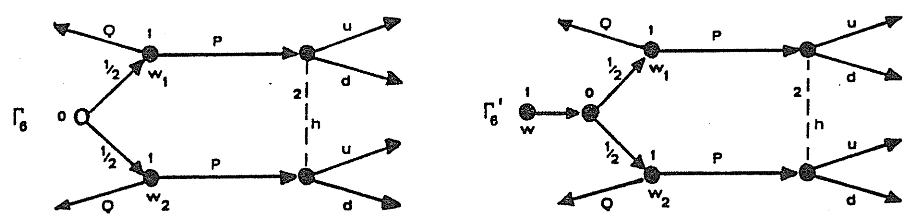

Figure 6.

Whenever a game does not satisfies G2, we can modify it by addition of some "dummy" information sets; the resulting game will be strategically equivalent to the former (see figure 6).

Note that Γ_6 is a game with chance moves. In such games assumption G2 eliminates the possibility that a conjecture depends on the state of nature. Otherwise, it may happen that an EFR-strategy is strictly dominated (see Battigalli [1988], p 720). It is proved in the appendix that in games without chance moves and in games with perfect information G2 is not at all restrictive.

A *strategy* of player i is a contingent plan $\sigma^i : H^i \to A$, where $\forall h \in H^i$ $\sigma^i(h) \in A(h)$. Σ^i denotes i's strategy set. Let $J \subseteq I$, then $\sigma^J = (s^j)_{j \in J}$ and $\Sigma^J = \prod_{j \in J} \Sigma^i$ (take the natural order for such $|J|$ – tuples and Cartesian products). I will often drop the superscript J, when J=I. Here I do not assume that players can choose mixed strategies. If I did, the difference would be immaterial. An expected utility maximizer chooses a mixed strategy only if all the pure strategies in its support yield the same expected utility and I do not need the mixed extension to get existence of "ex post rational" equilibria.

In this context, mixed strategy profiles are interpreted as conjectures. Let $\Delta(Y)$ denote the set of probability measures over Y. An element $c^i \in \Delta(\Sigma^i)$ is a *conjecture about player i.* Here I adopt the classical interpretation that (1) any event, which could be used to correlate the actions of two

different players, is adequately represented in the extensive form, and (2) this is common knowledge. Therefore each player should regard the strategies played by two different opponents as stochastically independent events. This seems to be the relevant interpretation, for example, if one wants to analyze the credibility properties of a correlated equilibrium implemented as a Nash equilibrium of a suitable extensive game. With this in mind, it makes sense to represent a conjecture about a group of players $J \subseteq I$ as a $|J|$ – *tuple* $c^J = (c^j)_{j \in J} \in \prod_{j \in J} \Delta(\Sigma^j)$. When $J \cap K = \emptyset$, I write $(c^J, c^K) = (c^i)_{i \in J \cup K}$. When $J = I \backslash \{i\}$ I write $c^J = c_i$, or $c^J = c_h$ if $i = \iota(h)$. c_i is a complete *conjecture of player i* and C_i is the space of such conjectures.

A conjecture c^I induces a probability measure $P(.; c^I) \in \Delta(Z)$ in the usual way (see e.g. Kuhn [1953]). As a function, $P(.; c^I)$ is extended to 2^T *via* the identity $P(h; c^I) = P(Z(h); c^I), h \subseteq T$. With an innocuous abuse of notation a strategy σ^i is identified with the corresponding measure c^i, where $c^i(\sigma^i) = 1$. Therefore $P(.; c_j, \sigma^i)$ is the subjective probability measure of player i, when she plans to play σ^i and has a conjecture c_i.

Let $\emptyset \neq J \subset I$. c^J *reaches* $h \subseteq T$ if and only if there exists a complementary conjecture $c^{I \backslash J}$ such that $P(h; c^J, c^{I \backslash J}) > 0$. By convention, $c^\emptyset$ does not reach any $h \subseteq T$. Let $J \neq \emptyset \neq K$. Note that "c^J reaches h and c^K reaches h" is a necessary but not sufficient condition for "$c^{J \cup K}$ reaches h". A J-strategy profile σ^J reaches h if and only if the corresponding degenerate conjecture does. $\Sigma|h$ denotes the set of J-strategy profiles reaching h.

It is now possible to produce a formal restatement of assumptions 1, 2, 4, 5, 6 (assumptions 3 and 7 are interpretative and cannot be formalized in this context, while assumption 8 – common knowledge – is represented through the use of iterative procedures).

DEFINITION 1. Let $c_i \in C_i$ reach $h \in H^i$. Then σ^i is a *best response to* c_i *at* h if and only if

$$\sigma^i \in \underset{\sigma \in \Sigma^i|h}{\mathrm{Argmax}} \sum_{z \in Z(h)} P(z; c_i; \sigma) u^i(z)$$

REMARK 1. By perfect recall (G1), there is a unique sequence of i's actions and information sets leading to $h \in H$; therefore all the strategies in $\Sigma^i|h$ coincide at any h'<h. Moreover the constraint $z \in Z(h)$ implies that choices at information sets not following h are irrelevant. Thus σ^i maximizes i's expected utility conditional on h among all the modifications of σ^i which are still possible at h (Pearce [1984] call them h-*replacements* for σ^i).

DEFINITION 2. An *updating process* is a collection of conjectures, one for each information set, $\{c_h; h \in H\} \subseteq \cup_{i \in I} C_i$, such that $\forall h \in H$ $c_h \in C_{\iota(h)}$ and c_h reaches h. $\mathcal{U}$ denotes the set of updating processes.

To be precise, an element of $\mathcal{U}$ is an n-*tuple* of different updating processes, since *a priori* no consistency condition between $\{c_h; h \in H^i\}$ and $\{c_h; h \in H^j\}$ ($i \neq j$) is required. I use one mathematical object instead of n just to simplify notation.

REMARK 2. The formal restatement of assumption 1 (interim rationality) is:

$\exists \{c_h; h \in H\} \in \mathcal{U} : \forall h \in H$ $\iota(h)$ actually chooses $a = \sigma^{\iota(h)}(h)$, where $\sigma^{\iota(h)}$ is a best response to c_h at h.

This is a rationality condition about *actions*; the updating process has to satisfy the intertemporal consistency condition to get rationality of strategies.

U1 (*Intertemporal consistency*). $\{c_h; h \in H\}$ is such that

$\forall h', h" \in H\, (\iota(h') = \iota(h")\&h' < h"\&c_{h'}\text{reaches } h") \Rightarrow c_{h'} = c_{h"}$.

REMARK 3. U1 is strictly related with Bayesian updating. U1 and interim rationality imply that agents behave as if they maximized conditional expected utility and updated probabilities via Bayes rule whenever possible (see also the appendix and Battigalli [1988]). From now on $B^J = \prod_{j \in J} B^J$ is a subset of Σ^J and is to be interpreted as a prior belief about the group of players $J \subseteq I$: "no player $j \in J$ will play a strategy outside B^j". Assumption 4 (updating of arbitrary conjectures) is formalized as a family of axioms indexed by prior beliefs. Let $\text{Supp}(c^J) = \{\sigma^J \in \Sigma^J | \prod_{j \in J} c^j(\sigma^j) > 0\}$ be the set of J-strategy profiles having positive probability according to c^J. A conjecture c^J is consistent with belief B^J if and only if $\text{Supp}(c^J) \subseteq B^J$; it is arbitrary w.r.t. B^J when the inclusion is strict. In all the algorithms considered here it is essential that a player may hold arbitrary conjectures. When a belief is not a singleton, a player is free to hold any conjecture consistent with it. Moreover a conjecture may be arbitrary at step t of a reasoning process, but not at step t+k. If non-arbitrariness were imposed at each step, a contradiction could arise: *i* does not play σ^i because σ^j has a positive probability, but *j* does not play σ^j because σ^i is not played (see Pearce [1982]). Axiom U2 says that if player *i* observes an event which falsifies her (arbitrary) conjecture, but not her belief, she continues to hold her belief.

