Available online at www.sciencedirect.com
SCIENCE DIRECT®

Games and Economic Behavior 45 (2003) 38–72

GAMES and Economic Behavior

www.elsevier.com/locate/geb

Rationalizable bidding in first-price auctions ☆

Pierpaolo Battigalli [a] and Marciano Siniscalchi [b,*]

[a] *Bocconi University, Milan, Italy*
[b] *Princeton University, Department of Economics, 309 Fisher Hall, Princeton, NJ 08544-1021, USA*

Received 12 January 2001

Abstract

We analyze the consequences of strategically sophisticated bidding without assuming equilibrium behavior. In particular, we characterize interim rationalizable bids in symmetric first-price auctions with interdependent values and affiliated signals. We show that (1) every nonzero bid below the equilibrium is rationalizable, (2) some bids above the equilibrium are rationalizable, (3) the upper bound on rationalizable bids of a given player is a nondecreasing function of her signal. In the special case of independent signals and quasi-linear valuation functions, (i) the least upper bound on rationalizable bids is concave; hence (ii) rationalizability implies substantial proportional shading for high valuations, but is consistent with negligible proportional shading for low valuations. We argue that our theoretical analysis may shed some light on experimental findings about deviations from the risk-neutral Nash equilibrium.

JEL classification: C72; D44; D82

Keywords: Auctions; Rationalizability; Overbidding; Proportional shading

☆ This material is based upon work supported by the National Science Foundation under Grant No. 9911490. Financial support from the European University Institute and Bocconi University (Battigalli) and IGIER-Università Bocconi (Siniscalchi) are also gratefully acknowledged.
* Corresponding author.
E-mail addresses: pierpaolo.battigalli@uni-bocconi.it (P. Battigalli), marciano@princeton.edu (M. Siniscalchi).

0899-8256/$ – see front matter
doi:10.1016/S0899-8256(02)00543-2

1. Introduction

The analysis of simultaneous bidding games generally builds upon the notion of (Bayesian) Nash equilibrium. Implicit in the latter solution concept are the assumptions that players are rational and hold *correct* beliefs about the play of their opponents.

This paper represents a first step toward the analysis of simultaneous bidding games under the assumption that bidders' beliefs are *strategically sophisticated*, but not necessarily correct; the rationality hypothesis is maintained.

Strategic sophistication is defined as the conjunction of the following assumptions about beliefs:

(1) Bidders expect positive bids to win with positive probability;
(2) Bidders are certain that their opponents are rational and certain of (1);
(3) Bidders are certain that their opponents are certain of (2); and so on.

We focus on first-price sealed-bid auctions with private or interdependent valuations and independent or correlated signals, and adopt the notion of (interim) rationalizability to capture strategic sophistication.

Our approach is motivated by the following considerations. In our opinion, the equilibrium assumption that beliefs are correct should be justified in terms of more fundamental hypotheses about the bidders' belief formation process. In particular, one may attempt to find a justification based on either introspection or learning in the *specific* context of auction games.

This paper provides an analysis based on beliefs that are strategically sophisticated, and hence consistent with a careful introspective analysis of the game. We show that, in first-price auctions, although strategic sophistication has nontrivial implications for bidding behavior, it is consistent with a wide range of nonequilibrium beliefs. Thus, introspection alone does not provide a justification for equilibrium analysis.

One may then argue that, even if bidders initially hold heterogeneous nonequilibrium beliefs, a learning process should nevertheless lead to an equilibrium.[1] This argument, however, is subject to important qualifications. First, it applies only to situations where bidders repeatedly play similar auction games with different competitors (a fixed set of bidders could give rise to collusion). Second, whether convergence to an equilibrium occurs at all, as well as the speed of convergence, crucially depend on how much feedback each player obtains about the decision rules adopted by his competitors in previous plays. In auctions games, this feedback is typically very poor: only the actual bids, and not the private information that induced such bids, can typically be observed.[2]

Therefore, we find no compelling reasons to expect approximate equilibrium behavior in the short run. Not surprisingly, experimental evidence shows significant and persistent deviations from the risk-neutral Nash equilibrium in first-price auctions (cf. Kagel, 1995).

[1] On learning in games see, for example, Fudenberg and Levine (1998).
[2] In a Dutch auction, whose reduced normal form is like a first-price auction, only the winning bid is observed.

These considerations suggest that it may be interesting to ascertain the extent to which the predictions of "textbook" auction theory (cf. Milgrom and Weber, 1982; Myerson, 1981; Riley and Samuelson, 1981; Vickrey, 1961) are dependent on the assumption that bidders' beliefs are correct.

Our analysis addresses this issue. We find that *bid shading* (bidding below the expected value of the good conditional on private information) is a robust phenomenon in first-price auctions. In settings with common values, shading is a consequence of rational reaction to the winner's curse, which is not present in private values settings. However, we also find substantial shading in settings with private values. Moreover, our results are qualitatively consistent with the empirical finding (cf. Kagel and Roth, 1992) that higher types tend to shade proportionally more than lower types.

On the other hand, the equilibrium assumption appears to be crucial for *revenue equivalence* in auctions with independent private values (and risk neutral bidders). In particular, our analysis of first-price auctions shows that

(i) for every type, every positive bid below the corresponding equilibrium bid is rationalizable, and
(ii) for almost every type, the highest rationalizable bid is above the equilibrium bid.

