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IMPLEMENTABLE STRATEGIES, PRIOR INFORMATION AND THE PROBLEM OF CREDIBILITY IN EXTENSIVE GAMES

by
PIERPAOLO BATTIGALLI *

1. *Introduction*

It is well known that, in extensive games, Nash equilibria can be sustained by threats or promises which are not credible. The classical way to deal with such equilibria is to suppress them, showing that they are not "perfect" (Selten, 1975) or "sequential" (Kreps, Wilson, 1982).

These equilibrium concepts are defined in order to eliminate irrational behavior at "unreached" information sets.

A *perfect equilibrium* is the limit of some sequence of perturbed game equilibria, in which each action has at least a small positive probability to be chosen.

A *sequential equilibrium*

"is not simply a strategy, but consists instead of two types of probability assessments by the players: the *beliefs* of a player concerning where in the game tree he is whenever it is his turn to choose an action, and his conjecture concerning what will happen in the future as given by the strategy" (Kreps, Wilson, 1982, p. 863).

Beliefs are induced by equilibrium strategies with Bayes formula, or, at least, they are justified by a limiting process in which Bayes formula is applied; while equilibrium strategies prescribe an expected utility maximizing action for each information set.

The sequential equilibrium concept turns out to be only a slight generalization of the perfect equilibrium one.

* Università Bocconi, Milano.

Robust counterexamples exist which show that each of them fails to eliminate all "unreasonable" equilibria. This fosters an effort to define stronger refinements of the Nash equilibrium concept for games in extensive form; see for example Cho, Kreps (1987), Kohlberg, Mertens (1986), Grossmann, Perry (1986). This line of research is very interesting, but does not constitute the main subject of this paper.

My approach focuses on the elements of "ex ante" and "ex post" rationality involved in Nash equilibrium and in its refinements such as perfect and sequential equilibrium.

"Ex ante" rationality implies the maximization of expected utility with respect to some conjecture, which in turn is compatible with the prior information of the player. This information can regard the rules of the game and the other players' behavior. This latter kind of information corresponds to different levels of rationality and mutually expected rationality in the behavior of other players.

In this respect, my approach is closely related to Pearce's paper on rationalizable strategic behavior (Pearce, 1984; see also Bernheim, 1984 and 1986).

Pearce assumes common knowledge of the rules of the game and of the players' rationality. A strategy is *rationalizable* if it is a best response to a conjecture c and c is compatible with such knowledge. This means that each player figures out the range of possible conjectures and learning processes of the other players and checks if a strategy profile can be justified. That is, if it prescribes a maximazing action of the relevant player for each relevant information set with respect to some plausible conjectures. Of course a plausible conjecture is based on an analogous reasoning. Circularity is avoided defining the set of rationalizable strategies through a step by step procedure, which can be interpreted as a reasoning process or as a flow of higher and higher prior information about rationality and mutually expected rationality.

My procedure is similar to Pearce's one. Hence it is worth noting two main differences.

1) Within my approach, complete information and common knowledge are only possible values of a "parameter", i.e. players' "a priori" knowledge.

2) For "almost all" games with perfect information, Pearce's rationalizability is equivalent to the backward induction procedure. This shows that a "no correlation in errors" hypothesis is embedded in Pearce's inductive definition. I will argue that such a hypothesis cannot be based on mutually expected rationality alone (a similar argument can be found in Binmore, 1986).

"Ex post" rationality is an equilibrium condition: it asserts that each player has no regret about the action he has chosen. Obviously, a Nash equilibrium always fulfils this condition. This property is not unimportant at all, because it implies that such an outcome is a steady state of some learning process involving conjectures and confutations. Elsewhere I argue that this steady state is a conjectural equilibrium and that it does not necessarily fulfil Nash's best response property. This argument will be simply sketched here. It seems to me that the "no regret property" is a sufficient reason to consider Nash equilibria interesting (anyway, one can always recur to the "parable" of preplay communication to justify Nash property as a necessary condition). Therefore I define inductively a sequence of equilibrium concepts which somehow combine the "refinements of Nash equilibrium concept" approach with the rationalizability approach.

I say that a strategy s_j of a player j is ***implementable*** with respect to his prior information I_j, if an initial conjecture ς_j and a coherent learning rule L_j exist, so that ς_j is compatible with I_j and s_j maximizes expected utility for each relevant information set h, ς_j and L_j being given. The learning rule L_j assigns to each piece of information h, which confutes ς_j, a new conjecture $\varsigma_j' = L_j(\varsigma_j, h)$, with respect to which h is possible. If n is the number of players, the $2n$-tuple $(\varsigma_1 \dots \varsigma_n; L_1 \dots L_n)$ generates a system of beliefs in the sense of Kreps and Wilson.

Credibility is clearly related to implementability. Roughly speaking, a strategy is not credible if no one believes that a rational player will implement it. My point is that such beliefs depend on the information about that player's knowledge, because the rationality principle is not sufficient to decide credibility matters.

An equilibrium is *semisequential* if every player knows that the others are rational and fully informed, i.e. each player knows that the others' behavior is coherent with at least one strategy that is implementable with respect to the knowledge of the rules of the game.

It turns out that "almost every" perfect equilibrium is semisequential. As a consequence "almost every" sequential equilibrium is semisequential too. Exceptions consist of pathological cases in which it is irrational to use consistent assessment for expected utility calculation at unreached information sets.

A *second order semisequential* equilibrium is such that every player knows that other players know that everyone is rational and completely informed. Higher order prior information and semisequential equilibria can inductively be defined.

Perfect and sequential equilibria fail to be second order semisequential for quite a large class of games. This shows that such equilibria can be "unreasonable" because they do not assume a sufficient amount of prior information about reciprocal rationality and knowledge.

Another flaw in these equilibrium concepts is that they are not robust with respect to inessential transformations of the game structure. I will argue that this is a consequence of the fact that these concepts assign a crucial role to purely formal aspects in the mathematical definitions of the strategies.

2. *Main Concepts and Definitions*

I follow Kreps and Wilson in analizing finite extensive games with perfect recall, in which nature moves first and once for all, choosing a "state of the world" and then the play takes place. Therefore I simply sketch the formal definition and present the symbolism I will use throughout the paper. More formal and detailed definitions of the standard concepts can be found in Kuhn (1953) and Kreps and Wilson (1982). Particular attention should be given to some unusual concepts, such as "conjecture", "prior information", "learning rule" and so on.

2.1. *Games in extensive form.* – A game in extensive form is defined by the following elements:

1) an *arborescence* $(T, <)$, i.e. a set of topological trees, one for each "state of the world", representing the order of the moves;

2) a rule $\iota : X \rightarrow N$ (X is the set of non-terminal nodes and $N = \{1, 2, ..., n\}$ is the players' set), which determines whose turn it is to move;

3) a *pattern of information* $H : X \rightarrow 2^x$, which specifies the information available to each player for each situation in which it is his turn to move;

4) a rule $\alpha : (T-W) \rightarrow A$ (W is the set of initial nodes and A is the set of the actions), which specifies the *actions* available to the relevant player for every possible situation;

5) an "*objective*" *probability* distribution $r : W \rightarrow [0, 1]$ over the set of the state of the world;

6) an n-tuple $\mathrm{u} = (u_1, u_2, ..., u_n)$ of Von Neumann-Morgenstern utility functions: $\mathrm{u} : Z \rightarrow R^n$ (Z is the set of terminal nodes).

It is assumed that each player has perfect recall, i.e. at each information set h player $\iota(h)$ knows whatever he previously knew, including his previous actions.

TABLE I

Notation	Definition	Terminology
$P(x)$	$(t \in T: t < x)$	set of the predecessors of x
W	$(w \in T: P(w) = \emptyset)$	set of the initial nodes or states of the world
$p_1(x)$	$\max P(x)$ for $x \in T-W$	first predecessor of x
$p_k(x)$	$p_1(p_{k-1}(x))$ for x such that $p_{k-1}(x) \in T-W$; $p_0(x) = x$	k^{th} predecessor of x
$S(x)$	$p_1^{-1}(x)$	set of the immediate successors of x
Z	$(z \in T: S(z) = \emptyset)$	set of the outcomes or terminal nodes
$Z(x)$ $Z(h)$	$(z \in Z: x < z)$ $\bigcup_{x \in h} Z(x)$ for $h \subseteq T$	set of terminal successors of x set of terminal successors of h
$\mathbf{H}$	$H(X)$	family of all the information sets
X_j	$\iota^{-1}(j)$	set of player j's decision nodes
$\mathbf{H}_j$	$H(X_j)$	set of player j's information sets
$A(h)$	$\alpha\left(\bigcup_{x \in h} S(x)\right)$ for $h \in \mathrm{H}$	available action for player $\iota(h)$ at information set h
A_j	$\bigcup_{h \in \mathrm{H}_j} A(h)$	set of player j's actions
$l(x)$	a positive integer such that $p_{l(x)}(x) \in W$	number of predecessors of x
$h < h'$	$Z(h') \subsetneq Z(h)$, $h, h' \in \mathrm{H}_j$	j knows at h' at least as much as he knew at h

Notice that W may contain impossible events (w: $r(w) = 0$) and these events can be relevant. In fact it is not assumed that the players know the "random machine", generating the state of the world; therefore they can believe that "objectively impossible" states are possible and consequently choose certain actions. This is a basic way to represent the players' uncertainty about the rules of the game and the point will be discussed later.