U2(B) (*Updating of arbitrary conjectures*). Given $B \subseteq \Sigma$, $\{c_h; h \in H\}$ is such that $\forall h \in H$ $(B^{-\iota(h)} \cap \Sigma^{-\iota(h)}|h) \neq \emptyset \Rightarrow \text{Supp}(c_h) \subseteq B^{-\iota(h)}$.

Since the meaning of this formulation is "if the players have beliefs B, then ...", U*i*(.) – where beliefs are not fixed – is informally called axiom form.

A *proper k-hierarchy of beliefs* is a sequence $\{B(\nu)\}_{\nu=0}^{\nu=k}$, where $B(k) \subseteq B(k-1) \subseteq \ldots \subseteq B(1) \subseteq B(0) = \Sigma$, but in the following formalization I just need $B(0) = \Sigma$. Let μ_h^k denote the highest index ν for which belief $B^{-\iota(h)}(\nu)$ in a k-hierarchy is compatible with information h:

$$\mu_h^k = \max\{\nu \in \{0, 1 \ldots k\} | (B^{-\iota(h)}(\nu) \cap \Sigma^{-\iota(h)}|h) \neq \emptyset\}$$

Then assumption 5 is restated as follows.

U3' (B(0),...,B(k)) (*Hierarchies of beliefs*). Given B(0),...B(k), $\{c_h; h \in H\}$ is such that $\forall h \in H$ $\text{Supp}(c_h) \subseteq B^{-\iota(h)}(\mu_h^k)$.

REMARK 4. U3'(.) makes sense for any sequence of subsets of Σ with Σ as an element. But if the sequence is a proper hierarchy, U3(B(0), ...B(k-1),B(k)) $\Rightarrow$ U3(B(0), ...B(k-1)), since $B(k) \subseteq B(k-1)$. Moreover U3(B(0), ...B(k)) $\Rightarrow$ U2(B(k)), since $(B^{\iota(h)}(k) \in \Sigma^{\iota(h)}|h) \neq \emptyset \Rightarrow \mu_h^k = k$. By induction, U3(B(0) ...B(k)) $\Rightarrow$ U2(B(0)) & ... & U2(B(k)).

But also the reverse implication holds true. Therefore U3(B(0)...B(k)) $\Leftrightarrow$ U2(B(0)) & ... & U2(B(k)) whenever $B(k) \subseteq B(k-1) \subseteq \ldots \subseteq B(0)$. I will show in Lemma 1 that hierarchies of beliefs are proper by construction. This justifies the following alternative formulation.

U3 (B(0), ..., B(k)) (*Hierarchies of beliefs*).

Given B(0),...B(k), $\{c_h; h \in H\}$ is such that U2(B(0))&...&U2(B(k)) holds.

Independent updating (assumption 6) concerns both conjectures and beliefs, therefore it is formalized as a stronger version of U1, U2(.) and U3(.). From now on a star (*) denotes the fact that independent updating is at work. P(h) denotes the collection of partitions of the set $-\iota(h)$ of opponents of the player moving at h.

U1* (*Intertemporal consistency with independent updating*). $\{c_h; h \in H\}$ is such that $\forall h', h" \in H\, \forall \{J, K\} \in P(h')\, (\iota(h') = \iota(h")\&h' < h"\&c_{h'}^J \text{ reaches } h" \&c_{h'}^K \text{ does not reach } h") \Rightarrow c_{h'}^J = c_{h"}^J$.

U2*(B) (*Independent updating of arbitrary conjectures*). Given $B \subset \Sigma, \{c_h; h \in H\}$ is such that $\forall h \in H \; \forall \{J,K\} \in P(h) \; (B^J \cap \Sigma^J|h \neq \emptyset \,\&\, B^K \cap \Sigma^K|h = \emptyset) \Rightarrow \text{Supp}(c_h) \subseteq B^J$.

To keep the formulation of U3* simple, I rely again on remark 4.

U3*(B(0)...B(k)) (*Hierarchies of beliefs with independent updating*).

Given $B(0), \ldots B(k), \{c_h; h \in H\}$ is such that $U2^*$ (B(0)) & ... & $U2^*$ (B(k)) holds.

REMARK 5. $U1^* \Rightarrow U1, U2^*(B) \Rightarrow U2(B)$, and $U3^*(B(0)\ldots B(k)) \Rightarrow U3(B(0)\ldots B(k))$ (just take $K = \emptyset$ and note that $c^{\emptyset}$ does not reach any h in H and $\Sigma^{\emptyset}|h = \emptyset$).

Note that independent updating requires the proviso that it is possible to identify an "innocent" component of a conjecture or a belief. Thus, the fact that the J-component is compatible with current information is not enough, since the same may be true of the complementary K-component. This happens in game Γ_7, where the independent updating hypothesis is useless. In this game player 2 and 3 have to guess the past choice of 1. Player 2 has also to guess which action will be chosen by 3. Player 3 observes whether 1 and 2 "agreed" $(L, l$ or $R, r)$ or "disagreed" $(L, r$ or $R, l)$. L is strictly dominated; thus, by forward induction, 3 should play always right (r, r'); by forward and backward induction 2 should play r. Therefore the "natural" solution of the model yields a conjecture (belief) (R, r) for 3. But a typical counterfactual problem arises: 2 may deviate in order to signal "irrationality" of player 1, inducing 3 to play l. He may hope to obtain this result because 3 cannot identify who deviated from the proposed solution. Of course, a proper hierarchy of beliefs eliminates the problem, but independent updating does not help.

Γ_7

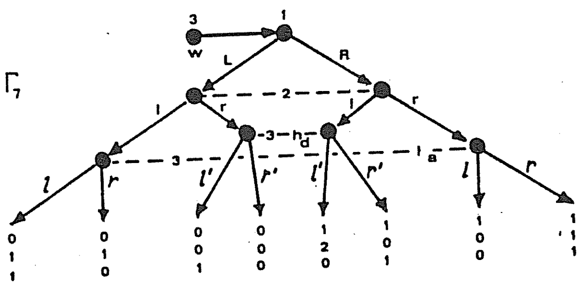

Figure 7.

One may also wonder whether U1*, combined with U2(.), is enough to represent what is meant here by "independent updating". The following example shows that this is not the case.

By forward and bakward induction, one obtains the proposed solution (belief) $\{L, R\} \times \{r\} \times \{r, r'\}$.

Γ_8

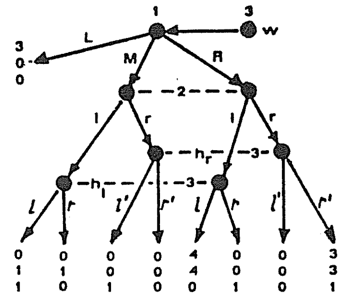

Figure 8.