Note that our assumptions about beliefs and behavior imply that players do not use weakly dominated bids. In a second-price auction with private values, analogous assumptions imply that each player bids its valuation, as in the dominant-strategy equilibrium. Therefore, in light of the standard (i.e., equilibrium) revenue equivalence results, we conclude that the expected revenue of a seller with rationalizable beliefs in a second-price auction may be lower or higher than the rationalizable expected revenue in a first-price auction.

A further motivation for our work does not directly apply to this paper, but rather to the general approach we are attempting to develop. In recent years, many novel auction designs have been implemented in practice. When faced with such "novelties," bidders cannot be expected to have learned to play equilibrium strategies—even if, say, they may be reasonably expected to have learned the shape of each other's valuation functions, each other's signal distribution, and so on.

In such situations, we find the case for an analysis based on strategic sophistication alone particularly compelling. We hope that the methodology of this paper can be extended to more complex bidding games.

This paper employs an interim notion of rationality: different types of the same player are allowed to hold different beliefs about the bidding behavior of his opponents. Correspondingly, our results characterize interim rationalizability.

The latter solution concept involves the iterative deletion, for each possible type, of bids that cannot be justified by beliefs consistent with progressively higher degrees of strategic sophistication. A direct application of this procedure to bidding games would be analytically cumbersome and numerically intractable.

Our main technical contribution is to provide a more efficient implementation of interim rationalizability in the setting under consideration. The methodology we propose entails constructing bounds on the set of rational(izable) bids for a given type (valuation, signal) then proving that every bid within these bounds is rational(izable).[3]

The remainder of the paper is organized as follows. Section 2 illustrates the steps of our analysis, as well as the main ideas, by means of an example. Section 3 introduces our basic characterization result for symmetric auctions with interdependent values and affiliated signals. This result is used in Section 4 to obtain an iterative characterization of interim rationalizable bids in auctions. Section 5 discusses the relationship with experimental evidence and some extensions of our results. Appendix A contains some proofs and ancillary results.

2. An illustration: the two-bidder uniform IPV case

In order to develop the main ideas, we consider the following simple setting: two bidders (denoted $i = 1, 2$) participate in the auction; each bidder's valuation is fully determined by an independent draw $\mathbf{s}_i$ from the uniform distribution on $[0, 1]$. We reserve boldface letters for random variables, and italics for their realizations. Bidders are risk-neutral: thus, if bidder i wins the object for a price \$$b$ when her valuation is $\mathbf{s}_i = s_i$, her payoff equals $s_i - b$.

A bidding function is a map $\mathbf{b} : [0, 1] \to \mathbb{R}_+$. Recall that, in this setting, there exists a unique, symmetric Bayesian Nash equilibrium, characterized by the bidding function

$$\mathbf{b}^{\mathrm{eq}}(s_i) = \frac{1}{2} s_i. \tag{1}$$

To simplify the exposition, throughout this section we assume that a bidder's conjecture about her opponent's behavior may be represented by a continuous, increasing[4] bidding function (these restrictions are relaxed in the main text). In light of our distributional assumptions, the expected payoff to bidder i when her type is s_i, her conjecture is $\mathbf{b}_j$, and she bids $b \in [\mathbf{b}_j(0), \mathbf{b}_j(1)]$ is

$$\pi(b, s_i; \mathbf{b}_j) = (s_i - b)\mathbf{b}_j^{-1}(b). \tag{2}$$

The first objective of this paper is to characterize bids that survive finitely many steps of interim rationalizability, under the additional assumption that (it is common belief that) bidders expect positive bids to win with positive probability. The set of bids for type s_i that survive k steps of the procedure is denoted by $\mathcal{R}(s_i; k)$.

In the present setting, it turns out that interim rationality, with the additional assumption just indicated, eliminates all weakly dominated bids: thus, if bidder i's signal is $s_i > 0$, she may only place bids in the interval $(0, s_i)$. A bidder with signal $\mathbf{s}_i = 0$ will only bid 0.

This is the first step of the procedure, i.e., the characterization of the set of interim 1-rationalizable bids for each bidder: we may write $\mathcal{R}(s_i; 1) = (0, s_i)$ for $s_i \in (0, 1]$ and $\mathcal{R}(0; 1) = \{0\}$.

---

[3] Our techniques also provide upper bounds on the set of *ex-ante rationalizable bids*, although these bounds may not be tight. See Section 5 for a brief discussion of ex-ante and interim rationalizability.

[4] Throughout the paper we call a function h *increasing* if $x' > x''$ implies $h(x') > h(x'')$, and we call h *nondecreasing* if $x' > x''$ implies $h(x') \geqslant h(x'')$.

The inductive step entails characterizing the sets $\mathcal{R}(s_i; k)$ for each $s_i \in [0, 1]$ and $k > 1$; by definition, these are the collections of bids that are best responses for bidder i to conjectures consistent with the following assumptions:

(1) positive bids win with positive probability; and
(2) for all s_j, bidder j's type s_j places bids drawn from the set $\mathcal{R}(s_j; k-1)$.