Some useful notations and definitions are compiled in Table I; not all of them have been mentioned above.

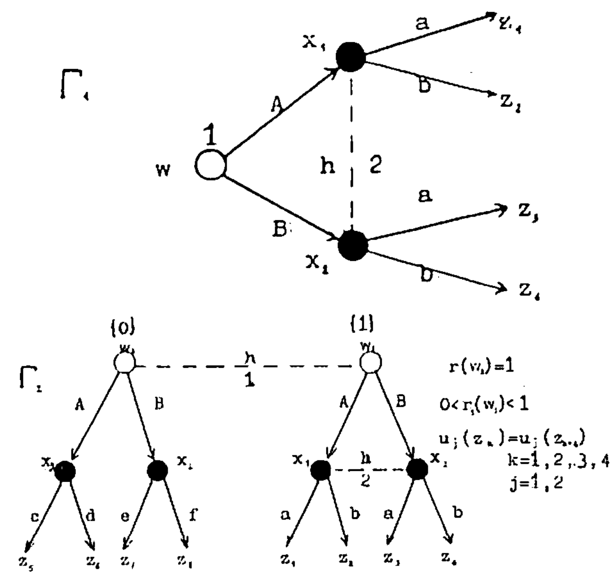
If Γ is the game in extensive form at hand, its dependence on the above mentioned parameters is represented by the expression

$$\Gamma = (T, <; N, i; H; \alpha; u; r).$$

I call *game structure* the "discrete" part of Γ, i.e. that part represented by $(T, <; N, i; N; \alpha)$.

The basic assumption about the players' knowledge is that *they know the game structure and their own utility function*, but they do not necessarily know the continuous parameters (u, r). For example player j certainly knows u_j, but perhaps does not know $(u_1, \ldots u_{j-1}, u_{j+1}, \ldots u_n)$ and r. Therefore incomplete information is the basic situation and stronger hypothesis have to be formally specified. Note the difference between the information of this kind and that specified by the information pattern H. The former has an "a priori" and general character, while the latter is a particular information about place and time. A richer model than Γ is needed to formalize players' a priori knowledge. As it will be seen, the incomplete information basic assumption allows a partial formalization of the complete information assumption, as opposed to Harsanyi's method of representing incomplete information as imperfect information in a higher order game, for which complete information is informally assumed. However the transformation of incomplete into imperfect information can be used to see that the knowledge of the game structure is not a really restrictive assumption. Consider the game Γ_1 in Fig. 1: player 1 moves first and he does not know whether player 2 can observe his choice or not, but the latter case is the real one. Therefore 1 does not know the game structure. But this situation can be represented through the game Γ_2, the structure of which is known to the players. Γ_2 is formed by two trees, corresponding to the two cases. 1 does not know in which tree he is, but the objective probability distribution r (unknown to 1) assigns zero probability to the "wrong" tree.

2.2. *Pure, mixed and behavior strategies.* – A pure strategy s_j is a plan prescribing an action for each piece of information h that player j can



Legenda

Graphic	Numeric	Alphabetic	Terminology
○		w	initial node, state of the world
●		x	other decision nodes
		z	outcomes
		A, B, a, b ...	actions
● - - - ●		h	information set
	1, 2		players
	{1} {0}	$r(w_1)$, $r(w_2)$	object prob.

FIGURE 1.

receive. A mixed strategy m_j is a probability mixture of pure strategies and a behavior strategy b_j is a plan which assigns to each information set h a probability mixture of the available actions $a \in A(h)$. A mixed strategy is a prior randomization, while a behavior strategy is a local randomization.

DEFINITION 1. *A pure strategy* for player $j \in N$ is an assignment $s_j : \mathrm{H}_j \to A_j$ such that $s_j(h) \in A(h)$, $h \in \mathrm{H}_j$. The set of j's pure strategies is denoted by S_j.

A *mixed strategy* for j is a probability distribution

$$m_j : S_j \to [0, 1], \quad \sum_{s \in S_j} m_j(s) = 1.$$

A *behavior strategy* for j is a function

$$b_j : A_j \to [0, 1], \quad \sum_{a \in A(h)} b_j(a) = 1, \qquad h \in \mathrm{H}_j.$$

The sets of j's mixed and behavior strategies are respectively denoted by M_j and B_j. The sets S_{-j}, S, B_{-j}, B, M_{-j} and M are so defined:

$$Y_{-j} = \mathop{\times}_{k \neq j} Y_k, \quad Y := \mathop{\times}_{j \in N} Y_j, \quad Y = S, B, M, \quad Y_k = S_k, B_k, M_k.$$

Pure strategies will be identified with the corresponding mixed or behavior strategies. The letter σ will be used for reference to either mixed or behavior strategies. Strategy profiles, i.e. n-tuple of strategies, are denoted by σ: $\mathbf{m} := (m_1, \ldots, m_n)$, $\mathbf{b} := (b_1, \ldots, b_n)$. Each behavior strategy profile $\mathbf{b} \in B$ induces a function $b : A \to [0, 1]$ defined by

$$b(a) = b_j(a), \qquad \text{if } a \in A_j.$$

Each strategy profile σ induces an *"objective" probability distribution* Pr (.; σ) *over* Z. Kuhn defined an n-tuple of transformations

$$\beta : M \to B, \quad \beta = (\beta_1, \ldots, \beta_n), \quad \beta_j : M_j \to B_j$$

and showed that $Pr(.; \mathbf{m}) = Pr(.; \beta(\mathbf{m}))$ for an arbitrary $\mathbf{m}$, iff (if and only if) Γ is a game with perfect recall. The strategy $b_j = \beta_j(m_j)$ is called the *behavior* of the mixed strategy m_j (Kuhn, 1953, def. 16, Lemma 3, theor. 4).

The probability distribution $Pr(.; \mathbf{b})$ is so defined:

$$(1) \qquad Pr(z; \mathbf{b}) := r(p_{l(z)}(z)) \prod_{k=0}^{l(z)-1} b(\alpha(p_k(z))).$$

For each strategy profile σ and for each player j an *"objectively" expected utility* is defined:

$$(2) \qquad V_j(\sigma) := \sum_{z \in Z} u_j(z)\, Pr(z; \sigma).$$

2.3. *Conjectures, potential falsifiers and relevant information sets.* – A conjecture ς_j for player j is a subjective probability assessment about the state of the world and other players' actions. Without a conjecture j cannot assign a probability distribution over the outcome space Z to each strategy σ_j and no expected utility maximization is possible.

DEFINITION 2. A *conjecture* for player j is a couple of functions $\varsigma_j = (r_j, c_j)$ so defined:

a) $r_j : W \to [0, 1]$, $\sum_{w \in W} r_j(w) = 1$,

b) $c_j \in B_{-j}$.

The set of j's conjectures is denoted by $\mathcal{C}_j$. $\boldsymbol{\varsigma}$ denotes the n-tuple of the players' conjectures and $\mathcal{C} := \mathcal{C}_1 \times \ldots \times \mathcal{C}_n$ is the set of such n-tuples.

c_j can be interpreted as an $(n-1)$-tuple of behavior of other players' mixed strategies and these mixed strategies can in turn be interpreted as probability assessments over the other players' pure (or even mixed) strategy sets.

Condition b) is imposed for the sake of simplicity. The assumed knowledge of the game structure allows a more general definition, unless $|W| = 1$ and $|N| = 2$ (for an explanation and exemplification of this point see Battigalli, 1987, pp. 165-167).