Again a counterfactual problem arises. A very sensible conjecture to hold at the dummy information set $h_0^3 = \{w\}$ is (L, r). With this initial conjecture, U1* is useless at h_l since both components of the conjecture are falsified. Therefore, if R is played – as allowed by the proposed solution – there is room for strategic manipulation of beliefs. 2 could play l hoping to mislead 3, signaling that he believes that 1 made an irrational move; if 3 believes this signal (actually a "lie"), l is her best response. But if U2* is common knowledge, all this cannot happen. At h, 3 would only change her belief about 2, adopting a conjecture which assigned positive probability to (R, l) and zero probability to M, so that the best response would still be r.

I now describe a generic algorithm to solve extensive games. Let U be a set of axioms and axiom forms about updating processes (for example $U = \{U1, U2(.)\}$). A U-algorithm is given by the following steps:

- *first step*: let $B(\{c_h; h \in H\})$ be the set of strategy profiles σ such that $\forall h \in H \; \sigma^{\iota(h)} \in \Sigma^{\iota(h)}|\, h \Rightarrow \sigma^{\iota(h)}$ best reply to c_h at h; then find the set of strategy profiles $\Sigma(1) = \{\sigma \in \Sigma | \exists u \in \mathcal{U}$ such that any axiom in U holds and $\sigma \in B(u)\}$;

- *inductive step*: let $\Sigma(\nu)(\nu = 0, 1, \ldots k)$ be given; set $B(\nu) = \Sigma(\nu)(\nu = 0, 1 \ldots k)$ and find the set $\Sigma(k+1) = \{\sigma \in \Sigma \,|\, \exists u \in \mathcal{U}$ such that any axiom in U holds and any axiom form in U holds at B(k) or (B(0), B(1), ... B(k)) (according to which expression applies) and $\sigma \in B(u)\}$.

The proviso $\sigma^{\iota(h)} \in \Sigma^{\iota(h)}|h$ (i.e. $\sigma^{\iota(h)}$ reaches h) is important in the definition of $B(u)$. If a strategy σ^i is regarded as a contingent plan, it is never necessary to consider those contingecies which are excluded by σ^i itself (remember that choices are assumed to be deliberate). If the updating process u satisfies intertemporal consistency (U1), $\sigma \in B(u)$ is necessary and sufficient for interim rationality.

This follows inductively by the equality

$$\sum_{z\in Z(\mathrm{h})} P(z;c_{\mathrm{h}},\sigma^{i})u^{i}(z) = \sum_{z\in Z(\mathrm{h})\setminus Z(h)} P(z;c_{\mathrm{h}},\sigma^{i})u^{i}(z) + \sum_{z\in Z(h)} P(z;c_{h},\sigma^{i})u^{i}(z),$$

where $\mathrm{h},h \in H^{i}$, $\mathrm{h}< h$, c_{h} and c_{h} are drawn form u. This also shows that $B(u) \neq \emptyset$ if u satisfies U1. A similar equality shows that, by G2, if $\sigma \in B(u)$, each σ^{i} is a best response to some conjecture c_i:

$$\sum_{z\in Z} P(z;c_i,\sigma^{i})u^{i}(z) = \sum_{z\in Z\setminus Z(h_0^i)} P(z;c_i,\sigma^{i})u^{i}(z) + \sum_{z\in Z(h_0^i)} P(z;c_i,\sigma^{i})u^{i}(z).$$

Let c_i be i's conjecture at h_0^i and note that the first element on the RHS depends only on c_i. Two strategies are ***structurally equivalent*** if and only if they reach the same collection of information sets and coincide on it (see Kuhn [1953] def 5). One can easily check that this restatement of interim rationality does not distinguish between structurally equivalent strategies. Note that, at each step, the space of eligible strategies is always Σ. Therefore it may happen that $\Sigma(k+1)$ is not a subset of $\Sigma(k)$. If $\Sigma(k+1) \subseteq \Sigma(k)(k = 1,2,3,\ldots)$ is not a theorem, I say that the algorithm is *reversible*; otherwise it is *irreversible*. If an U-algorithm is irreversible, then $\cap_{k>1}\Sigma(k)$ is a U-solution (a strong solution when the solution set is a singleton).

One may argue that an algorithm must always induce some sort of solution, for example $\limsup_{k\to\infty} \Sigma(k)$. However I feel that an instance of $\Sigma(k+1)\setminus\Sigma(k) \neq \emptyset$ would undermine the reasoning that the algorithm should represent. After all $\Sigma(k)$ is a conditional prediction at step k about strategies ("no player would play a strategy outside $\Sigma(k)$, if given the opportunity") and the prediction of the following step should not be weaker (see also the discussion of game Γ_9). Of course many algorithms may be defined by different choices of U, but I argue that a natural choice is given by the following table:

		independent updating: no	*independent updating*: yes
hierarchy of beliefs	no	U1, U2(.) IMP (Σ(k))	U1*, U2*(.) *-IMP (Σ^*(k))
hierarchy of beliefs	yes	U1, U3(.) EFR (R(k))	U1*, U3*(.) *-EFR (R*(k))

Table 1

In fact, it does not seem reasonable to assume independent updating only for conjectures or only for beliefs, while -as will be shown- the assumption of a hierarchy of beliefs characterizes irreversible algorithms. Since the {U1,U2(.)}-algorithm is essentially the same as in Battigalli [1988], I retain the same name: *implementability* (IMP). A k-implementable strategy is a contingent plan, which could actually be implemented by a player endowed with a (k-1)-prior belief. I will show that the {U1,U3(.)}-algorithm is equivalent to Pearce's *rationalizability* (EFR) and again I retain the same name. **-implementability* (*-IMP) and **-rationalizability* (*-EFR) are the companion algorithms with independent updating.

It should be clear by now that, though table 1 may seem reminiscent of an axiomatic approach, it is not possible to say a priori which algorithm is stronger, because the use of axiom forms, whose argument is determined by the previous step, prevents the application of the implications of remarks 4 and 5. The four algorithms are formally defined as follows:

DEFINITION 3. The k-*implementable* strategy set $\Sigma(k)$ is iteratively defined as follows:

$$\Sigma(0) = \Sigma,$$

$$\Sigma(k) = \{\sigma \in \Sigma \mid \exists u \in \mathcal{U} : \text{U1 and U2}(\Sigma(k\text{-}1)) \text{ hold and } \sigma \in B(u)\}.$$

The k-**-implementable* set $\Sigma^*(k)$ is defined in the same way, except that U1 is replaced by U1* and U2(Σ(k-1)) by U2*(Σ^*(k-1)). The k-*rationalizable* set R(k) is defined as follows:

$$R(0) = \Sigma,$$

$R(k) = \{\sigma \in \Sigma \mid \exists u \in \mathcal{U}: \text{U1 and U3}(R(0), R(1), \ldots R(k-1)) \text{ hold and } \sigma \in B(u)\}$

$R = \cap_{k \geq 1} R(k)$ is the *rationalizable* set.

The k-*-*rationalizable* set is defined in the same way, except that U1 is replaced by U1* and U3(R*(0)...R*(k-1)) by U3*(R (0)...R (k-1)). $R^* = \cap_{k \geq 1} R^*(k)$ is the *-*rationalizable* set.

REMARK 6. These four algorithms coincide for one-person games (where they select the set of ex ante rational strategies) and simultaneous games (where they coincide with normal form rationalizability). Implementability (rationalizability) coincides with *-implementability (*-rationalizability) for two-person games. Moreover by remark 4 and 5, $R(1) = \Sigma(1)$, $R^*(1) = \Sigma^*(1)$, $\Sigma^*(1) \subseteq \Sigma(1)$, $R^*(1) \subseteq R(1)$. The following lemma shows that rationalizability and *-rationalizability are irreversible and that they select a solution in a finite number of steps.