The inductive step can be understood by focusing on the characterization of the sets $\mathcal{R}(s_i; 2)$, $s_i > 0$. Note that, for $k = 2$, (2) actually implies (1); moreover, if bidder i's beliefs about bidder j are represented by the bidding function $\mathbf{b}_j$, (2) requires that, for (almost) all $s_j \in [0, 1]$,

$$0 < \mathbf{b}_j(s_j) < s_j. \tag{3}$$

That is, bidder i's belief (for any given signal) must be a positive function below the least upper bound $\mathbf{B}(s_j; 1) = s_j$.[5] It is plausible to conjecture that the sets $\mathcal{R}(s_i; 2)$ will also be intervals of the form $(0, \mathbf{B}(s_i; 2))$. Our task is then to *derive a new least upper bound* $\mathbf{B}(\cdot\,; 2)$ *from the preceding one, i.e.,* $\mathbf{B}(\cdot\,; 1)$. If successful, this approach generalizes to all finite iterations.

We now verify that this conjecture is correct, and derive the new least upper bound. Specifically, we first show that the "old" least upper bound may be used to construct an upper bound on the set of interim 2-rationalizable bids; then, we show that this upper bound is tight, i.e., that every positive bid below this new upper bound is interim 2-rationalizable.

The first key step is to note that the "old" least upper bound $\mathbf{B}(\cdot\,; 1)$, viewed as a conjecture that bidder i holds about bidder j, is *"more pessimistic" than any conjecture* $\mathbf{b}_j$ *that satisfies Eq.* (3).

More precisely, suppose that bidder i's type is $s_i > 0$, and consider any bid $b \in (0, s_i)$. Since $b < s_i$, bidder i strictly prefers to win the object than to lose it. Thus, a conjecture is "more pessimistic" than another if it implies a lower probability of winning, i.e., of placing the highest bid. In particular, if $\mathbf{b}_j$ satisfies Eq. (3), $\mathbf{B}(s_j; 1) < b$ implies $\mathbf{b}_j(s_j) < b$, but the converse is false. Thus, $\Pr[s_j\colon\ \mathbf{B}(s_j; 1) < b] \leqslant \Pr[s_j\colon\ \mathbf{b}_j(s_j) < b]$, and hence $\pi(b, s_i; \mathbf{B}(\cdot\,; 1)) \leqslant \pi(b, s_i; \mathbf{b}_j)$ for all $b \in (0, s_i)$.

We will be interested in the maximum expected payoff that bidder i can secure, given the conjecture $\mathbf{B}(\cdot\,; 1)$:

$$\pi^*\big(s_i; \mathbf{B}(\cdot\,; 1)\big) = \max_{b \geqslant 0}(s_i - b)\Pr\big[s_j\colon\ \mathbf{B}(s_j; 1) \leqslant b\big] = \max_{b \geqslant 0}(s_i - b)\cdot b = \frac{s_i^2}{4}.$$

The unique maximizer is $s_i/2$.

On the other hand, notice that, for any conjecture $\mathbf{b}_j$ and bid $b \in (0, s_i)$, $\pi(b, s_i; \mathbf{b}_j) \leqslant s_i - b$.

[5] Or a probability distribution over such functions (see Section 3).

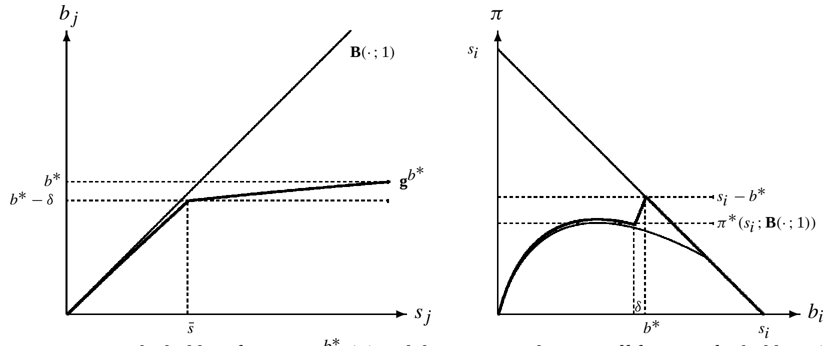

Fig. 1. The bidding function $\mathbf{g}^{b^*}$ (L) and the corresponding payoff function for bidder i (R).

The above observations allow us to place an upper bound on the set of interim 2-rationalizable bids. Suppose that b is such that $s_i - b < \pi^*(s_i;\mathbf{B}(\cdot\,;1)) = \pi(\frac{s_i}{2}, s_i;\mathbf{B}(\cdot\,;1))$. Consider any conjecture $\mathbf{b}_j$ that satisfies Eq. (3): then

$$\pi(b, s_i;\mathbf{b}_j) \leqslant s_i - b < \pi\left(\frac{s_i}{2}, s_i;\mathbf{B}(\cdot\,;1)\right) \leqslant \pi\left(\frac{s_i}{2}, s_i;\mathbf{b}_j\right).$$

Hence, *b cannot be a best reply to a conjecture that satisfies Eq.* (3). In other words, the quantity $s_i - \pi^*(s_i;\mathbf{B}(\cdot\,;1) = s_i - s_i^2/4$ is an upper bound on the set $\mathcal{R}(s_i;2)$.

We now prove that this upper bound is tight by exhibiting, for any bid $b^* \in (0, s_i - s_i^2/4)$, a conjecture $\mathbf{g}^{b^*}$ that satisfies Eq. (3) and such that b^* is a best reply to $\mathbf{g}^b$.