For a given ς_j, j can assign to each strategy σ_j a *subjective probability distribution* over Z:

$$(3) \qquad Pr(z; \varsigma_j, b_j) = r_j(p_{l(z)}(z)) \prod_{k=0}^{l(z)-1} g_j(\alpha(p_k(z))),$$

where g_j is the function $g_j : A \to [0, 1]$ for which if $a \in A_j$, then $g_j(a) = b_j(a)$, and if $a \in A_{-j}$, then $g_j(a) = c_j(a)$;

$$(4) \qquad Pr(z; \varsigma_j, m_j) = \sum_{s \in S_j} Pr(z; \varsigma_j, s)\, m_j(s).$$

If j is rational, he maximizes the *subjectively expected utility*

$$(5) \qquad U_j(\varsigma_j, \sigma_j) := \sum_{z \in Z} u_j(z)\, Pr(z; \varsigma_j, \sigma_j).$$

If a probability distribution $Pr(.; \theta)$ $(\theta = \sigma, (\varsigma_j, \sigma_j))$ over Z is defined, then it is possible to assign a probability to each subset $h \subseteq T$ (nodes are identified with the corrisponding singletones):

$$(6) \qquad Pr(h;\ \theta) := Pr(Z(h);\ \theta) := \sum_{z \in Z(h)} Pr(z;\ \theta).$$

With these definitions and formulae it is possible to formally identify and dispose of redundancies in the mathematical definition of a strategy. A strategy is a plan. If a plan prescribes an action $a \in A(h)$ and a makes impossible the realization of information h' $(h < h')$, then that plan need not prescribe any action $a' \in A(h')$. The only way to justify such prescription as necessary is Selten's "trembling hand": there is always a small positive probability of error in the implementation of a given decision. But if this is true, then the given game structure is not well specified, the real game is a perturbed game. The trembling hand argument is relevant in the analysis of robustness problems, not when one takes for granted that the game structure is correctly specified, as I do. Information sets such as h' are called irrelevant with respect to those strategies which prescribe a at h.

Definition 3. An information set $h \in H_j$ is called *relevant when playing* σ_j if a conjecture $\varsigma_j \in \mathcal{C}_j$ exists such that $Pr(h;\ \sigma_j,\ \varsigma_j) > 0$. The family of such sets is denoted by $Rel(\sigma_j)$. (Cfr. Khun, 1953, def. 6 and 9).

It is clear that if two strategies induce the same probability distribution for each conjecture, then there is no real difference between them, neither for the player who can choose them nor for other players. Thus they are called equivalent.

Definition 4. The strategies σ_j and σ_j' are *equivalent*, written $\sigma_j \equiv \sigma_j'$, iff $Pr(\cdot;\ \varsigma_j,\ \sigma_j) = Pr(\cdot;\ \varsigma_j,\ \sigma_j')$ for all conjectures $\varsigma_j \in \mathcal{C}_j$. (Cfr. Kuhn, 1953, def. 5).

It is intuitive to think that if two strategies define the same family Rel $(\cdot)$ and differ only on irrelevant information sets, then they must be equivalent. This intuition is in fact correct and can be the other way around.

Proposition 1. $b_j \equiv b_j'$ iff $Rel(b_j) = Rel(b_j')$ and $b_j(a) = b_j'(a)$ for all $h \in Rel(b_j)$ and all $a \in A(h)$.

Kuhn proved this proposition for pure strategies (Kuhn, 1953, theor. 1). Its extension to behavior strategies is straightforward (see the appendix). Of course the proposition is true for mixed strategies too, one need only recur to the transformation β.

If unexpected information h comes about, then the conjecture $\varsigma_{i(h)}$, on

which the probability assignment was based, must be false. In fact, even if $i(h)$ made an "error", at information set h he must know it (he has perfect recall); thus this error cannot be the cause for h to be unexpected: h is unexpected iff $Pr(h;\ \varsigma_{i(h)},\ \sigma_{i(h)}) = 0$ for some $\sigma_{i(h)}$ having $h \in Rel(\sigma_{i(h)})$. Such an h is called a potential falsifier of $\varsigma_{i(h)}$.

Definition 5. An information set $h \in H_j$ is a *potential falsifier* of the conjecture ς_j iff $Pr(h;\ \varsigma_j,\ \sigma_j) = 0$ for some σ_j having $h \in Rel(\sigma_j)$. The family of such information sets is denoted by $Ref(\varsigma_j)$. The complement family is denoted by $Coh(\varsigma_j) := H_j - Ref(\varsigma_j)$.

2.4. *Prior information.* – A typical example of prior information is the knowledge of the "objective" probability distribution r, i.e. of the random mechanism generating the state $w \in W$. A complete formalization of such information is not presented in this paper. Thus it is characterized through the sets of conjectures compatible with it. For example, j's knowledge of the distribution r is represented by the set

$$\{(r_j,\ c_j) \in \mathcal{C}_j : r_j = r\} := I_j(r)$$

In order to simplify the language, prior information will be identified with the corresponding set of conjectures. Of course this representation is imperfect, because sometimes different prior information corresponds to the same set of compatible conjectures. This is the case with complete information and the knowledge of r, because the additional knowledge of the other players' payoff function is useless without a behavioral theory (such a theory could be "no one chooses a strictly dominated strategy"). However this lack of one-to-one is not a problem from a decision theoretical point of wiew.

Definition 6. Conjecture ς_j is *compatible with complete information* iff $\varsigma_j \in I_j(\Gamma) := ((r_j,\ c_j) \in \mathcal{C}_j : r_j = r)$.

A generic prior information for j is denoted by I_j, and $I := X_{j \in N} I_j$ represents a prior information structure.

The family of potential falsifier of $I_j \subseteq \mathcal{C}_j$ is

$$Ref(I_j) := \bigcap_{\varsigma \in I_j} Ref(\varsigma_j);$$

as usual $Coh(I_j) := H_j - Ref(I_j)$.

2.5. *Learning rules and systems of beliefs.* – The expression "learning rules" does not refer to the Bayesian updating of a subjective probability distribution. This updating is already embedded in conjectures. A conjecture is sufficient to choose an "a priori" rational (that is maximizing) strategy. But the players must not and cannot commit themselves to follow an "a priori" chosen strategy and when they face a completely unexpected event they must resolve another strategic choice problem. Therefore they use a new conjecture (the old one being falsified) to compute the probabilities of still unknown events. In particular they compute the probability distribution over the unexpectedly reached information set.

The rationality principle, although insufficient to select a unique learning process, can be used to narrow the possible sequences of conjectures and confutations. One rationality condition is straightforward: the new conjecture must assign a positive unconditional probability to the just reached information set. The other condition is not so compelling, but seems extremely reasonable: an agent j does not reject his former conjecture ς_j for a new one, if ς_j is not confuted; and the same holds for his prior information I_j.

DEFINITION 7. A *learning rule* for j is a function $L_j : (\mathcal{C}_j \times \mathrm{H}_j) \to \mathcal{C}_j$ for which $h \in Cob\,(L_j\,(\varsigma_j, h))$ for all $h \in \mathrm{H}_j$ and $\varsigma_j \in \mathcal{C}_j$.

L denotes an array of players learning rules: $\mathbf{L} : = (L_1, ..., L_n)$. Given an $\mathbf{L}$ and an n-tuple ς of initial conjectures, the shorter notation ς_h^L will be used for $L_{i(h)}\,(\varsigma_{i(h)}, h)$, i.e. $\varsigma_h^L : = L_{i(h)}\,(\varsigma_{i(h)}, h)$.

DEFINITION 8. L_j is *coherent with respect to* I_j, iff the following conditions are fulfilled for all $\varsigma_j \in I_j$, $h, g \in \mathrm{H}_j$:

1) if $h \in Cob\,(\varsigma_j)$, then $\varsigma_j = \varsigma_h^L$;

2) if $h < g$ and $g \in Cob\,(\varsigma_h^L)$, then $\varsigma_h^L = \varsigma_g^L$;

3) if $h \in Cob\,(I_j)$, then $\varsigma_h^L \in I_j$.

The set of such learning rules is denoted by $L\,[I_j]$.

A system of beliefs assigns to each information set h a probability distribution over h conditional on h itself. A system of beliefs can always be obtained from a system $(\varsigma, \mathbf{L})$ of initial conjectures and learning rules.

The set of completely mixed behavior strategies for j is so defined: $B_j^0 : = \{b_j : A_j \to (0, 1)\}$. The set of completely mixed strategies profiles is $B^0 : = \underset{j \in N}{X} B_j^0$. Analogously $M_j^0 : = \{m_j : S_j \to (0,1)\}$, $\mathrm{M}^0 : = \underset{j \in N}{X} M_j^0$

DEFINITION 9. A *system of beliefs* is a function $\mu : X \to [0, 1]$ such that $\Sigma_{x \in h}\, \mu\,(x) = 1$ for all $h \in \mathrm{H}$. A system of beliefs is *induced* by $(\varsigma, \mathbf{L})$, written $\mu^{\varsigma, L}$, iff it is so defined:

$$(7) \qquad \mu^{\varsigma, L}\,(x) : = Pr\,(x;\ \varsigma_{H(x)}^L,\ b_{i(x)})/Pr\,(H\,(x);\ \varsigma_{H(x)}^L,\ b_{i(x)})$$

for all $x \in X$ and $\mathbf{b} \in B^0$.

