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Comment

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1 INTRODUCTION

Marco Mariotti has rendered a valuable service to the profession, making us aware of the problems arising when we try to develop a unified view of decision theory and non-cooperative game theory. Unlike von Neumann and Morgenstern (1944), the so-called Bayesian approach regards probability distributions over a player's strategies as opponents' subjective beliefs, which must be coherent with the implications of common knowledge of the game and of players' rationality. But formal theories yielding subjective probabilities, such as Savage (1954), have only been developed for one-person decision problems where consequences of choices are affected by an unknown state of nature. Mariotti argues that opponents' strategies cannot be interpreted as states of nature. He concludes that the decision-theoretic foundation of game-theoretic solution concepts is therefore undermined.

Mariotti's basic argument is the following. Savage-like theories of decision-making under uncertainty derive subjective probabilities of states from preferences over lotteries or *acts*. These preferences do not only concern actually possible actions in the given decision situation, but also *virtual* choices in hypothetical decision situations where the state of nature affects the consequences of an action in different ways. The underlying assumption is that the relative likelihood of different states is the same in all these situations. But this assumption cannot be maintained if the decision-maker is a player considering her choices in different games against a fixed set of opponents with a fixed set of possible strategies/acts. Indeed one should expect that the opponents' behaviour in different games be different, even though their payoff functions are fixed, as long as the game to be played is common knowledge.

This argument seems quite compelling, but it only shows that the extension of classical decision theory to multi-person decision situations is problematic. It does not show that such an extension is impossible. The core of this argument is that virtual acts in hypothetical decision situations should be identified with strategies of the decision-maker in hypothetical games. I will argue that a different interpretation of virtual acts as bets of an *external observer* eliminates the above-mentioned conceptual problem and is consistent with the Bayesian approach to non-cooperative game theory.

The remainder of this Comment is organized as follows. Section 2 presents a formal restatement of Mariotti's main proposition in order to further clarify the nature of the problem. Section 3 proposes an alternative interpretation of virtual acts in situations of strategic interaction.

2 ITERATED DOMINANCE AND SAVAGE POSTULATES: AN IMPOSSIBILITY RESULT

Mariotti presents two impossibility results concerning the inconsistency of Savage's postulates with iterated strict dominance and iterated weak dominance. Although the logic behind these results is basically the same, I think that the one concerning iterated strict dominance is more interesting.

Iterated *weak* dominance is a very controversial solution concept. It well known that the final solution may depend on the order of elimination, but this unpleasant fact is only a consequence of more fundamental theoretical problems (see, e.g., Samuelson, 1992; and Börgers and Samuelson, 1992). It has been recently pointed out that non-Archimedean preferences (Blume *et al.*, 1991) can justify the elimination of weakly dominated strategies in strategic environments where actions are chosen intentionally and there is complete information. But even this is not enough to justify the *iterative* deletion of weakly dominated strategies (see, e.g., Blume, *et al.* 1991). Strong and controversial assumptions about 'infinitesimals of different orders' of non-Archimedean probability measures are needed in order to provide a theoretical foundation to this algorithm (see, e.g., Stahl, 1991; Battigalli, 1993; and Rajan, 1993).

It can be reasonably argued that also the examples considered by Mariotti (including extensive games where forward induction arguments can be obtained by iterated weak dominance) display the controversial features of iterated weak dominance. Since the intrinsic difficulties of this solution concept could blur the main point to be discussed, here I focus on iterated *strict* dominance.

Since in the present context players' attitudes toward risk are not given, we consider an 'ordinal' normal form game, i.e. a game in strategic form specifying the consequences of each strategy profile and each player's ordinal preferences about consequences (see, e.g., Börgers, 1993). Formally, an n-person non-cooperative ordinal game is given by the following elements:

$C = \{a, b, c \ldots\}$: set of *consequences*
$N = \{1, 2, \ldots, n\}$: *players'* set; for each $i \in N$
S_i : player i's set of *strategies*
$S_{-i} = X_{j\in N\backslash\{i\}} S_j$: set of i's *opponents'* strategy profiles
$c_i : S_i \times S_{-i} \to C$: player i's *consequence function*
$\geq_i \subseteq C \times C$: player i's (complete, transitive) *preference relation*

It is informally assumed that the given ordinal strategic game and players' rationality are common knowledge. A minimal implication of this assumption is that the players do not choose interatively strictly dominated strategies. More formally, for all $i \in N$, $k = 1, 2, \ldots$, let

$$S_i(k) := S_i \backslash \{s_i \in S_i \mid \exists t_i \in S_i,\ \forall S_{-i} \in s_{-i}(k-1), c_i(t_i, s_{-i}) >_i c_i(s_i, s_{-i})\}$$

be i's set of k-undominated strategies. Note that $S_i(k) \subseteq S_i(k-1)$ for all k. By an obvious introspection argument no player i would choose a strategy outside the iteratively undominated set $S_i(\infty)\!:= \cap_k S_i(k)$. Therefore each player i should be indifferent between two strategies s_i' and s_i'' such that $c_i(s_i', s_{-i}) \simeq_i c_i(s_i'', s_{-i})$ for all $s_{-i} \in S_{-i}(\infty)$. This suggests that opponents' profiles $s_{-i} \in S_{-i}\backslash S_{-i}(\infty)$ should be

1 \ 2	L	R
U	a ; a	b ; a
M	c ; b	c ; c
D	b ; c	b ; b

1 \ 2	L	R
U	a ; a	b ; a
M	b ; b	b ; c
D	c ; c	c ; b

Figure 6.8 Two ordinal games.