The construction is illustrated in the left panel of Fig. 1. The point $\bar{s}$ is chosen so that $\mathbf{B}(\bar{s};1) = b^* - \delta$. For $s_j \in [0,\bar{s}]$, the bidding function $\mathbf{g}^{b^*}$ approximates the bound $\mathbf{B}(\cdot\,;1)$ from below.[6] For $s_j > \bar{s}$, the function $\mathbf{g}^{b^*}$ is defined as the line segment joining the points $(\bar{s}, \mathbf{g}^{b^*}(\bar{s}))$ and $(1, b^*)$. Note that Eq. (3) is satisfied for almost all s_j, as required.

The corresponding payoff function is shown in the right panel of Fig. 1. Notice that, for $b \in [0, b^*-\delta]$, by construction $\pi(b, s_i;\mathbf{B}(\cdot\,;1)) \approx \pi(b, s_i;\mathbf{g}^{b^*})$; moreover, $\pi(b^*, s_i;\mathbf{g}^{b^*}) = s_i - b^*$. Thus, since $b^* < s_i - \pi^*(s_i;\mathbf{B}(\cdot\,;1))$, we have

$$\pi\left(b^*, s_i;\mathbf{g}^{b^*}\right) = s_i - b^* > \pi^*\left(s_i;\mathbf{B}(\cdot\,;1)\right) \geqslant \pi\left(b, s_i;\mathbf{B}(\cdot\,;1)\right) \approx \pi\left(b, s_i;\mathbf{g}^{b^*}\right).$$

Hence, b^* is strictly more profitable than any bid $b \in [0, b^* - \delta]$. Furthermore, δ is chosen so as to ensure that $\pi(b, s_i;\mathbf{g}^{b^*})$ is increasing on the interval $(b^* - \delta, b^*]$, as shown in the figure. Since clearly bids above b^* are strictly worse than b^*, it follows that b^* *is a strict best reply to the conjecture* $\mathbf{g}^{b^*}$.

We conclude that the function $s_i \mapsto s_i - \pi^*(s_i;\mathbf{B}(\cdot\,;1)) = s_i - s_i^2/4$ provides the *least* upper bound on the sets $\mathcal{R}(s_i;2)$; we shall denote it by $\mathbf{B}(\cdot\,;2)$. The argument above shows that $\mathcal{R}(s_i;2)$ *is an interval*[7] *with interior* $(0, \mathbf{B}(s_i;2))$.

The preceding construction did not rely on the specific functional form of the "old" least upper bound $\mathbf{B}(\cdot\,;1)$, but only on some of its features—specifically, monotonicity and

[6] For instance, let $\mathbf{g}^{b^*}(s_j) = s_j^{\alpha}\mathbf{B}(s_j;1)$ for α close to 1.

[7] We call "interval" any convex subset of $[0,1]$.

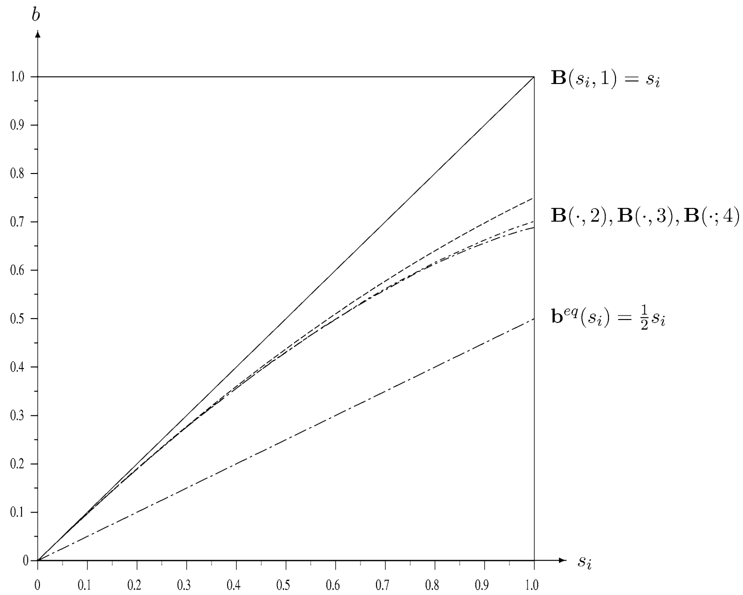

Fig. 2. The two-bidder uniform IPV model: bounds.

positivity. Thus, the construction can be repeated starting from the bound just derived, i.e., $\mathbf{B}(\cdot\,; 2)$, to conclude that the least upper bound $\mathbf{B}(\cdot\,; 3)$ on the set $\mathcal{R}(\cdot\,; 3)$ is given by

$$\mathbf{B}(s_i; 3) = s_i - \pi^*\big(s_i; \mathbf{B}(\cdot\,; 2)\big).$$

More generally, determining the maximum payoff against a given least upper bound $\mathbf{B}(\cdot\,; k-1)$ is sufficient to pin down the least upper bound $\mathbf{B}(\cdot\,; k)$ on the set of k-rationalizable bids. The set $\mathcal{R}(s_i; k)$ is an interval with interior $(0, \mathbf{B}(s_i; k))$.

Thus, the methodology we propose is particularly amenable to numerical computation. Figure 2 shows the first four bounds for the model under consideration; further bounds are not shown because they do not differ significantly from $\mathbf{B}(\cdot\,; 4)$.[8]

It can be easily proved by induction that $\mathbf{B}(s_i; k) \leqslant \mathbf{B}(s_i; k-1)$. Therefore the limit $\mathbf{B}(s_i; \infty) = \lim_{k\to\infty} \mathbf{B}(s_i; k)$ is well defined. In Section 4 we prove that the set of interim rationalizable bids for type s_i is an interval with interior $(0, \mathbf{B}(s_i; \infty))$.