Note that the *r.h.s.* of (7) is well defined and does not depend on $\mathbf{b}$ because of the perfect recall hypothesis and definition 6.

It is now possible to define expected utilities U_j and V_j conditional on $h \in \mathrm{H}_j$:

$$(8) \qquad U_j\,(\varsigma_j, \sigma_j|h) : = [Pr\,(h;\ \varsigma_j, \sigma_j)]^{-1} \sum_{z \in Z(h)} u_j\,(z)\,Pr\,(z;\ \varsigma_j, \sigma_j),$$

for $h \in Cob\,(\varsigma_j) \cap Rel\,(\sigma_j)$;

$$(9) \qquad V_j\,(\mu, \mathbf{b}|h) : = \sum_{z \in Z(h)} u_j\,(z)\,\mu\,(x_h^z) \prod_{k=1}^{l(z|h)} b\,(\alpha\,(p_{k-1}\,(z))$$

where $x_h^z \in h \cap P\,(z)$ (a singleton) and $l\,(z|h) : = l\,(z) - l\,(x_h^z)$. Note that a) $U_j\,(\varsigma_h^L, \sigma_j|h)$ is well-defined for all $h \in Rel\,(\sigma_j)$ and b) $U_j\,(\varsigma_h^L, b_j|h) = V_j\,(\mu^{\varsigma, L}, \mathbf{b}|h)$, if $\varsigma_h^L\,(a) = b\,(a)$ for all actions $a \in A_{-j}$ coming after h.

3. *Implementable and Rationalizable Strategies*

A strategy for player j can be considered as a plan or as a predictive means about j's contingent behavior. In the former case the strategy contains a certain amount of redundancy. In fact, j chooses it with a conjecture ς_j in mind; ς_j may assign zero probability to some of j's information sets. If h is one of such information sets ($h \in Ref\,(\varsigma_j)$), then it is completely useless planning the choice of an action $a \in A\,(h)$, because this action cannot affect the (subjectively) expected payoff and, therefore, is not relevant for the choice of actions preceding h. If h is reached, j must change his conjecture and choose another plan. But if a strategy s_j is considered as a predictive means, this kind of redundancy dissappears, because s_j is an answer to the question "how will j behave faced with the expected or unexpected event h?". Implementability is a relevant concept from this point of view.

DEFINITION 10. A mixed or behavior strategy σ_j^* is *implementable with respect to* I_j iff a couple $(\varsigma_j, L_j) \in I_j \times L\,[I_j]$ exists such that

$$U_j\,(\varsigma_h^L, \sigma_j^*|h) \geq U_j\,(\varsigma_h^L, \sigma_j|h)$$

for all $b \in Rel(\sigma_j^*)$ and all σ_j for which $b \in Rel(\sigma_j)$. The sets of such mixed or behavior strategies are denoted by $M[I_j]$ and $B[I_j]$ respectively.

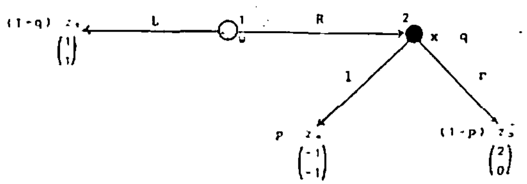
It is pointed out that only relevant information sets are considered. This means that the implementability property applies to equivalence classes of strategies as well as to single strategies.

PROPOSITION 2. If $b_j \equiv b_j'$ ($m_j \equiv m_j'$), then $b_j \in B[I_j]$ ($m_j \in M[I_j]$) iff $b_j' \in B[I_j]$ (m_j' $M[I_j]$).

Proposition 2 is a straightforward consequence of proposition 1. If two strategies are equivalent, then they define the same family of relevant information sets and they induce the same expected utility, conditional on such information sets, for every conjecture.

The meaning of definition 10 is clear enough: if it is known that prior information I_j is available to a rational player j, then no one believes j will implement a strategy b_j unless $b_j \in B[I_j]$.

The "threat game" of Figure 2 gives a simple illustration of this point.



1 \ 2	l	r
L	1 , 1	1 , 1
R	-1 , -1	2 , 0

FIGURE 2.

Player 2's strategy l is not implementable (with respect to whatever information $I_2 \in Ç_2$), because if x is reached, then 2 maximizes his payoff choosing r. Formally:

$$U_2(ç_x^L, l|x) < U_2(ç_x^L, r|x)$$

for all $(ç_2, L_2) \in Ç_2 \times L[Ç_2]$.

Definition 10 allows a simple formulation of the rationality postulate.

RATIONALITY POSTULATE (R): if prior information I_j is available to player j, then his contingent behavior can be predicted through (is coherent to) some strategy $b_j \in B[I_j]$.

The set M_j is isomorphic to the unit simplex in $R^{|S_j|}$, a convex set. Therefore linear convex combinations of mixed strategies are still mixed strategies. But the set $M[I_j]$ does not need to be convex. If player k knows that j is rational and has information I_j, k's conjecture about j action has to be deduced by a probability distribution over $M[I_j]$, but this is equivalent to picking up a particular mixed strategy in the convex hull of $M[I_j]$ (the convex hull of $M \subseteq R^s$ is the set

$$CoM := \bigcap_{C \in C(M)} C, \quad C(M) := \{C \subseteq R^s : C \text{ is convex and } M \subseteq C\},$$

i.e. the smallest convex set containing M).

If $N = \{1, 2\}$, then the deduced conjecture is in the set

$$\{(r_k, c_k) : c_k \in \beta_j(CoM[I_j])\},$$

i.e. $ç_j$ must be in the image set of $CoM[I_j]$ through the equivalence transformation β_j defined by Kuhn.

DEFINITION 11. The prior information set $I_j(k) \subseteq C_j$ ($j \in N$) is recursively defined as follows:

$$I_j(0) := I_j(\Gamma)$$

$$I_j(t) := \left\{(r, ç_j) : ç_j \in \left(\mathop{X}_{k \in N-\{j\}} \beta_k(CoM[I_k(t-1)])\right)\right\}, \quad t = 1, 2, \ldots$$

A conjecture $ç_j$ is t^{th} order reasonable iff $ç_j \in I_j(k)$.

Zero order reasonable conjectures are compatible with complete information. First order reasonable conjectures are compatible with complete information and the knowledge that other players are rational and fully informed, but need not be aware that their rationality and information is known.

Definition 11 is akin to Pearce's inductive definition of rationalizability (Pearce, 1984, def. 9) and it is also possible to translate the latter in the implementability language with some minor modifications [1].

DEFINITION 12. Let $M_j(t)$ ($j \in N$) be defined as follows:

[1] Pearce's extensive form definition of rationalizability does not consider an initial conjecture. This turns out to allow strictly dominated strategies to be rationalized, as one can verify

$$M_j(0) := M_j,$$
$$M_j(t) := M_j(t-1) \cap M[I_j^*(t-1)], \quad t = 1, 2, \dots$$
$$I_j^*(0) := I_j(0)$$
$$I_j^*(t) := \left\{(r, c_j) : c_j \in \left(\underset{k \in N-\{j\}}{\mathrm{X}} \beta_k (CoM_k(t))\right)\right\}, \quad t = 1, 2, \dots .$$

The mixed strategy m_j is *t-rationalizable* iff $m_j \in M_j(t)$, and is *rationalizable* iff it is t-rationalizable for all positive integers t, i.e. the set of rationalizable strategies for j is

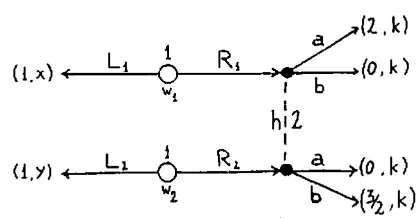
$$R_j := \bigcap_{t=1}^{\infty} M_j(t)$$

While definition 11 formalizes a sequence of possible prior information to the players, definition 12 is better interpreted as a reasoning process, at each step of which some conjectures and consequently some strategies can be discarded. Pearce proves that the number of steps is always finite, i.e. for some integer k and for all the players $M_j(k) = M_j(k+t)$, $t = 1, 2, \dots$.

The sets $M[I_j(t-1)]$ and $M_j(t)$ must be compared in order to understand the *difference between implementability and rationalizability*. It is always true that $M[I_j(0)] = M_j(1)$, but for $t > 1$ the two sets may be different. Counterfactuals are the source of this difference, as the following example will show.