LEMMA 1. $\forall k \in \mathcal{N}$ $R(k+1) \subseteq R(k) \& R^*(k+1) \subseteq R^*(k)$. Moreover $\exists K, K^* \in \mathcal{N}$: $k \geq K \Rightarrow R(k) = R \neq \emptyset, k \geq K^* \Rightarrow R^*(k) = R^*$ and R (R*) has the best response property, i.e. $\forall i \in I$ $\sigma^i \in R^i \Rightarrow (\exists c_i \in C_i : Supp(c_i) \subseteq R^{-i} \& \sigma^i$ is a best response to $c_i)$.

Proof. The proof is carried out for rationalizability only, since the case of *-rationalizability is almost identical. $R(1) \subseteq R(0)$ is trivially true. By induction, let $R(k) \subseteq R(k-1) \subseteq \ldots \subseteq R(0)$. If $\sigma \in R(k+1)$ then $\exists u \in \mathcal{U}$ such that U1, U3(R(0), ... R(k)) hold and $\sigma \in B(u)$; but u also satisfies U3(R(0),... R(k-1)), then $\sigma \in R(k)$. For the second part just note that it is always possible to find an updating process u satisfying U1 and U3(R(0)... R(k-1)) because the hierarchy of beliefs is proper, as it is proved above. Intertemporal consistency implies that $B(u) \neq \emptyset$. Therefore $\forall k \in \mathcal{N}$ $\emptyset \neq R(k)$. Definition 3 implies that R(k-1)=R(k)⇒R(k)=R(k+1). Since the sequence is decreasing and Σ is finite, for some K R(K)=R(K+1). For the best response property take a suitable $u = \{c_h; h \in H\}$ with $\sigma \in B(u)$. If $h_0^i = h$ is reachable by some profile in R^{-i}, set $c_h = c_i$; otherwise c_i is arbitrary provided that $Supp(c_i) \subseteq R^{-i}$. This completes the proof. □

Though my approach is not axiomatic, I am able to obtain one important relationship between algorithms: whenever a reversible algorithm selects a weak solution, it coincides with the corresponding irreversible algorithm.

THEOREM 1. $(\forall k \in \mathcal{N}\ \Sigma(k+1) \subseteq \Sigma(k)) \Leftrightarrow (\forall l \in \mathcal{N}\ \Sigma(l) = R(l))$. The same holds true for *-implementability and *-rationalizability.

Proof. (⇒) Let $\{\Sigma(k)\}_0^\infty$ be decreasing. By induction, let Σ(k)=R(k), $k = 0, 1 \ldots, l-1$. If $\sigma \in R(l)$, then $\exists u \in \mathcal{U}$ such that $\sigma \in B(u)$ and U1, U3(R(0),...R(*l-1*)) hold. Then U2(R(*l-1*)) holds. U2($\Sigma(l-1)$) is the same as U2($R(l-1)$); thus $\sigma \in \Sigma(l)$. If $\sigma \in \Sigma(l)$, by assumption $\sigma \in \cap_{k=0}^{k=l} \Sigma(k)$. Then one can find l updating processes $u^0 \ldots u^{l-1}$ such that $\sigma \in B(u^k)$ and U1, U2(Σ(k)) hold (k=0,...$l-1$). Let H(k) be the collection of information sets h reached by some conjecture consistent with belief $\Sigma^{-\iota(h)}(k)$, or equivalently $H(k) = \{h \in H \mid \Sigma^{-\iota(h)}(k) \cap \Sigma^{-\iota(h)} | h \neq \emptyset\}$ $(k = 0, \ldots l-1)$. Since $\{\Sigma(k)\}_0^{l-1}$ is decreasing, also $\{H(k)\}_0^{l-1}$ is decreasing. Now define $H^+(k) = H(k) \backslash H(k+1)$ for k=0,...$l-2$, and $H^+(l-1) = H(l-1)$. $\{H^+(k)\}$ is such that $\cup_{k=0}^{k=l-1} H^+(k) = H$ and either $H^+(s) \cap H^+(t) = \emptyset$ or $H^+(s) = H^+(t)$. Another property of $\{H^+(k)\}$ is that if h, $h \in H^i$, h < h, $h \in H^+(s)$ and $h \in H^+(t)$ (with $H^+(s) \cap H^+(t) = \emptyset$), then no conjecture consistent with $\Sigma^{-i}(s)$ may reach h and s>t. In fact, h contains more information than h and it falsifies the higher level belief, which has a higher predictive power. Now construct a new updating process $u = \{c_h; h \in H\}$ as follows: if $h \in H^+(k)$, then $c_h = c_h^{(k)} \in u^k$. By the inductive hypothesis, u^k satisfies U2(R(k)). Hence, by definition of {H+ (k)}, c_h is consistent with the highest belief compatible with h, i.e. $R^{\iota(h)}(\mu_h^{l-1})$. Therefore u satisfies U3(R(0),...R($l-1$)). By the properties of $\{H^+(k)\}$ (see above) the only instances in which U1 applies are those with h, $h \in H^+(s)$ for some s. Then c_h and c_h are drawn from the same updating process u^s, which satisfies U1. Therefore ($\iota(h) = \iota(h) \in I \&$ h< $h \& c_h$ reaches $h) \Rightarrow c_h = c_h$, i.e. u satisfies also U1. Moreover, by construction, u is such that $\sigma \in B(u)$. Thus $\sigma \in R(l)$. This completes the proof of the implication "⇒". (⇐) This implication follows immediately by lemma 1. The proof for the case of independent updating is almost identical. □

REMARK 7. One may easily check that $\Sigma(2) \subseteq \Sigma(1) \subseteq \Sigma(0)$. Then the proof of theorem 1 shows that Σ(k) = R(k), k=0,1,2. The same holds true for the case of independent updating.

The meaning of theorem 1 is that implementability may be used to check the robustness of rationalizability with respect to problems due to counterfactuals. One may object, however, that $\Sigma(k) \subseteq \Sigma(k-1)$ is only a sufficient condition for such robustness. After all, only predictions about the outcome should matter. Let Z(k) be the set of terminal nodes reached by some $\sigma \in \Sigma(k)$. It may happen that Σ(k) is not included in Σ(k-1), but Z(k)⊆Z(k-1). The follwing example, due to

Van Damme, shows that, though this objection is reasonable, one should be suspicious whenever $\Sigma(k)$ is not included in $\Sigma(k\text{-}1)$.

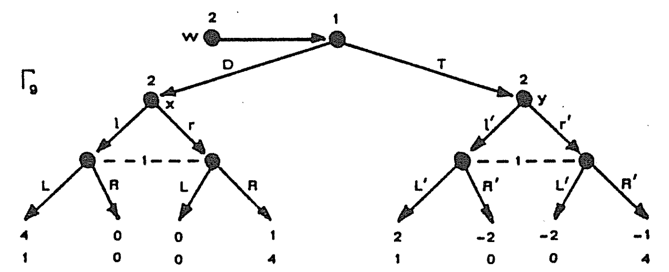

Figure 9.

Γ_9 is a version of the battle of the sexes, in which player 1 may signal her conjecture, and hence her choice between left and right, by throwing away two "utils" (T). In fact, this is rational only if she expects that l' would be played with sufficiently high probability. Taking this into account, it is easy to understand the following steps (note that structurally equivalent strategies are not distinguished):

Σ (1)=R(1)= $\{TL', DL, DR\} \times \{ll', lr', rl', rr'\}$

Σ (2)=R(2)= $\{TL', DL, DR\} \times \{ll', rl'\}$

Σ (3)=R(3)= $\{TL', DL\} \times \{ll', rl'\}$

Σ (4)=R(4)= $\{TL', DL\} \times \{ll'\}$

Σ (5)=R(5)= $\{DL\} \times \{ll'\} \Rightarrow$ R(k)=R(5), k$\geq$5.