To further illustrate our techniques, we sketch an argument showing that, as suggested by Fig. 2, there are rationalizable bids *strictly above the Nash equilibrium* for almost every type. First note that $\mathbf{b}^{\mathrm{eq}}(s_i) = s_i/2$ must be rationalizable for type s_i because the bidding function $\mathbf{b}^{\mathrm{eq}}$ is a best reply to itself. Therefore $\mathbf{B}(s_i; \infty) \geqslant \mathbf{b}^{\mathrm{eq}}(s_i)$. Since the set of rationalizable bids for s_i is an interval with interior $(0, \mathbf{B}(s_i; \infty))$, every bid in the interval $(0, \mathbf{b}^{\mathrm{eq}}(s_i))$ is rationalizable. This implies that all the best replies to conjectures $\mathbf{b}_j$ such that $0 < \mathbf{b}_j(s_j) < \mathbf{b}^{\mathrm{eq}}(s_j)$ (for almost all s_j) are rationalizable. The least upper bound on

[8] Details and code are available from the authors upon request.

such best replies can be derived from $\mathbf{b}^{\text{eq}}$, regarded as a least upper bound on conjectures, using our method:

$$\mathbf{b}^{\text{new}}(s_i) = s_i - \pi^*(s_i; \mathbf{b}^{\text{eq}}) = s_i - \frac{s_i^2}{2}.$$

It is easily verified that $\mathbf{b}^{\text{new}}(s_i) > \mathbf{b}^{\text{eq}}(s_i)$ for all $s_i \in (0, 1)$. Therefore $\mathbf{B}(s_i; \infty) > \mathbf{b}^{\text{eq}}(s_i)$ for almost all s_i. In Section 4 we show that this is a general result.

In this example, the least upper bound $\mathbf{B}(s_i; 2) = s_i - s_i^2/4$ is a nondecreasing and concave function of s_i; also, it lies very close to the 45° line near the origin. This is not specific to the case $k = 2$, or indeed to this particular example: our results in Section 4 show that, in all symmetric auctions with independent private values as well as in "generalized wallet games" (cf. Klemperer, 1998), each bound $\mathbf{B}(\cdot; k)$ computed as indicated above has these properties, for all $k = 2, \dots, \infty$. Therefore, rationalizability implies substantial proportional shading (bidding below the valuation) for high types, but is consistent with negligible proportional shading for low types.

3. Characterizing best responses

Consider the following game with asymmetric information representing a single-object, first-price auction with (possibly) interdependent values and risk-neutral bidders. There are n players, or bidders. Each bidder i observes a random signal with realizations s_i in the compact interval $S_i = [0, 1]$. Signals are distributed according to the joint c.d.f. $F : S \to [0, 1]$, where $S = \prod_{i=1}^n S_i$. We shall often refer to signals as *types*.

After observing her signal, each player chooses a bid $b \geqslant 0$. The object is assigned to one of the high bidders, breaking ties at random. The winner pays her bid, losers do not pay anything. Bidder i's value for the object is given by a function (random variable) $\mathbf{v}_i : S \to \mathbb{R}$.

3.1. Notation

Random variables and beliefs. From the point of view of a bidder, her competitors' bids are random variables. We use **boldface** letters to denote random variables. A function (random variable) $\mathbf{b}_j : [0, 1] \to \mathbb{R}_+$ can be interpreted as a conjecture of player i about the bidding behavior of player j—a description of how player j would bid for any possible signal (or type) s_j.[9] To allow for the possibility that a player is uncertain about the bidding behavior of her competitors, we model beliefs as probability distributions over $(n - 1)$-*tuples* of bidding functions (random variables). Let $\mathcal{B}_j$ denote the jth copy of the set of bounded functions with domain $[0, 1]$ and range $\mathbb{R}_+$, interpreted as the set of conjectures about j. The set of possible conjectures for bidder i about her competitors is $\mathcal{B}_{-i} = \prod_{j \neq i} \mathcal{B}_j$. A *belief* for player i is a probability measure on $\mathcal{B}_{-i}$, that is, an element

[9] We need *not* interpret $\mathbf{b}_j$ as a bidding strategy chosen *ex-ante*.

μ of the set $\Delta(\mathcal{B}_{-i})$.[10,11] With a slight abuse of notation we identify a belief μ assigning probability one to a *tuple* $\mathbf{b}_{-i} \in \mathcal{B}_{-i}$ with $\mathbf{b}_{-i}$ itself. As a matter of terminology, we refer to elements $\mathbf{b}_{-i} \in \mathcal{B}_{-i}$ as *conjectures*. Thus, in our setting conjectures are degenerate beliefs.

Inequalities. Inequalities between random variables are interpreted as pointwise inequalities which hold almost everywhere. For example, $\mathbf{b}_j < \mathbf{B}_j$ if and only if the set of s_j such that $\mathbf{b}_j(s_j) \geqslant \mathbf{B}_j(s_j)$ has (Lebesgue) measure zero. Similarly, inequalities between tuples of random variables are interpreted as coordinate-wise inequalities: $\mathbf{b}_{-i} < \mathbf{B}_{-i}$ if and only if $\mathbf{b}_j < \mathbf{B}_j$ for all $j \neq i$. Degenerate random variables and collections of identical degenerate random variables are represented by the corresponding real numbers.