Consider the two-person zero sum game with perfect information in Figure 3 (only 1's payoff is dispayed). The set $M[I_1(0)] = M_1(1)$ contains only three of the four 1's pure strategies, i.e. the two equivalent strategies (L, H'), (L, L') and the strategy (H, L'), while $M_2(1) = M_2$. In the second step (or with the higher information) the situation is the same for 1, but not for 2, because now he knows that, if x is reached, it is better to choose l

looking at the figure. Strategy (R_1, R_2) is strictly dominated, but it is also rationalizable, if one relies on Pearce's extensive form definition (PEARCE, 1984, p. 1042).



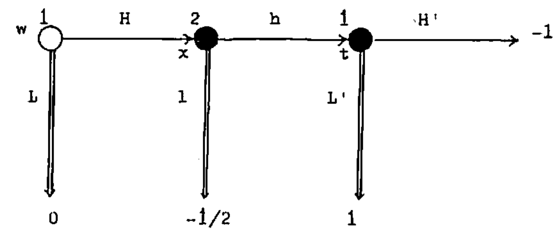


FIGURE 3.

than h. In the third step 1 knows that if he chooses H, he loses 1/2, then L is better, while nothing changes for 2. Up to this point the two approaches give the same results ($M_j(t) = M[I_j(t-1)]$, $t = 1, 2, 3$), but at the fourth step player 2 faces a counterfactual problem: if x were reached, then prior information $I_2(3)$ would be falsified ($\{x\} \in Ref(I_2(3))$) and 2 would have to point the "*modus tollens* arrow" against some components of the prior information. He could think that perhaps 1 is not completely rational and has made an error, but certainly 1 would choose the right action if t were reached. This is the "no correlation in error hypothesis", typical of backward induction. At least for this game, Pearce implicitly assumes this hypotesis holds and is common knowledge among the players. In fact definition 12 yields: $M_j(t-1) \subseteq M_j(t)$ (discarded strategies cannot be recovered). Thus l, the only strategy in $M_2(3)$, is the only strategy in $M_2(4)$ and $M_1(5)$ still contains only the equivalent strategies (L, H'), (L, L') and their mixtures. But other hypotheses are equally possible about the players' learning processes. 2 could think that 1 is irrational or equivalently that the game payoff does not represent his preferences. Knowing this, 1 can try to bluff, choosing H and hoping that the disoriented player 2 will choose h. Formally: $M[I_2(3)] = M_2$, $M[I_1(4)] = M[I_1(0)]$.

It transpires that rationalizability is indeed equivalent to backward induction for a large class of games with perfect information.

PROPOSITION 3. If Γ is a game with perfect information and if $u_j : Z \to R$ is one to one for each player $j \in N$, then the set of the razionalizable outcomes following w and the set of the backward induction outcomes following w are equal and contain only one element for all $w \in W$. (See Appendix).

4. *Semisequential, Sequential and Perfect Equilibria*

Nash's definition of equilibrium, as such, makes no sense in games with incomplete information. If the Nash equilibria are interpreted as possible "solutions" of the game, then incomplete information prevents the player from computing them. Anyway, the "objectively" expected payoff functions V_j need not be relevant, because the players' subjective distributions, r_j, can differ from the "objective" one, r. The standard device is to define a new game with complete (though imperfect) information, in which the initial game is embedded. But this device is arbitrary and is not the only way to give sense to Nash equilibrium. To see this it is sufficient to think of the "ex ante-ex post" interpretation of equilibrium in standard economic theory. Following Hahn (1973), we can define an *equilibrium* as a situation in which agents choose rationally with respect to their conjectures and the chosen actions do not induce information which falsify any conjecture. Situations of this kind, i.e. *conjectural equilibria*, can be seen as steady states of some learning process involving conjectures and confutations. Of course Nash equilibria are conjectural equilibria, but other conjectural equilibria exist.

Let us consider again the "threat game" of figure 2. 1 could fear the "threat" and choose L, but in this way he prevents himself from knowing that 2, being rational, would choose r. Thus the following situation arises: 2's real plan is to choose r if x were reached, but he threatens 1 and hopes to be believed; 1 choosing L cannot discover 2's real plan and the conjectures of both are not falsified. Therefore (L, r) is a conjectural equilibrium, but it is not a Nash equilibrium, because 1's best reply to r is R. However (L, r) and the Nash equilibrium (L, l) induce the same outcome, z_1. If 1 knew 2's payoff function and that 2 is rational, then outcome z_l could not be produced, because l is not an implementable strategy for 2 and L is not implementable for 1 with respect to this prior information.

This example suggests that the differences between Nash equilibria and other conjectural equilibria may be inessential, while prior information really matters.

Nash best reply property is a necessary condition which any solution concept and any "agreed upon" mode of behavior must fulfil. On the other hand, for a lot of games no real difference exists between Nash and conjectural equilibria. For these reasons it seems to me that the Nash equilibrium is an interesting concept in itself. Therefore, in what follows I will use the implementability approach in order to suppress "unplausible" Nash equilibria, assuming that adequate prior information is available to the players.

But the same approach could be applied as well to more general conjectural equilibria.

DEFINITION 13. An *equilibrium* is a $2n$-tuple $(\boldsymbol{\varsigma}, \mathbf{b})$ such that

a) $\varsigma_j = \mathbf{b}_{-j} := (b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n)$, $r_j = r$,

b) $b_j \in \operatorname{Argmax}_{y \in B_j} U_j(\varsigma_j, y)$

for all $j \in N$.

Definition 13 stresses the conjectural character of the Nash equilibrium, but it is equivalent to the usual definition, because condition a) yields $U_j(\varsigma_j, b_j) = V_j(\mathbf{b})$.

The "threat game" shows that condition a) and b) may be too loose, when they are interpreted as rationality constraints on conjectures, at least if it is implicitly assumed that players are well informed. In order to make explicit the assumptions about prior information, a third constraint is imposed: each conjecture must be compatible with proper prior information. Using definition 11, a countable class of equilibrium concept results.

DEFINITION 14. An equilibrium $(\boldsymbol{\varsigma}, \mathbf{b})$ is k^{th} *order semisequential* iff ς_j is k^{th} order reasonable $(\varsigma_j \in I_j(k))$ for all $j \in N$.

For 1^{st} order semisequential equilibrium the shorter expression "semisequential equilibrium" will be used.

The "threat game" allows a straightforward illustration: only the "good" equilibrium (R, r) is semisequential, because if 1 knows 2's payoff function and 2's rationality, then he does not believe 2 will ever implement strategy l and then he chooses R, i.e. the best reply against r.

As we shall see, the use of the attribute "semisequential" is not at all casual. Let us turn now to sequential equilibria.

An *assessment* is a couple $(\mu, \mathbf{b})$, where μ is a system of beliefs (definition 9). As we are interested in equilibria, it is only natural to impose that μ is induced by $\mathbf{b}$, whenever possible, through Bayes rule. That is, if $\varsigma_j = \mathbf{b}_{-j}$, $r_j = r$ for all j (equilibrium condition a)), then μ must be induced by $(\boldsymbol{\varsigma}, \mathbf{L})$ $(\mu = \mu^{\varsigma, L})$ for some coherent $\mathbf{L}$. Assume now that $r > 0$, as in the standard definition of extensive games (this assumption will be mantained throughout the remaining part of the paper). Then if b_j is completely mixed $(b_j \in B_j^0)$ and $\varsigma_j = \mathbf{b}_{-j}$ for all j, learning rules play no role at all (no unexpected information exists) and μ is determined by $\mathbf{b}$ in the obvious way:

$$\mu(x) = \mu^{\varsigma, L}(x) = \frac{Pr(x;\ \varsigma^{L}_{H(x)},\ b_{i(x)})}{Pr(H(x);\ \varsigma^{L}_{H(x)},\ b_{i(x)})} = \frac{Pr(x;\ \mathbf{b})}{Pr(H(x);\ \mathbf{b})}$$

Define now the set of assessments (μ, **b**) such that **b** is completely mixed and induces μ through Bayes rule:

$$\Omega^0 := \{(\mu, \mathbf{b}) : (\mathbf{b} \in B^0_1 x \ldots B^0_n,\ \mu(x) = Pr(x;\ \mathbf{b})/Pr(H(x);\ \mathbf{b}),\ x \in X\}.$$

The consistency criterion proposed by Kreps and Wilson is that an assessment must be a limit point of Ω^0.

DEFINITION 15. An assessment (μ, **b**) is *consistent* iff $\lim_{m \to \infty} (\mu_m, \mathbf{b}_m) = (\mu, \mathbf{b})$ for some sequence $\{(\mu_m, \mathbf{b}_m)_{m=1}^{\infty}\} \subseteq \Omega^0$. A *sequential equilibrium* is a consistent assessment (μ^0, $\mathbf{b}^0$) such that

$$V_{i(h)}(\mu^0, \mathbf{b}^0|h) \geq V_{i(h)}(\mu^0, b_{i(h)}, \mathbf{b}^0_{-i(h)}|h)$$

for all $h \in$ H and all $b_{i(h)} \in B_{i(h)}$.