In words, the mere opportunity to signal her intention by throwing away two utils is enough for 1 to get the preferred equilibrium in *each* subgame. Then there is no reason to throw away two utils. Note, however, that information set {y} is a counterfactual at this point. According to implementability, 2 is free to hold any conjecture consistent with {y}. Thus, $\Sigma^2(6) = \{ll', lr'\}$. But Z(6)=Z(5), because there is no incentive for strategic manipulation of beliefs by 1. In my opinion, one should be uneasy with the solution provided by rationalizability. This solution concept implicitly argues that (DL, ll') should be played because if {y} is reached, since 2 believes 1 rational, 2 expects L'. But the solution implies that if {y} is reached, 1 is "irrational"! Of course, a suitable hierarchy of beliefs solves the problem, but the point is that *there is a problem* to be solved. Theorem 1 shows that irreversible algorithms (rationalizability) are stronger than reversible ones (implementability). This

is not surprising, since they rely on stronger hypotheses about updating processes. But one must be careful with such inferences, because my approach is not really axiomatic. This is well-illustrated by the hypothesis of independent updating. Intuition suggests that an algorithm which relies on independent updating should be stronger than a corresponding algorithm which does not.

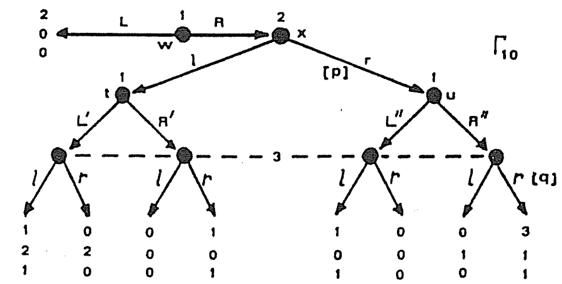

Figure 10.

This intuition is illustrated by game Γ_{10}. Letters in brackets denote the transition probability assigned to a particular action by a conjecture of player 1. It is rational for 1 to play R only if c_w – her initial conjecture – yields p>0, q$\geq$2/3. Now assume that 1 expects r with probability 1 (c_w =(1,q)). If U1 holds, but U1* does not hold and 2 plays l, 1 may change her entire conjecture. Her new conjecture could be c_t=(p,q), where p<1 and q<1/2. Therefore the choice L' is justified by rationalizability. Player 3 needs to know wether 1 played left (L or L') or right (R or R'), but rationalizability does not provide any help. Player 2 is sure to get 1 if he plays r, but he might try to make 1 play L', by playing l. If succesful, he would get 2. This explains why:

$\Sigma(k) = R(k) = \{L, RR'R", RL'R"\} \times \{l,r\} \times \{l,r\},\ k \geq 1.$

Now assume that U1* holds. Then c_t = (p,q) must be such that p<1 and q$\geq$2/3 if 1 chose R, and only the choice R' is justified at {t}. Therefore

$\Sigma^*(1) = R^*(1) = \{L, RR'R"\} \times \{l,r\} \times \{l,r\}$

$\Sigma^*(2) = R^*(2) = \{L, RR'R"\} \times \{r\} \times \{r\}$

$\Sigma^*(3) = R^*(3) = \{RR'R"\} \times \{r\} \times \{r\}$

But the intuition according to which *-algorithms are stronger is not correct in terms of strategies. Consider game Γ_{11}. In this game player 2 is just a sort of unpredictable dummy. Player 1 rationally chooses R only if p>0 and q$\geq$1/2 according to c_w. If U1* holds q$\geq$1/2 also at x. Hence only L and $RR'R"$ may be justified. Assume that player 3 has a prior belief $B^1=\{L, RR'R"\}$ about player 1; if h_{LM} were reached, this belief would be falsified and 3's conjecture at h_{LM} would be unrestricted.

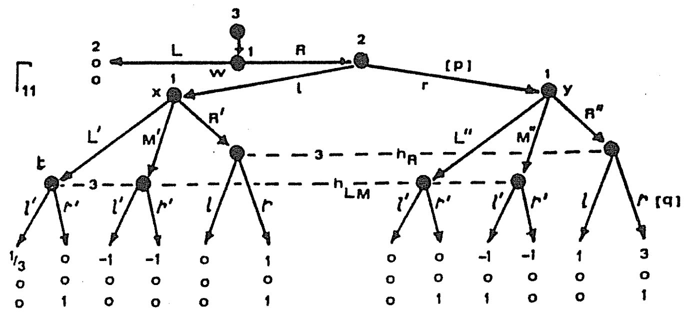

Figure 11.

This conjecture might assign a positive probabilty to M' or M'', two strictly dominated actions, and both l' and r' may be justified. This shows that

$$\Sigma^*(k) = R^*(k) = \{L, RR'R''\} \times \{l, r\} \times \{rr', rl'\},\ k \geq 1$$

But if U1* does not hold, while U1 holds, L' (not L'') may be justified. In this case h_{LM} is consistent with player 1 rational and 3 has no reason to assign a positive probability to M' or M'' at h_{LM}. On the contrary, she should be sure to be at node t, corresponding to R, l, L'. This explains why

$$\Sigma(1) = R(1) = \{L, RR'R'', RL'R''\} \times \{l, r\} \times \{rr', rl'\}$$

$$\Sigma(2) = R(2) = \{L, RR'R'', RL'R''\} \times \{l, r\} \times \{rr'\}.$$

Note that $R^3(k)$ is strictly included in $R^{*3}(k)$ for $k \geq 2$. This happens because *-rationalizability is "too strong" in the first step. But note also that the *-rationalizable set of outcomes is strictly included in the rationalizable set. In fact, *-rationalizability may be weaker only at unreached information sets. Though I strongly suspect that this result is general, I have not yet been able to prove it.

4. Rationalizability and Backward Induction.

In this section I prove the equivalence between the characterization of extensive form-rationalizability (EFR) given in definition 3 and Pearce's definition. Then I will present some results about the relationships between algorithmic solutions and backward induction. I give here for completeness the relevant part of Pearce's definition of EFR (see Pearce [1984] def.9) with only some changes in notation and a completely innocuous modification. Pearce assumes that players may use mixed strategies. As ususal, I identify pure strategies with the corresponding degenerate measures. Let

$M^i(0)=\Delta(\Sigma^i)$ be i's mixed strategy set at step 0. Assume that $M(t-1) = \prod_{i\in I} M^i(t-1)$ is defined. Then $H^i(\sigma^i, t)$ is the collection of i's information sets which are consistent with step $(t-1)$ and are reached by σ^i, or $H^i(\sigma^i, t) = \{h \in H^i | \exists c_i \in \prod_{j\neq i} M^j(t-1) : (c_i, \sigma_i) \text{ reaches } h\}$

REMARK 8. Let $H(t-1)$ be the collection of information sets reached by some conjecture consistent with step $(t-1)$ of rationalizability as defined in def. 3 (see the proof of theorem 1) and let CH(Y) denote the convex hull of Y. If $\{c_i \in C_i | \text{Supp}(c_i) \subseteq R^{-1}(t-1)\} = \prod_{j\neq i} CH(M^j(t-1))$, then $H^i(t-1) \cap \{h \in H^i | (\sigma^i \in \Sigma^i|h)\} = H^i(\sigma^i, t)$.