Conditional expectations and probabilities. The expected value of a random variable $\mathbf{x}: S \to \mathbb{R}$ conditional on realization s_i is denoted $\mathrm{E}[\mathbf{x} \mid s_i]$ and the expected value of $\mathbf{x}$ conditional on s_i and event $C_{-i} \subseteq S_{-i}$ is denoted $\mathrm{E}[\mathbf{x} \mid s_i, C_{-i}]$. For example,

$$\mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{b}_{-i} \leqslant b] = \int\limits_{[\mathbf{b}_{-i} \leqslant b]} \mathbf{v}_i(s_i, s_{-i})\, \mathrm{d}F_{-i|i}(s_{-i} \mid s_i)$$

is the expected valuation for bidder i conditional on the signal and the event that b is the high bid.

A similar notation is used for conditional probabilities: the probability of event $A_{-i} \subseteq S_{-i}$ given s_i and $C_{-i} \subseteq S_{-i}$ is denoted $\Pr[A_{-i} \mid s_i, C_{-i}]$, and C_{-i} is omitted if $C_{-i} = S_{-i}$. For example,

$$\Pr[\mathbf{b}_{-i} \leqslant b \mid s_i] = \int\limits_{[\mathbf{b}_{-i} \leqslant b]} \mathrm{d}F_{-i|i}(s_{-i} \mid s_i)$$

is the conditional probability that b is the high bid given conjecture $\mathbf{b}_{-i}$.

We shall only need to consider beliefs assigning zero probability to ties. Given such a belief $\mu \in \Delta(\mathcal{B}_{-i})$, the expected payoff of bidding b conditional on signal s_i is

$$\pi(b, s_i; \mu) = \int\limits_{\mathcal{B}_{-i}} \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{b}_{-i} \leqslant b] \Pr[\mathbf{b}_{-i} \leqslant b \mid s_i] \mu(\mathrm{d}\mathbf{b}_{-i}).$$

Let

$$\pi^*(s_i; \mu) = \sup_{b \geqslant 0} \pi(b, s_i; \mu).$$

Bid b is a best response to belief μ for type s_i if $\pi(b, s_i; \mu) = \pi^*(s_i; \mu)$.

---

[10] Our results do not depend on the choice of a specific sigma-algebra of measurable subsets of $\mathcal{B}_{-i}$. We only require that singletons are measurable, so that degenerate beliefs belong to $\Delta(\mathcal{B}_{-j})$.

[11] Note that we allow for *correlated* choices of bidding functions, and hence spurious correlation among opponents' bids. However, the formulation in the text does entail a mild restriction: player i cannot believe that player j's bid is a function of the valuation of competitor k.

3.2. Assumptions

We assume that the environment is symmetric (cf. (Milgrom and Weber, 1982)). More precisely:

Assumption 1. The cumulative distribution F is symmetric: that is, for any permutation $\{\pi(1), \ldots, \pi(n)\}$ of $\{1, \ldots, n\}$, and for any $s \in [0, 1]^n$, $F(s_1, \ldots, s_n) = F(s_{\pi(1)}, \ldots, s_{\pi(n)})$.

Assumption 2. The valuation functions are symmetric: that is, there exists a function $\mathbf{v} : [0, 1] \times [0, 1]^{n-1} \to \mathbb{R}$ such that:

(i) For every $s_1 \in [0, 1]$, $s_{-1} \in [0, 1]^{n-1}$, and permutation $\{\pi(2), \ldots, \pi(n)\}$ of $\{2, \ldots, n\}$, $\mathbf{v}(s_1, s_{-1}) = \mathbf{v}(s_1, (s_{\pi(j)})_{j \neq 1})$.
(ii) For every $i \in N$, and $s = (s_i, s_{-i}) \in S$, $\mathbf{v}_i(s) = \mathbf{v}(s_i, s_{-i})$.

For example, in an auction with *private values*, $\mathbf{v}(s_1, s_{-1}) = s_1$. In an auction with pure common values, $\mathbf{v}(s_1, \ldots, s_n)$ is the expected value of the object conditional on the realization $(s_1, \ldots, s_n)$.

In the following, we shall drop player indices whenever no confusion can arise.

Assumption 3. The cumulative distribution function F is differentiable, with continuous density f bounded away from zero.

Assumption 4. The function $\mathbf{v} : [0, 1] \times [0, 1]^{n-1} \to \mathbb{R}$ is continuous, increasing in the first argument, and nondecreasing in all the other arguments. Moreover, it satisfies $\mathbf{v}(0, 0, \ldots, 0) = 0$.[12]

As a consequence of Assumption 4, the expected valuation conditional on a player's signal is positive:

Remark 1. For each player i and each signal $s_i > 0$,

$$\mathrm{E}[\mathbf{v}_i \mid s_i] > 0. \tag{4}$$

We also assume that *signals are affiliated*. As is well known from Milgrom and Weber (1982) (MW henceforth), this is equivalent to the supermodularity of $\log f$. For any pair of vectors $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ let $x \vee y$ and $x \wedge y$ denote the componentwise maximum and minimum, respectively, i.e., $x \vee y = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$ and $x \wedge y = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$.