Therefore sequential equilibria are defined by two conditions: consistency and sequential rationality.

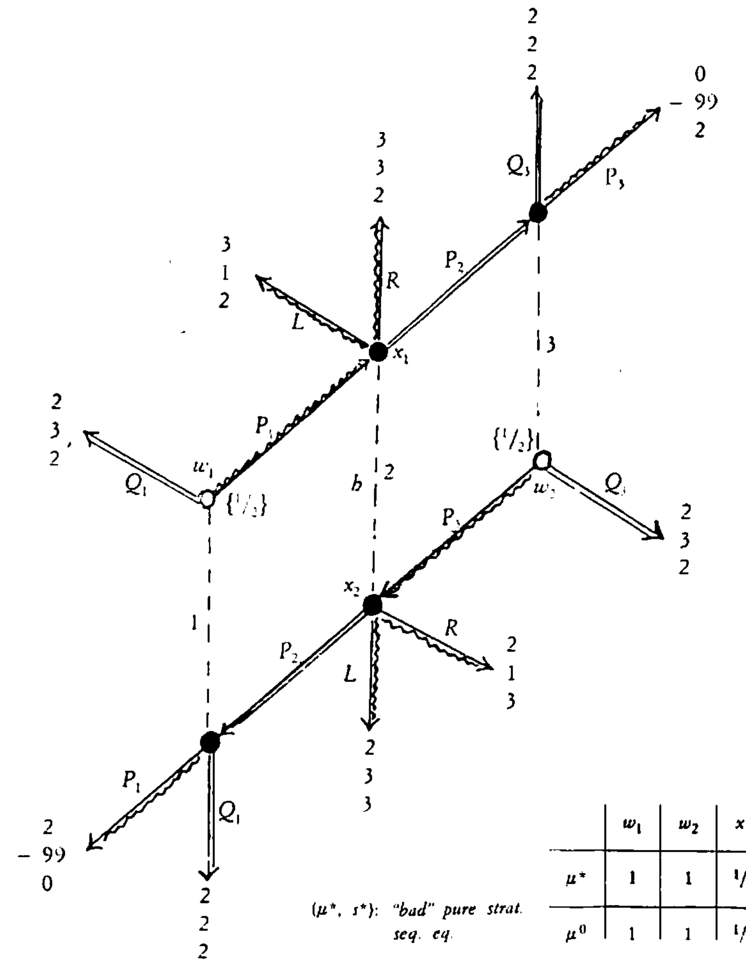
For a given equilibrium pair (ς, **b**), consistency is a far more restrictive condition than "$\mu = \mu^{\varsigma, L}$ for some n-tuple of coherent (with respect to complete information) learning rules". Kreps and Wilson point out that the players are "epistemically conservative" (i.e. the "no correlation in error hypothesis" holds) and all use the *same* learning rule (Kreps, Wilson, 1982, pp. 874-75).

Sequential rationality seems to be a much stronger property than implementability with respect to complete information, because the latter involves maximization at relevant information sets only, while the former involves all the information sets. But sequential rationality conceals a flaw: in the conditional expected utility computation the probability of "future" actions is always given by the equilibrium strategy profile b. As the following example will show, this can be self-contradictory [2]. Formally: given a sequential equilibrium (μ, b), it is possible that for an information set h no learning rule $L_{i(h)}$ exist, for which

$$U_{i(h)}(\varsigma^{L}_{h},\ b_{i(h)}|h) = V_{i(h)}(\mu,\ \mathbf{b}|h).$$

The game represented in Figure 4 has the same extensive form of an

[2] Between the first revision of this paper (September, 1987) and the final revision, Kreps' and Ramey's article has come to my attention (KREPS and RAMEY, 1987). In this article the authors discuss the same point mentioned above.



(μ^*, s^*): "bad" pure strat seq. eq.

	w_1	w_2	x_1
μ^*	1	1	$^1/_2$
μ^0	1	1	$^1/_2$

(μ^0, s^0): "good" pure strat seq. eq

	1	2	3
s^*	Q_1	P_2	Q_3
s^0	P_1	L or R	P_3

FIGURE 4.

example presented by Kreps and Ramey (Kreps, Ramey, 1987, fig. 2), where only payoff functions differ. The peculiar characteristic of this game is that information set h is preceded by two actions, which also follow it. If an assessment assigns zero probability to h, then player 2 should change his conjecture at h about the following actions. The expected payoff, conditional on h, must take this into account. The game has two sequential equilibria in pure strategies: in the "bad" one, (μ^*, s^*), player 2 threatens to "play" if 1 or 3 "plays", because he prefers that they "quit" (player j "plays" if he chooses P_j, the only alternative choice of 1 and 3 is "to quit", i.e. Q_j). This threat is intuitively incredible, because if he has to move and has a boundary belief, it is better to choose L or R ($\mu(x_1) = 1 \Rightarrow R$, $\mu(x_1) = 0 \Rightarrow L$); if he has an interior belief and chooses to "play", he must assign a positive probability to the very bad payoff − 99, hence his best response is L or R or a convex combination $\alpha L + (1 - \alpha) R$. Knowing this, players 1 and 3 choose to "play" because they are sure to get a higher payoff, and the result is the "good" equilibrium (μ^0, s^0).

The discussion above shows that (μ^*, s^*) is not semisequential, because strategy P_2 is not implementable with respect to $I_2(0)$ (nor it is rationalizable), but 1 and 3 believe that it would be carried out, were 2 given the opportunity. Hence 1's and 3's conjectures are not 1st order reasonable.

Why is (μ^*, s^*) a sequential equilibrium? Because in the computation of the conditional expected payoff $V_2(\mu^*, Q_1, \ldots, Q_3|h)$ a zero probability is still attached to P_1 and P_3, so that

$$V_2(\mu^*, Q_1, P_2, Q_3|h) = V_2(\mu^*, Q_1, L, Q_3|h) = V_2(\mu^*, Q_1, R, Q_3).$$

Consistency is fulfilled, and so is sequential rationality, because for all information sets h all actions in $A(h)$ are indifferent for player $i(h)$, given the other players' strategies.

Note that:

a) the source of implausibility of sequential equilibrium assessment (μ^*, s^*) is not in the system of beliefs μ^*, but in the contradictory use of (μ, b) in formula (9);

b) "bad" equilibria dissappear if perfection is imposed, because this implies that each strategy is a sequential best reply against all completely mixed b, provided that they are sufficiently close to the proposed equilibrium b*.

The former example shows that semisequential equilibrium is not a rigorous generalization of the sequential equilibrium concept. However, sequentiality was intended as an easy and intuitive approach to perfectness, and it will be shown that every perfect equilibrium of the normal form is

semisequential. This implies that "almost all" perfect equilibria of the extensive form are semisequential.

DEFINITION 16. An assessment (μ^*, b^*) is a *perfect equilibrium in the extensive form* (PEE) iff $\lim_{m\to\infty}(\mu_m, b_m) = (\mu^*, b^*)$ for some sequence $\{(\mu_m, b_m)\} \subseteq \Omega^0$ and $V_j(\mu_m, b_m/b_h'|h) \geq V_j(\mu_m, b_m/b_h|h)$ [3] for all $h \in H$, $j = i(h)$, $b_j \in B_j$.

A mixed strategy profile m* is a *perfect equilibrium in the normal form* (PEN) iff $\lim_{n\to\infty} m_n = m^*$ for some sequence $\{m_n\} \subseteq M^0$ and $V_j(m_j^*, m_{-jn}) \geq V_j(m_j, m_{-jn})$ for all $j \in N$ and all $m_j \in M_j$. (See Selten, 1975, theor. 4 and 7).

The PEE equilibrium concept strengthens the sequential rationality condition, so that every perfect equilibrium is sequential. Kreps and Wilson proved that the reverse statement is "almost true", i.e. for almost every game almost every sequential equilibrium is perfect (Kreps, Wilson, 1982, pp. 882-83). Normal form perfectness is generically a weaker condition than extensive form perfectness, i.e. for almost every game every PEE is a PEN (see Van Damme, 1987, p. 113). Proposition 4 states that every PEN is semisequential. Hence semisequentiality is "almost" a generalization of the sequentiality concept.

PROPOSITION 4. If m is a perfect equilibrium in the normal form and $\varsigma_j = (r, \beta_{-j}(m_{-j}))$ for all $j \in N$, then $(\varsigma, \beta(m))$ is a semisequential equilibrium. (The proof is in the appendix).

COROLLARY. Every extensive game with perfect recall has at least one semisequential equilibrium.