Let $R_P(t)$ denote the set of t-rationalizable *pure* strategies according to Pearce's definition. $R_P(t)$ is defined as follows:

$\sigma^i \in R^i_P(t)$ if and only if $\sigma \in R^i_P(t-1)$ and $\exists\{c_h; h \in H^i(\sigma^i, t)\}$ such that

(iii) $(h, h'' \in H^i(\sigma^i, t) \& h' < h'' \& c_{h'} \text{ reaches } h'') \Rightarrow c_{h'} = c_{h''}$,

(iv) $\forall h \in H^i(\sigma^i, t)$ c_h reaches h

(v) $c_h \in \prod_{j\neq i} CH(M(t-1))$

(vi) $\forall h \in H^i(\sigma^i, t)$ σ^i is a best response to c_h among all h-replacements for σ^i in $R^i_P(t-1)$ ((iii)-(vi) above correspond (iii)-(vi) in Pearce's definition).

Note that, at each step, additional restrictions are placed on conjectures and actions only at information sets that can be reached by strategy profiles which were not previously eliminated. Moreover (vi) requires utility maximization over a restricted set included in $R_P(t-1)$ and it is stipulated by definition that $R_P(t) \subseteq R_P(t-1)$. By contrast, definition 3 requires utility maximization at each information set h reached by σ^i and *all* strategies reaching h are eligeble, not only those in $R_P(t-1)$. Nonetheless theorem 2 states that these two definitions are equivalent.

THEOREM 2. $\forall t \in \mathcal{N}$ $R(t) = R_P(t)$.

Proof. By definition $R_P(0)=R(0)$. By induction, assume $R_P(k)=R(k)$ for $k = 0, 1 \ldots (t-1)$. Let $\sigma \in R(t)$, then $\sigma \in R(t-1) = R_P(t-1)$ by lemma 1. Since $M(t-1)$ has the pure strategy property (a mixed strategy c^j belongs to $M^j(t-1)$ only if all the pure strategies in its support belong to $M^j(t-1)$, see Pearce [1984]), the proviso of remark 8 holds true, i.e. $\{c_i \in C_i | \text{Supp}(c_i) \subseteq R^{-i}(t-1)\} = \prod_{j\neq i} CH(M^j(t-1))$, because $R^j_P(t-1)$ is the set of extreme points of $M^j(t-1)$.

Then $H^i(\sigma^i, t) = H(t-1) \cap \{h \in H^i | (\sigma^i \in \Sigma^i | h)\}$. Take $u = \{c_h; h \in H\} \in \mathcal{U}$ such that $\sigma \in B(u)$ and U1, U3(R(0)...R(t − 1)) hold. Then (iii)-(vi) are certainly satisfied for all $i \in I$ (for (vi) see remark 1). Let $\sigma \in R_P(t)$. First I show that the restriction in maximization (vi) is not binding. Let σ^i be a best response to $c_h \in \prod_{j \neq i} CH(M^j(t-1))$ among all h-replacements for $\sigma^i \in R^i_P(t-1)$. Let σ^* be a best response to c_h among all h-replacements for σ^i. Then σ^i and σ^* yield the same expected utility (given c_h). Assume not, i.e. σ^* is strictly better than σ^i as a response to c_h. Then $\sigma^* \notin R_P(t-1)$. But $\exists k \in \{1, \ldots t-2\} : \sigma^* \in R_P(k)$ (recall $R^i_P(0)=\Sigma^i$). Since $M(t-1) \subseteq M(k), c_h \in \prod_{j \neq i} CH(M^j(k))$. Thus $\sigma^* \in R^i_P(k+1)$. The same argument may be repeated, if necessary, to show $\sigma^* \in R^i_P(t-1)$, a contradiction. Since $R_P(k) \subseteq R_P(k\text{-}1)$, $\sigma \in \cap_{k=1}^{k=t} R_P(k)$. Taking this into account, together with the previous argument, remark 1 and 6, it is possible to find t updating processes $u^k = \{c_h^{(k)}; h \in H\}, k = 0, \ldots (t-1)$, such that U1,U2(R(k)) hold and $\sigma \in B(u)$. Now construct the desired updating process u as in the proof of theorem 1, to show that $\sigma \in R(t)$. □

Theorem 2 provides an alternative characterization of Pearce's EFR, showing that it is possible to defend it, at least to some extent, from the critique by Reny of being contradictory for those games in which some information sets are not reached by the solution set R. Of course, EFR is "contradictory" if it is interpreted as an algorithm which requires common knowledge of the theory at *each* information set. But this "contradiction" is eliminated if EFR is interpreted as an algorithm which assumes common knowledge *at the beginning of the game* of a theory specifying a hierarchy of beliefs. Theorem 2 allows me to state a proposition, already proven elsewere, relating rationalizability ((*-) EFR as defined in def. 3) to backward induction (BI). Let $T^R(T^{R*})$ be the set of (*-) rationalizable nodes ($T^R = \{t \in T | \exists \sigma \in R : \sigma$ reaches t$\}$, T^{R*} is analogous).

PROPOSITION 1. If Γ is a game of perfect information with a unique subgame-perfect equilibrium σ, then the equilibrium outcome and the (*-) rationalizable outcome coincide, i.e. σ reaches $t \in T \Leftrightarrow t \in T^R$ (σ reaches $t \in T \Leftrightarrow t \in T^{R*}$). This may be slightly generalized as follows: let Γ_x be a perfect information subgame with root x of an extensive game Γ and assume that Γ_x has a unique subgame-perfect equilibrium, then either $x \in T \backslash T^R$ or the subgame-perfect path and the EFR path coincide on Γ_x; the same holds true for *-EFR. (Battigalli [1988] provides a proof for the case of EFR in perfect information games with one-to-one payoff functions, which works as well for proposition 1).

Proposition 1 provides a foundation for the principle of BI, but at a price: one has to assume the existence of a proper and sophisticated hierarchy of beliefs. One may wonder how far one can go without them. An easy answer is that any BI-solvable game in which the solution reaches each information set (or at least any information set h at which $\iota(h)$ does not have a strictly dominant strategy) may be solved by a reversible algorithm (implementabilty). The following theorem extends the class of solvable games. From now on I drop assumption G2, since theorem A1 (see the appendix) shows that it is not necessary in games with perfect information. Following Bonanno [1989] I call *non-recursive* those games in which no player (except possibly the chance player) moves more than once in a play: $\forall i \in \{1, \ldots, n\}\ h, h \in H^i \Rightarrow Z(h) \cap Z(h) = \emptyset$.

THEOREM 3. Let Γ be a non-recursive game of perfect information with a unique subgame-perfect equilibrium, σ^*. Then $\forall t \in \mathcal{N}\ \Sigma^*(t+1) \subseteq \Sigma^*(t)$ and $\cap_{t \geq 1} \Sigma^*(t) = R^* = \{\sigma^*\}$.