[12] The assumption that $\mathbf{v}_i(0, \ldots, 0) = 0$ is made for expositional simplicity only. It ensures that, in the symmetric equilibrium constructed in Theorem 14 of (Milgrom and Weber, 1982), positive bids win with positive probability; thus, equilibrium bidding functions are admissible conjectures. Our analysis goes through unchanged if one instead assumes that $\mathbf{v}_i$ is nonnegative, and players only expect bids above $\mathbf{v}_i(0, \ldots, 0)$ to win with positive probability (so that, again, equilibrium bidding functions are admissible conjectures).

Assumption 5. For all $s, s' \in S$,

$$f(s \vee s')f(s \wedge s') \geqslant f(s)f(s'). \tag{5}$$

Clearly, statistical independence is a special case of affiliation. Two key properties of affiliated random variables will be employed here:

Result 1 (cf. MW, Theorem 4). For every nonempty $J \subseteq N$, and for every $K \subseteq N$ disjoint from J, the random variables $\{\mathbf{s}_j\}_{j \in J}$ are affiliated conditional upon the realizations of the (possibly empty) collection of random variables $\{\mathbf{s}_k\}_{k \in K}$.

Result 2 (MW, Theorem 5). For every random variable $\mathbf{H} : S \to \mathbb{R}$, if $\mathbf{H}$ is nondecreasing in each argument, then the conditional expectation function

$$h(x_1, y_1, \dots, x_n, y_n) = \mathrm{E}[\mathbf{H} \mid x_1 \leqslant \mathbf{s}_1 \leqslant y_1, \dots, x_n \leqslant \mathbf{s}_n \leqslant y_n]$$

is nondecreasing in each argument.

3.3. Upper bounds on best responses

It is often noted that a player bidding close to her expected valuation $\mathrm{E}[\mathbf{v}_i \mid s_i]$ is subject to the *winner's curse*, because she is not taking into account the fact that, if she wins the object, it must be the case the competitors have observed low signals. This adverse-selection argument relies on the assumption that player i thinks that her competitors are using *increasing* bidding functions. To see this, note that if bidder i has conjecture $\mathbf{b}_{-i}$ and $\mathbf{b}_{-i}$ is increasing (in each component), then the expected valuation *conditional on* (the signal and) *the event of winning the object* is $\mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{b}_{-i} \leqslant b] \leqslant \mathrm{E}[\mathbf{v}_i \mid s_i]$, where the inequality is strict in nondegenerate cases. However, if the conjecture $\mathbf{b}_{-i}$ is not increasing, then it may be the case that $\mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{b}_{-i} \leqslant b] > \mathrm{E}[\mathbf{v}_i \mid s_i]$, and the best response to $\mathbf{b}_{-i}$ may lie above $\mathrm{E}[\mathbf{v}_i \mid s_i]$.

We find it interesting to carry out our analysis of strategically sophisticated bidding focusing on beliefs for which the adverse-selection argument mentioned above is valid; therefore, we mainly restrict our attention to beliefs that assign positive weight only to increasing bidding functions.[13] Also, we shall show that above-equilibrium bids are interim rationalizable for almost all bidder types; in light of the preceding discussion, this conclusion would be uninteresting if we allowed for nonmonotonic beliefs.

Formally, let $\mathcal{M}_j$ denote the set of monotone increasing bidding functions for player j and let $\mathcal{M}_{-i} = \prod_{j \neq i} \mathcal{M}_j$. Then, in our analysis of auctions with interdependent values, we consider beliefs $\mu \in \Delta(\mathcal{M}_{-i})$. In the special case of private values we consider more general beliefs.

In the same spirit, we wish to avoid the possibility that a player may bid either zero or above her conditional valuation only because she is certain that she is not going to get the object. Therefore we assume that a player believes that every positive bid yields a positive

[13] Including nondecreasing bidding functions with flat segments in the support of beliefs involves technical complications that are dealt with in (Battigalli and Siniscalchi, 2000).

probability of winning the object. Formally, the set of bidder i's beliefs we restrict our attention to is

$$\Delta^+(\mathcal{B}_{-i}) = \left\{ \mu \in \Delta(\mathcal{B}_{-i})\colon\ \forall b > 0,\ \int\limits_{\mathcal{B}_{-i}} \Pr[\mathbf{b}_{-i} \leqslant b \mid s_i]\mu(\mathrm{d}\mathbf{b}_{-i}) > 0 \right\},$$

where s_i on the right-hand side is arbitrary.[14] We record two immediate consequences of these restrictions on beliefs.

Remark 2. For any signal s_i and conjecture $\mathbf{b}_{-i} \in \mathcal{M}_{-i}$, the function $b \mapsto \pi(b, s_i; \mathbf{b}_{-i})$ is continuous, and there exists $b^* \in \mathbb{R}_+$ such that $\pi(b^*, s_i; \mathbf{b}_{-i}) = \pi^*(s_i; \mathbf{b}_{-i})$.

Proof. See Lemma 14 in Appendix A. □

Remark 3. Fix $s_i > 0$ and $\mu \in \Delta(\mathcal{M}_{-i}) \cap \Delta^+(\mathcal{B}_{-i})$. Then $\pi^*(s_i; \mu) > 0$. Moreover, neither $b = 0$ nor any $b \geqslant \mathrm{E}[\mathbf{v}_i \mid s_i]$ are maximizers.