This is an immediate consequence of proposition 4 and Selten's existence theorem (Selten, 1975, theor. 5).

The former discussion shows that the perfect and sequential equilibrium concepts implicitly assume 1st order prior information for the players. But this is not the case for 2nd order information.

The two games in Figure 5 are presented and discussed in Kohlberg, Mertens (1986). A game similar to the first of them, $\Gamma^1(k)$, is discussed in Pearce (1984). For these games implementability coincides with rationalizability, i.e. $M[I_j(t)] = M_j(t + 1)$.

Consider the game $\Gamma^1(k)$: z_1 is a sequential equilibrium outcome for

[3] b_h is the probability distribution over $A(h)$ (i.e. the local strategy) given by b. b/b_h' is the behavior strategy profile which results when b_h is replaced by b_h'.

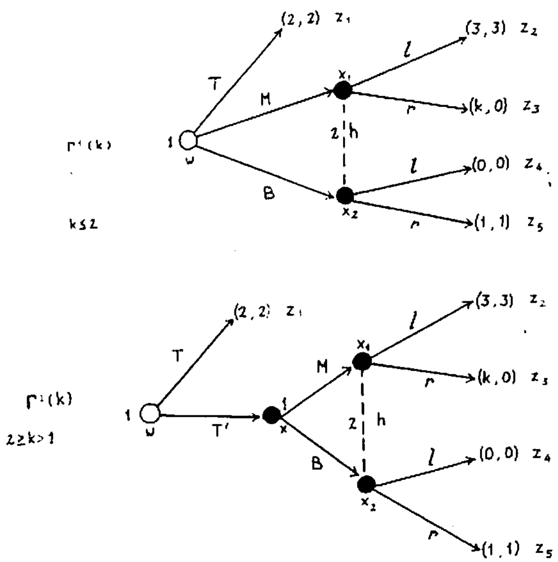


FIGURE 5.

all $k \leq 2$ and is a perfect equilibrium outcome for all $k < 2$. But z_1 is an unplausible outcome, if a certain amount of mutually expected rationality is assumed. Strategy B is strongly dominated by T: if h were reached player 2 would think he is in x_1 and he would choose l. Knowing this, player 1 has to choose M, which yields a higher expected payoff than T. T can be a rational choice only if 1 does not know that 2 knows 1's rationality. Formally: $r \in M[I_2(0)]$, but $M[I_2(1)] = \{l\}$; thus $I_1(2) = \{l\}$ and $M[I_1(2)] = \{M\}$. When 2^nd^ order prior information is assumed, only the outcome z_2 is admissible.

Consider now the game $\Gamma^2(k)$. The only difference between the two games is that in the second two binary choices substitute a ternary choice. This is an intuitively irrelevant detail in the specification of the game structure. But, if $k > 1$, z_2 is the only perfect and sequential equilibrium outcome for $\Gamma^2(k)$. This

"highlights a basic flaw in the concept of sequential (and perfect) equilibrium: it depends on all the arbitrary details with which the tree was drawn" (Kohlberg, Mertens, 1986, p. 1008).

In contrast to this, note that 1st and 2nd order semisequential equilibrium outcomes are the same for $\Gamma^1(k)$ and $\Gamma^2(k$. The reason of this difference is simple: sequential rationality imposes maximization at all the information sets, relevant and irrelevant, while implementability considers relevant information sets only. Maximization at information set $\{x\}$ (irrelevant when 1 chooses T) implies that M is chosen (M is strongly dominant in the subgame with root x) and only the belief $\mu(x_1) = 1$ is allowed. But (T, M) and (T, B) in $\Gamma^2(k)$ are equivalent strategies (proposition 1) corresponding to a unique plan: to choose T. Thus sequential rationality depends on redundant details in the formal definition of a strategy, while implementability (and rationalizability), considering relevant information sets only, applies to equivalence classes of strategies (proposition 2).

5. *Appendix*

PROPOSITION 1. Let b and b^* be two behavior strategies for player j $(b, b^* \in B_j)$. then the following statements are equivalent:

(a) $b \equiv b^*$;

(b) $Rel(b) = Rel(b^*)$ and $b(a) = b^*(a)$ for all $h \in Rel(b)$ and all $a \in A(h)$.

PROOF. (b) $\Rightarrow$ (a).

If $h \in Rel(b)$, it follows from (b) that $Pr(z; \varsigma, b) = Pr(z; \varsigma, b^*)$ for all $\varsigma \in Ç_j$ and all $z \in Z(h)$. If $h \in H_j - Rel(b) = H_j - Rel(b^*)$, then $Pr(z; \varsigma, b) = 0 = Pr(z; \varsigma, b^*)$ for all $\varsigma \in Ç_j$ and all $z \in Z(h)$. If $z \in (Z - \bigcup_{h \in H_j} Z(h)$, then $Pr(z; \varsigma, b')$ depends only on ς. Therefore $Pr(z; \varsigma, b) = Pr(z; \varsigma, b^*)$ for all $\varsigma \in Ç_j$ and all $z \in Z$, i.e. $b \equiv b'$.

(a) $\Rightarrow$ (b)

Reductio ad absurdum is used in the following proof.

Let's assume that an $h \in (Rel(b) - Rel(b^*))$ exists. Then $Pr(h; \varsigma, b) > 0 = Pr(h; \varsigma, b^*)$ for some $\varsigma \in Ç_j$. This yields $Pr(z; \varsigma, b) > = Pr(z; \varsigma, b^*)$ for some $z \in Z(h)$. Therefore b and b^* are not equivalent. Of course the same conclusion follows if an $h' \in (Rel(b^*) - Rel(b))$ exists. This proves that (a) implies $Rel(b) = Rel(b^*)$.

Let us assume now that b and b^* differ at some relevant information set. The perfect recall hypothesis implies that a partial order on H_j exists, which is coherent with $<$. Hence it makes sense to write $h < h'$ for some $h, h' \in H_j$. Therefore a first relevant information set, h^*, at which b and b^* differ, exists; i.e.

$$h^* \in \{h \in Rel(b): b(a) \text{ differs from } b^*(a) \text{ for some } a \in A(h)\} = D$$

$$\text{and } \{h \in D: h < h^*\} = \varnothing.$$

For example $b(a) > b^*(a)$ for some $a \in A(h^*)$. $h^* \in Rel(b) = Rel(b^*)$ and $b(a') = b^*(a')$ for all $h' < h^*$ and all $a' \in A(h')$. Hence $Pr(x; \varsigma, b) = Pr(x; \varsigma, b^*) > 0$ for some $x \in h^*$ and some $\varsigma \in Ç_j$. If $t \in S(x)$, then $Pr(t; \varsigma, b') = Pr(x; \varsigma, b')\, b'(\alpha(t))$ for all $b' \in B_j$.

Hence $Pr(t; \varsigma, b) > Pr(t; \varsigma, b^*)$ for $t \in \alpha^{-1}(a) \cap S(x)$ and $Pr(z; \varsigma, b) > Pr(z; \varsigma, b^*)$ for some $z \in Z(t)$. The hypothesis (a) is contradicted. From this follows (b). Q.E.D.

Some symbols are useful in the formulation and proof of proposition 3.

T^R is the set of rationalizable nodes:

$T^R = \{t \in T : Pr(t; m) > 0$ for some $m \in R_1 x \dots R_n\}$;

s^* is an n-tuple of pure strategies selected by backward induction (in a game with perfect information);

$s(x)$ is the action prescribed by $s_{i(x)}$ at $\{x\} = H(x)$: $s(x) = s_{i(x)}(\{x\})$;

$R^P(x) := \{s_{i(x)} \in S_{i(x)} \cap R_{i(x)} : H(x) \in Rel(s_{i(x)})\}$ is the set of pure rationalizable strategies for which $\{x\}$ is relevant;

$T_x := \{x\} \cup \{t \in T : x < t\}$ is the subtree with root x;

$T^*_x := \{x\} \cup \{t \in T_x - \{x\}: \alpha(t) = s^*(p_1(t))$ for some $s^*\}$ is the backward induction path from x;

$l_\Gamma := \max_{z \in Z} l(z)$ is the length of Γ.

PROPOSITION 3. If $\Gamma := (T, <; I, i; H; A; u; r)$ is a game with perfect information and u_j is one to one for all j, then the backward induction algorithm selects only one n-tuple s^*, $T^R \cap T_w = T^*_w$ and $|Z \cap T^*_w| = 1$ for all $w \in W$.