Savage-null from the point of view of i.

Now fix a player, say player i, and interpret S_{-i} as the set of states of nature for decision-maker i. According to this interpretation the set of *acts* is $\mathscr{A}_i\!:= \{a_i \mid a_i\!:S_{-i} \to C\}$. Note that i's strategies in the given game correspond to the following subset of acts:

$$\mathscr{A}_i(c_i) := \{a \in \mathscr{A}_i \mid \exists s_i \in S_i, \forall s_{-i} \in S_{-i}, a(s_{-i}) = c_i(s_i, s_{-i})\}$$

But the decision-maker's subjective probability distribution is recovered from her preference relation over the whole $\mathscr{A}_i$. Preferences concerning acts outside $\mathscr{A}_i(c_i)$ are interpreted as virtual choices in hypothetical ordinal games $G(c_i')$ obtained by replacing i's consequence function c_i with another consequence function c_i'.

Note that i's opponents' consequence functions do not change. Nonetheless each player j's set of *iteratively* undominated strategies in each game $G(c_i')$ depends on i's consequence function c_i'. Consider, for example, the games in Figure 6.8 where both players' ranking of consequences is $a < b < c$. In the left game M is strictly dominant for player 1. Thus player 2's iteratively undominated strategy is R. In the right game D is strictly dominant and player 2's interatively undominated strategy is L. The set of player j's iteratively undominated strategies in the game corresponding to c_i' is denoted $S_j(\infty, c_i')$.

In this formal framework Mariotti's main result (proposition 2.1) can be restated as follows:

Proposition. Consider the class of ordinal games $\{G(c_i') \mid c_i' : S_i \times S_{-i} \to C\}$. Let $\mathscr{R}_i$ be the set binary relations $\leq\cdot$ on $\mathscr{A}_i$ satisfying the following properties:

(SA) axioms P1–P5 in Savage (1954);
$(\leq_i)$ for every pair of constant acts (a_i', a''_i) where $a_i'(s_{-i}) \equiv c'$ and $a_i''(s_{-i}) \equiv c''$, $a_i' \leq . a_i''$ if and only if $c' \leq_i c''$;
(ID) for all $c_i : S_i \times S_{-i} \to C$, $s_i \in S_{-i'}$, if $s_{-i} \in S_{-i} \backslash S_{-i}(\infty, c_i')$
then s_{-i} is $\leq\cdot$ –null (i.e. two acts are $\leq\cdot$.- equivalent if they yield the same consequences for all $s_{-i} \in S_{-i}(\infty, c_i')$)

Then, for some specifications of $\{G(c_i') \mid c_i'\!:S_i \times S_{-i} \to c\}$, $\mathscr{R}_i$ is empty, i.e. (S), $(\leq_i)$ and (ID) are mutually inconsistent.

Remark. Property $(\leq_i)$ says that the preference relation $\leq\cdot$ concerning acts must be consistent with the given preference relation $\leq_i$ concerning consequences. Property (ID) says that, for every game $G(c_i')$, $\leq .$ must 'reveal' a zero probability for opponents' strategies which are iteratively dominated. This is justified by the informal assumption that the decision-maker bets must be 'public'. Since such bets correspond to hypothetical games $G(c_i')$, the player/decision-maker should take into account the implications of common knowledge of $G(c_i')$ and of players' rationality.

Proof. Consider again the games in Figure 6.8. Applying (ID) to the left game we obtain $(a,b) \simeq .(b,b)$, while applying (ID) to the right game we obtain $(a,b) < .(b,b)$. (I am maintaining Mariotti's notation where acts are identified by ordered *tuples* of consequences.)

As the simple example used in the proof makes clear, once it is appropriately formalized, Mariotti's impossibility result is quite trivial. This, however, does not mean that the result is not interesting. (Actually one could argue that quite a few famous impossibility results are trivial, if triviality is identified with simplicity of the proof). This result makes crystal clear that, if we want to take a Savage-like approach to decision-making in a situation with strategic interaction, we cannot identify hypothetical decision situations with hypothetical games where the decision-maker's choices affect her opponents' consequences.