Proof. Assume that $x \in X$ ($\iota(x) \neq 0$ is always implicit in this proof) is not reached by any strategy profile in $\Sigma^{*-\iota(x)}(t)$. Let $J = \{j \in I | \exists y > x : j = \iota(y)\}$ be the set of players following $\iota(x)$ and $K = -\iota(x) \backslash J$. By perfect information, only conjectures about $j \in J$ are relevant for expected utility maximization conditional on x. By assumption, players in J have not yet played when x is reached, then $\Sigma^{*J}(t)$ is consistent with x and $\Sigma^{*K}(t)$ is not. The same is true of some conjecture $(c^J, c^K) \in \Sigma^{*-\iota(x)}(t)$. Therefore U1* and U2*($\Sigma^{*J}(t)$) apply and $\iota(x)$ has a conjecture (c^J_x, c^K_x) such that $c^J_x \in \Sigma^{*J}(t)$ and the fact that $\Sigma^*(t)$ is not consistent with x is not relevant. This shows that $\Sigma^*(t+1) \subseteq \Sigma^*(t)$. Then, by theorem 1, $\Sigma^*(t) = R^*(t)$ and $\cap_{t \geq 1} \Sigma^*(t) = R^*$. Let $X(1) = \{x \in X | S(x) \subseteq Z\}$, $X(k) = \{x \in X \backslash \bigcup_{s=1}^{s=k-1} X(s) | S(x) \subseteq Z \cup \bigcup_{s=1}^{s=k-1} X(s)\}$. Note that, by non-recursivity, any strategy σ^i reaches all the information sets in H^i. Therefore implementability requires expected utility maximization at each $h \in H^i$. Let $\sigma \in R^*$. Take $x \in X(1)$. Clearly $\sigma \in \Sigma^*(1)$ and, by uniqueness of the subgame perfect equilibrium (SPE), $\sigma^{\iota(x)}(x) = \sigma^{*\iota(x)}(x)$, since both coincide with the unique optimal action (if the optimal action were not unique, there would exist at least two SPEs, a contradiction). By induction, assume that $\sigma^{+\iota(y)}(y) = \sigma^{*\iota(y)}(y)$ for $y \in \cup_{k=1}^{k=t} X(k)$ and any $\sigma^+ \in R^*$. Take $x \in X(t+1)$. By lemma 1, $\exists K : \Sigma^*(K) = \Sigma^*(K+1) = R^*$. Since falsification plays no role (see above), $\sigma^{+\iota(x)}(x)$ must be a best response at x to some conjecture c^J with $Supp(c^J) \subseteq \Sigma^{*J}(K) = R^{*J}$. But c^J must be equivalent to σ^{*J} for all nodes following x, by the induction hypothesis. Thus, since the SPE is unique, $\sigma^{+\iota(x)}(x) = \sigma^{*\iota(x)}$. Since $\{X(t)\}_{t \geq 1}$ is a partition of X, the proof is complete. □

A concequence of theorem 3 is that *-implementability is quite similar, with respect to its performance, to a solution concept proposed by Bonanno [1989]. Bonanno uses a logic approach and defines a local strategy $\sigma^{\iota(h)}(h) = \alpha$ as a material implication: "if h, then α". Therefore $\sigma^{\iota(h)}(h) = \alpha$ may be a false proposition only if the solution implies that h occurs. This is quite different from the solution concepts considered here, since they implicitely model a local strategy as a counterfactual statement, which may be "false" even if the information set is not reached. A second difference is that Bonanno's solution either yields a unique outcome or does not exist. Finally, Bonanno does not assume anything about the preference order over lotteries and many games with chance moves or imperfect information cannot be solved for this reason. Nonetheless, there are important similarities with *-implementability. First, if a game is not solvable by *-IMP, Bonanno's solution does not exist due to counterfactuals. For example, games Γ_3 and Γ_4 (if regarded as a two-person game) are not solvable (see Bonanno [1989], fig. 4 and 5), essentially for the same reason for which *-IMP does not provide a solution. Second, Bonanno's solution selects the backward induction outcome for all one-person and generic non-recursive games of perfect information without chance moves. By remark 6 and theorem 3 the two solution concepts are realization-equivalent for such games.

5. Concluding remarks.

In this paper I have defined four algorithmic solution concepts: EFR, IMP, *-EFR and *-IMP. EFR is shown to be equivalent to Pearce's notion of extensive form-rationalizability, while IMP was proposed in Battigalli [1988] as an algorithm to refine game theoretic equilibrium concepts. My characterization of EFR provides a partial defense against the critique of Reny that, if the solution excludes some information sets, it is contradictory. I argue that EFR should be interpreted as a solution concept which assumes common knowledge of the theory at the beginning of the game. This theory include a method to construct a hierarchy of beliefs, which places suitable constraints on the process of conjecture formation for any play of the game. Theorem 1 shows that whenever IMP provides a solution, this is the rationalizable set and the counterfactuals problem does not arise. The last two solution concepts are new. They are based on the assumption that a player updates her conjectures and beliefs about different opponets independently of one another whenever her current information makes it possible (independent updating). Unfortunately I have not yet been able to characterize the relationships with the first two solution concepts, except that there are games in

which they are not stronger (in terms of strategies) and that *-IMP (*-EFR) is stronger than IMP (EFR) in all perfect information, non-recursive games with a unique subgame-perfect equilibrium, where it selects the backward induction strategy profile (Theorem 3). This also shows that *-IMP turns out to be quite similar to a logic solution concept proposed by Bonanno. Proposition 1 and Theorem 3 provide some foundation for the principle of backward induction, showing that it relies either on a proper hierarchy of beliefs or on the principle of independent updating.

APPENDIX: initial conjectures.

Assumption G2 was included in section 2 in order to obtain ex ante rationality from interim rationality and intertemporal consistency, without directly assuming that players have initial conjectures at the beginning of the game. This allows a more direct comparison with Pearce's approach, but restricts the class of games to which the analysis applies. Note that no definition in this paper needs assumption G2 in order to make sense, and G2 is never used in the proof of a theorem, except the last part of Lemma 1 on the best response property of the rationalizable set. In this appendix I show that assumption G2 is innocuous, except possibly for some games with chance moves and imperfect information (e.g. signaling games).

THEOREM A1. Let Γ be a game of perfect information and/or a game without chance moves, then for any updating process $u = \{c_h; h \in H\} \in \mathcal{U}$ satisfying U1 there exists an n-tuple of conjectures $(c_1, \dots c_n) \in \prod_{i \in I} C_i$ such that

$$\sigma \in B(u) \Rightarrow \forall i \in I \ \sigma^i \text{ is a best response to } c_i.$$

Before proving theorem A1 some additional notation is necessary:

- $H^{-i} = H \backslash H^i$: collection of information sets of players different from i;
- $A^{-i} = \{a \in A | a \in A(h), h \in H^{-i}\}$: action set for players different from i.
- $A[h] = \{a \in A^{-\iota(h)} | \exists h \in H^{-\iota(h)} \exists (x,t) \in h \times h : x < t \& a \in A(h)\}$: set of actions "following" h for players different from $\iota(h)$; let c_i reach $h \in H^i$, then c_i yields a unique updated probability assignment

$$\mu(x; h, c_i) = [P(h; c_i, \sigma^i)]^{-1} P(x, c_i, \sigma^i), \ x \in h, \sigma^i \in \Sigma^i | h$$

(note that by perfect recall (G1), the RHS is independent of σ^i);

– c_i also yields a transition probability assignement

$$\pi(.;c_i) : A^{-i} \to [0,1], \forall h \in H^{-i} \sum_{a \in A(h)} \pi(a, c_i) = 1$$

(see Kuhn [1953]).

Player i at h needs only $\mu(.;\ h,\ c_i)$ and $\pi(.;\ c_i)\ |A[h]$,the restriction of $\pi(.;c_i)$ on $A[h]$, to compute expected utility conditional on h. Formally, let c_i and c_i' be two conjectures of i reaching h, if $\mu(.;h,c_i) = \mu(.;h,c_i')$ and $\pi(.;c_i)|A[h] = \pi(.;c_i')|A[h]$, then there exists a constant $K(h,\ c_i,\ c'_i)>0$ such that

$$\forall z \in Z(h)\ P(z;c_i,\sigma^i) = K(h,c_i,c_i')P(z;c_i',\sigma^i)$$

Proof of theorem A1.