PROOF. Let a sequence of subsets of X $\{X^k\}$ be defined in the following way:

$$X^1 = \{x \in X: S(x) \subseteq Z\}$$

$$X^k = \{x \in X: S(x) \subseteq (Z \cup X^{k-1})\}. \qquad k = 2, 3, \dots l_\Gamma.$$

If u_j is one to one for all j and perfect information prevails, then a unique action $s^*(x)$ is chosen by the backward induction algorithm for each step $k = 1, 2 \dots l_\Gamma$ and for all information set $\{x\} \subseteq X^k$. But $\cup_k X^k = X = \cup_{j \in N} X_j$, hence the backward induction solution is unique and is a pure strategy n-tuple s^*. Thus the set $Z \cap T^*_t$ has only one element, z^*_t, for all $t \in T$ and it makes sense to write $u^*_j(t) := u_j(z^*_t)$.

X^k is the set of all the x for which $\max_{z \in Z(x)}(l(z) - l(x)) = k$. Hence $X^k = W$ for $k = l_\Gamma$.

It will be proved that if $x \in X^k \cap T^R$, then $T^R \cap T_x = T^*_x$ and for $k = l_\Gamma$ the thesis is obtained. The proof is by induction.

If $x \in X^1 \cap T^R$ then $R^P(x)$ is not void. $R^P(x) \subseteq M_{i(x)}(1)$, hence $s(x) = \alpha(\text{argmax}_{z \in Z(x)} u_{i(x)}(z)) = s^*(x)$ for all $s_{i(x)} \in R^P(x)$. The set $R_{i(x)}$ has the pure strategy property (see Pearce, 1984, prop. 4), hence $t \in S(x) \cap T^R$ iff $\alpha(t) = s(x)$ for some $s_{i(x)} \in R^P(x)$. This latter proposition and the former together yield $T^R \cap T_x = T^*_x$. Thus it is proved that: $x \in T^R \cap X^1 \Rightarrow T^R \cap T_x = T^*_x$.

Let us assume $(x \in T^R \cap X^k \Rightarrow T^R \cap T_x = T^*_x)$ is true and $x \in T^R \cap X^{k+1}$, we have to prove that from this $T^R \cap T_x = T^*_x$ follows.

If $x \in T^R$, the rationalizability hypothesis is not falsified (i.e. the conjecture of $i(x)$ at x, ς^L_x, is such that $\varsigma^L_x \in (\text{X}_{j \in N - i\{x\}} \beta_j(CoR_j))$). This means that $i(x)$ expects the play will follow a path in T^R from x onward for all the strategies $s_{i(x)} \in R^P(x)$ that $i(x)$ can implement at x. If $x \in T^R \cap X^{k+1}$ and $t \in T^R \cap S(x)$, then either $t \in T^R \cap X^k$ or $t \in T^R \cap Z$ (see the definition of X^k). If $t \in Z \cap T^R$, the expected utility (conditional on $\{x\}$) of action $\alpha(t)$ is simply $u_{i(x)}(t) = u^*_{i(x)}(t)$. If $t \in T^R \cap X^k$ the expected utility is $u^*_{i(x)}(t)$ for all strategies $s_{i(x)} \in R^P(x)$ such that $s(x) = \alpha(t)$. These two statements follow from the fact that $i(x)$ expects a path in T^R and $T^R \cap T_t = T^*_t$ when $t \in T^R \cap X^k$. But the utility function $u^*_{i(x)}(t)$

is one to one on $S(x)$. Hence $s(x) = \alpha(\text{argmax}_{t \in S(x)} u^*_{i(x)}(t)) = s^*(x)$ for all $s_{i(x)} \in R^P(x)$. This statement and the conditional statement $(t \in T^R \cap X^K \Rightarrow T^R \cap T_t = T^*_t)$ together imply $T^R \cap T_x = T^*_x$. Q.E.D.

PROPOSITION 4. If m is a perfect equilibrium and $\varsigma_j = (r, \beta_{-j}(\text{m}_{-j}))$ for all $j \in N$, then $(\varsigma, \beta(\text{m}))$ is a semisequential equilibrium.

PROOF. Since all perfect equilibria are Nash equilibria, $(\varsigma, \beta(\text{m}))$ certainly is an equilibrium. If $m_i \in M_i[I_i(0)]$ for all $i \in N$, then $\varsigma_j \in I_j(1)$ for all $j \in N$ (all the conjectures are 1st order reasonable) and $(\varsigma, \beta(\text{m}))$ is semisequential. Thus it will be proved that $m_j \in M_j[I_j(0)]$ or equivalently $\beta_j(m_j) \in B_j[I(0)]$ for an arbitrary j.

Define $\text{b} := \beta(\text{m})$, $b_m := \beta(\text{m}_n)$ and $\varsigma_{jn} := (r, \text{b}_{-jn})$, where m_n is a suitable test sequence (see definition 16).

Since $\text{m}_n \in M^0$ ς_{jn} has no potential falsifier (i.e. $Coh(\varsigma_{jn}) = \text{H}_j$) and $U_j(\varsigma_{jn}, b_j)|b)$ is well defined for all $b \in Rel(b_j')$ (see identity (8)).

Now define L_j as follows

[1]
$$L_j(\varsigma_j, b) = \varsigma_j, \text{ if } b \in Coh(\varsigma_j)$$
$$L_j(\varsigma_j, b) = \varsigma_{jn}, \text{ if } b \in Ref(\varsigma_j)$$

It is easy to check that L_j is coherent with respect to $I_j(0)$.

From def. 16 it follows

[2] $$U_j(\varsigma_{jn}, b_j|b) \geq U_j(\varsigma_{jn}, b_j'|b), \text{ if } b \in Rel(b_j) \cap Rel(b_j').$$

Taking the limit as $n \rightarrow \infty$

[3] $$U_j(\varsigma_j, b_j|b) \geq U_j(\varsigma_j, b_j'|b), \text{ if } b \in Rel(b_j) \cap Rel(b_j') \cap Coh(\varsigma_j).$$

[1], [2] and [3] imply

[4] $$U_j(\varsigma^L_b, b_j|b) \geq U_j(\varsigma^L_b, b_j'|b), \text{ if } b \in Rel(b_j) \cap Rel(b_j').$$

Since $\varsigma_j \in I_j(0)$ and $L_j \in L[I_j(0)]$, from [4] and def. 10 it follows that $b_j \in B_j[I_j(0)]$, Q.E.D.

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STRATEGIE AMMISSIBILI, INFORMAZIONI A PRIORI E CREDIBILITÀ NEI GIOCHI IN FORMA ESTENSIVA

In questo saggio si esamina il problema della credibilità ad un livello astratto, utilizzando l'apparato formale dei giochi in forma estensiva. In tale contesto una strategia implica, di norma, la minaccia o la promessa di compiere certe azioni, se si verificano certe circostanze. Con che criterio gli agenti scelgono la propria strategia? La risposta tradizionale della teoria dei giochi è che le strategie scelte da degli agenti razionali costituiscono un *equilibrio perfetto*. Ma tale concetto è almeno implicitamente basato su due diverse nozioni di "razionalità": una razionalità *ex ante*, secondo cui si massimizza l'utilità attesa sulla base di qualche ipotesi sul comportamento altrui e sullo stato di natura, e una razionalità *ex post*, secondo cui gli agenti – a giochi fatti – non sono pentiti delle loro scelte. Una analisi metodologicamente corretta dovrebbe considerare separatamente queste due componenti. La razionalità *ex post* caratterizza gli stati stazionari di processi

d'apprendimento per congetture e confutazioni; mentre la caratteristica cruciale della razionalità *ex ante* è che le congetture, su cui gli agenti basano le loro decisioni, devono essere coerenti con le loro informazioni a priori. Una strategia S dell'agente A è credibile, solo se può essere giustificata supponendo che A sia razionale e dotato di certe informazioni a priori, cioè solo se S è *ammissibile*. La credibilità dipende quindi dalle informazioni a priori possedute dagli agenti. Poiché tali informazioni possono riguardare il comportamento altrui e in particolare la altrui "razionalità" (opportunamente definita), esistono diversi livelli di credibilità, ovvero di ammissibilità. Si può mostrare che l'equilibrio perfetto implica un livello di ammissibilità del tipo "A sceglie razionalmente, subordinatamente all'ipotesi che l'agente B sia razionale", ma talvolta non soddisfa i requisiti di ammissibilità al livello superiore (cioè "A sceglie razionalmente subordinatamente all'ipotesi che B sia razionale e sappia che A è razionale"). D'altra parte certe condizioni imposte dall'equilibrio perfetto sono troppo restrittive e hanno implicazioni assai controintuitive. Sembra quindi più opportuno definire l'appropriato concetto di equilibrio in relazione alle informazioni a priori possedute dagli agenti, determinando l'insieme delle strategie razionali rispetto a tali informazioni e selezionando quelle combinazioni di strategie che rispettano la condizione "di non rimpianto" imposta dalla razionalità *ex post*.