3 SUBJECTIVE PROBABILITIES AND BETS OF AN EXTERNAL OBSERVER

Orthodox game theory offers two kind of solutions for non-cooperative games: (i) set-valued solution concepts such as *rationalizability*, whereby the players may have heterogeneous probabilistic expectations, but they share a common view about 'strategically irrational' choices, and (ii) point-valued solution concepts such as the Nash equilibrium whereby the players have common probabilistic expectations (for a review see, e.g., Myerson, 1991, ch. 3). In both cases the solution concept is meant to describe the behaviour of rational and intelligent players where

> We say that a player in the game is *intelligent* if he knows everything that we know about the game and he can make any inferences about the situation that we can make. (Myerson, 1991, p. 4)

An intelligent player analyzes a game from the point of view of an *external observer*. This suggests an interpretation of states of nature, acts, and preferences over acts that is very different from the one considered in the previous section.

Consider player/decision-maker i in a given n-person game G with a set of virtual acts $\mathscr{A}_i := \{a_i \mid a_i : S_{-i} \to C\}$. Her preference relation over $\mathscr{A}_i$ should represent her choices in hypothetical $(n+1)$-games where i is in the role of player $n+1$, the 'observer', the consequence functions of players 1, ..., n are constant with respect to the observer's choice and correspond to game G, and the observer consequence function c_{n+1} is constant with respect to the choice of player i (a copy of player/decision-maker i in the actual game G).[1] Note that the set of iteratively undominated strategies of players 1, ..., n in G and in all $(n+1)$-extensions of G is obviously the same. Therefore Savage axioms are no longer inconsistent with the requirement that

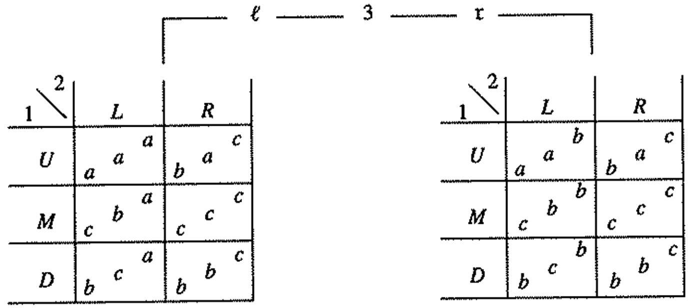

Figure 6.9 A 3-person extension of the left game in Figure 6.8. Player 3 is an 'observer'.

iteratively dominated opponents' strategies should be Savage-null.

For example, let G be the left game of Figure 6.8 and consider the 3-person extension depicted in Figure 6.9. According to (ID), player 3 is indifferent between L and R_i, i.e. $(a,c) \simeq .\ (b,c)$ and L is $\geq .$-null.

The fact that a transitive preference over $\mathscr{A}_i$ is recovered from virtual choices in such hypothetical games relies on the natural assumption that the player\decision-maker regards the other players' behaviour in each $(n+1)$-extension as independent of the consequence function c_{n+1}' and there is no relevant distinction between these games as far as players 1, ..., n are concerned.[2] Analogously, i's preferences over S_i in G are recovered from preferences over $\mathscr{A}_i$ because there is no relevant distinction between G and its $(n+1)$-extension as far as i's opponents are concerned.

The decision-maker's choices in the $(n+1)$-extension of G can be interpreted as isolated bets about the outcome of separate games identical to G. Mariotti argues that, according to the common knowledge assumption of orthodox game theory, bets should be public. But it is easy to see that in the present context the difference between private and public bets is immaterial, since the players of the hypothetical games are not concerned with the observer's behaviour.

Mariotti has convincingly argued that game theory cannot be regarded as a mere extension of decision theory to a multi-person setting. But in my opinion he goes too far when he claims that Savage-like representation theorems are not applicable to a game-theoretic context. On the contrary, I have just argued that there is no intrinsic inconsistency between decision theory and classical game theory.

Notes

1. The latter assumption is not really necessary, but it simplifies the comparison with Mariotti's framework.

2. Of course this would not be true if side payments were possible. But side payments are excluded in non-cooperative games.

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7 Nash Equilibrium and Evolution by Imitation*

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1 INTRODUCTION

The Nash equilibrium criterion is usually justified on rationalistic grounds, in terms of the involved players' 'rationality', and, in some way or other, shared knowledge or beliefs about each other's rationality and/or strategies etc. (see, e.g., Tan and Werlang, 1988; and Aumann and Brandenburger, 1991).

1.1 Nash's Rationalistic Interpretation

In his unpublished Ph.D. dissertation, John Nash provided the following rationalistic interpretation of his equilibrium criterion:[1]

> We proceed by investigating the question: what would be a 'rational' prediction of the behaviour to be expected of rational playing the game in question? By using the principles that a rational prediction should be unique, that the players should be able to deduce and make use of it, and that such knowledge on the part of each player of what to expect the others to do should not lead him to act out of conformity with the prediction, one is led to the concept of a solution defined before.
>
> If $S_1, S_2, ..., S_n$ were the sets of equilibrium strategies of a solvable game, the 'rational' prediction should be: the average behaviour of rational men playing in position i would define a mixed strategy s_i in S_i if an experiment were carried out.
>
> In this interpretation we need to assume the players know the full structure of the game in order to be able to deduce the prediction for themselves. It is quite strongly a rationalistic and idealizing interpretation. (1950, p. 23)

Nash used the phrase *position i* as we today would use the phrase 'player *i*'.

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