(i) (*No chance moves*). For any $i \in I$ let $h_0^{i1},...h_0^{ik(i)} \in H^i$ be her initial information sets, i.e. information sets without predecessors in H^i. If $k(i)=1$ (i.e. G2 holds for i) set $c_i=c_h$, where $h=h_0^{i1}$. If $k(i)>1$ let c_i be such that

[1] $P(h;\ c_i,\ \sigma^i) = 1$

[2] $\forall \sigma^i \in \Sigma^i|h\ \forall z \in Z(h)\ P(h;c_h,\sigma^i)P(z;c_i,\sigma^i) = P(z;c_h,\sigma^i)$.

[1] is possible because any edge leading to h corresponds to a personal choice and its transition probability may assume the extreme values zero and one. Since $h=h_0^{i1}$ is neither a predecessor nor a successor of h_0^{ik} ($k\neq1$), $Z(h)\cap Z(h_0^{ik})=\emptyset$ and [1] imply $P(h_0^{ik};c_i,\sigma^i) = 0$ for $k\neq1$. Since the two measures are proportional over $Z(h)$, for all $h \in H^i$ reached by c_i σ^i is a best response to c_h at h if and only if it is a best response to c_i at h (recall that, by U1, $c_h = c_h$ for all these information sets).

(ii) (*Perfect information*). I use the same symbol to denote a decision node and the corresponding information set. Call $x_0^{i1} \ldots x_0^{ik(i)}$ "initial" nodes of player $i \in I$. If $k(i)>1$, choose c_i so that

[3]$\pi(a;c_i) = \pi(a;c_{x_0}ik)$, if $a \in A[x_0^{ik}]$, $k = 1 \ldots k(i)$.

[3] is possible because, by perfect information and notational convention, $A[x_0^{ik}] \cap A[x_0^{il}] = \emptyset$ for $k \neq l$. Perfect information also trivially implies that $\mu(.;h,c_i) = \mu(.;h,c_h)$ for all $h \in H^i$ reached by c_i. Hence for all these information sets c_i and c_h induce proportional measures over $Z(h)$ and σ^i is a best response to c_h at h if and only if it is a best response to c_i at h. Repeat the same operation of (i) or (ii) (as applicable) for all $i \in I$. Then straightforward dynamic programming shows that

$$\sigma \in B(u) \Rightarrow \forall i \in I\ \sigma^i \text{ is a best response to } c_i.$$

REFERENCES

AUMANN,R.,[1987],"Correlated Equilibrium as an Expression of Bayesian Rationality", *Econometrica*, 55, 1-18.

BACHARACH,M.O.L.,[1987],"A Theory of Rational Decision in Games", *Erkenntnis*, 27, 17-55.

BASU, K., [1988], "Strategic Irrationality in Extensive Games", *Mathematical Social Sciences*, 15, 247-260.

BATTIGALLI, P., [1988], "Implementable Strategies, Prior Information and the Problem of Credibility in Extensive Games", *International Review of Economics and Business*, 35, 705-733.

——, GUAITOLI, D., [1988], Conjectural Equilibria and Rationalizability in a Macroeconomic Game with Incomplete Information, working paper no. 5, *Istituto di Economia Politica*, Università "L. Bocconi".

BERNHEIM,D.,[1984], "Rationalizable Strategic Behavior", *Econometrica*, 52, 1007-1028.

——, [1986], "Axiomatic Characterization of Rational Choice in Strategic Environments", *Scandinavian Journal of Economics*, 88, 473-488.

BICCHIERI,C., [1987a], "Self-refuting Theories of Strategic Interaction: a Paradox of Common Knowledge", forthcoming in *Erkenntnis*.

——, [1987b], "Strategic Behavior and Counterfactuals", forthcoming in *Synthese*.

BINMORE, K., [1987a], "Modeling Rational Players: Part I", *Economics and Philosophy*, 3, 179-214.

——, [1987b], "Modeling Rational Players: Part II", *Economics and Philosophy*, 4, 9-55.

BONANNO, G., [1989], The Logic of Rational Play in Games of Perfect Information, mimeo, University of California, Davis.

DAMME, E.VAN, [1987], Stability and Perfection of Nash Equilibria, Berlin: Springer Verlag.

FUDENBERG, D., KREPS, D., LEVINE, D., [1988], "On the Robustness of Equilibrium Refinements", *Journal of Economic Theory*, 44, 354-380.

GUL, F., [1989], Rational Strategic Behavior and the Notion of Equilibrium, mimeo, *Graduate School of Business*, Stanford University.

HARSANYI, J., [1967], "Games with Incomplete Information Played by Bayesian Players: Part I", *Management Science*, 14, 159-182.

KANEKO, M., NAGASHIMA T., [1988], Players' Deductions and Deductive Knowledge, and Common Knowledge of Theorems, mimeo, Hitotsubashi Universtity, Kunitachi, Tokyo.

KOHLBERG, E., MERTEN, F., [1986], "On the Notion of Stability of Equilibria", *Econometrica*, 54, 1003-1037.

KREPS, D., RAMEY, G., [1987], "Structural Consistency, Consistency and Sequential Rationality", *Econometrica*, 55, 1331-1348.

——, WILSON R., [1982], "Sequential Equilibria", *Econometrica*, 50, 863-894.

KUHN, H.W., [1953], "Extensive Games and the Problem of Information", in Contribution to the Theory of Games, vol.2, Kuhn, Tuker (eds.), Princeton: Princeton University Press.

MOULIN, H., [1979], "Dominance Solvable Voting Schemes", *Econometrica*, 47, 1351-1373.

MYERSON, R., [1978], "Refinements of the Nash Equilibrium Concept", *International Journal of Game Theory*, 7, 73-80.

PEARCE, D., [1982], Ex Ante Equilibrium: Strategic Behaviour and the Problem of Perfection, Econometric Research Program, RM 301, Princeton University.

——, [1984], "Razionalizable Strategic Behaviour and the Problem of Perfection". *Econometrica*, 52, 1029-1050.

PETTIT, P., SUDGEN, R., [1989], "The Backward Induction Paradox", *The Journal of Philosophy*, 86, 169-182.

RENY, P., [1987], Rationality, Common Knowledge and the Theory of Games, mimeo, University of Western Ontario.

——, [1988a], Explicable Equilibria, mimeo, University of Western Ontario.

——, [1988b], Backward Induction, Normal Form Perfection and Explicable Equilibria, mimeo, University of Western Ontario.

——, [1989], On the strategic equivalence of Extensive Games, mimeo, University of Western Ontario, Columbia University.

ROSENTHAL, R.W., [1981], "Games of Perfect Information, Predatory Pricing and the Chain-Store Paradox", *Journal of Economic Theory*, 25, 92-100.

SELTEN, R., [1975], "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games", *International Journal of Game Theory*, 4, 25-55.

TAN, T., WERLANG, S., [1988], "The Bayesian Foundation of Solution Concepts of Games", *Journal of Economic Theory*, 45, 370-391.

THOMPSON, F., [1952], Equivalence of Games in Extensive Form, RM 759, The Rand Corporation.

ZERMELO, E., [1912], "Uber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels", in *Proceedings of the Fifth International Congress of Mathematicians*, E.W. Hobson and A.E.H. Love (eds.).