Proof. By Assumption 4, $\mathbf{v}_i(s_i, 0, \ldots, 0) > \mathbf{v}_i(0, \ldots, 0) = 0$. Let $b' = \frac{1}{2}\mathbf{v}_i(s_i, 0, \ldots, 0) > 0$; then $\int_{\mathcal{M}_{-i}} \Pr[b_{-i} \leqslant b' \mid s_i]\mu(\mathrm{d}\mathbf{b}_{-i}) > 0$, so $\pi(b', s_i; \mu) > 0$. On the other hand, $\pi(0, s_i; \mu) = 0$. Also, by Result 2,

$$\begin{aligned}\pi(b, s_i; \mathbf{b}_{-i}) &= \big(\mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{b}_{-i} \leqslant b] - b\big)\Pr[\mathbf{b}_{-i} \leqslant b \mid s_i]\\ &\leqslant \big(\mathrm{E}[\mathbf{v}_i \mid s_i] - b\big)\Pr[\mathbf{b}_{-i} \leqslant b \mid s_i] \leqslant 0\end{aligned}$$

for all $\mathbf{b}_{-i} \in \mathcal{M}_{-i}$ and $b \geqslant \mathrm{E}[\mathbf{v}_i \mid s_i]$. Thus, $\pi(b, s_i; \mu) \leqslant 0$ for all $b \geqslant \mathrm{E}[\mathbf{v}_i \mid s_i]$. □

Let $\mathbf{B}_{-i} = \{\mathbf{B}, \mathbf{B}, \ldots\}$ be an arbitrary (symmetric) upper bound on the bids of player i's competitors. The main result of this section characterizes the set of interim best replies to "monotonic" beliefs assigning probability one to bids below this upper bound. The set of such beliefs is

$$\Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i}) = \big\{\mu \in \Delta(\mathcal{M}_{-i}) \cap \Delta^+(\mathcal{B}_{-i})\colon\ \mu\big(\{\mathbf{b}_{-i}\colon\ \mathbf{b}_{-i} < \mathbf{B}_{-i}\}\big) = 1\big\}.$$

For the special case of private values, where adverse-selection considerations play no role, we are able to characterize best responses to arbitrary beliefs below the upper bound, i.e., beliefs in the set

$$\Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i}) = \big\{\mu \in \Delta^+(\mathcal{B}_{-i})\colon\ \mu\big(\{\mathbf{b}_{-i}\colon\ \mathbf{b}_{-i} < \mathbf{B}_{-i}\}\big) = 1\big\}.$$

Theorem 6. *Let $\mathbf{B}\colon [0,1] \to \mathbb{R}_+$ be a nondecreasing function such that $\mathbf{B} > 0$, and define $\mathbf{B}_{-i} = \{\mathbf{B}, \mathbf{B}, \ldots\}$. For every bid $b^* > 0$ and signal $s_i \in [0,1]$,*

(1) *if $\mathrm{E}[\mathbf{v}_i \mid s_i] - b^* < \inf_{\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})} \pi^*(s_i; \mu)$, then b^* is not a best reply to any belief $\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$ for s_i;*

[14] By Assumption 3, for every s_i, the conditional density $f_{-i|i}(\cdot \mid s_i)$ is bounded away from zero; hence the expression on the right-hand side is independent of s_i.

(2) *if* $\mathrm{E}[\mathbf{v}_i \mid s_i] - b^* > \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi^*(s_i;\mu)$, *then* b^* *is a strict best reply to some belief* $\mu \in \Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})$ *for* s_i;

(3) *furthermore,*

$$\begin{aligned}&\inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi^*(s_i;\mu)\\ &\quad= \sup_{b\geqslant 0}\Big\{\big(\mathrm{E}[\mathbf{v}_i \mid s_i,\ \mathbf{B}_{-i} < b] - b\big)\Pr[\mathbf{B}_{-i} < b \mid s_i]\\ &\qquad + \max\Big\{0, \big(\mathrm{E}\big[\mathbf{v}_i \bigm| s_i, \mathbf{B}_{-i}^{\max} = b\big] - b\big)\Pr\big[\mathbf{B}_{-i}^{\max} = b \bigm| s_i\big]\Big\}\Big\}\end{aligned}$$

and the supremum is attained;

(4) *if the auction game has private values*

$$\inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi^*(s_i;\mu) = \pi^*(s_i;\mathbf{B}_{-i}) = \inf_{\mu\in\Delta^+(\mathcal{B}_{-i};\mathbf{B}_{-i})} \pi^*(s_i;\mu).$$

Therefore parts (1) and (2) hold with $\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})$ *replaced by* $\Delta^+(\mathcal{B}_{-i};\mathbf{B}_{-i})$.

Note that, by (3) above and Remark 2, if $\mathbf{B}$ is increasing, then

$$\inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi^*(s_i;\mu) = \pi^*(s_i;\mathbf{B}_{-i})$$

as in the case of private values. Note also that the given bound $\mathbf{B}_{-i}$ can be chosen so high that

$$\inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi^*(s_i;\mu) = 0.$$

Therefore, in view of Remark 3, we obtain:

Corollary 7. *The set of best replies for* $s_i > 0$ *to beliefs in* $\Delta^+(\mathcal{M}_{-i})$ *is the interval* $(0, \mathrm{E}[\mathbf{v}_i \mid s_i])$.

Theorem 6 shows that for every (common) least upper bound $\mathbf{B}: [0,1] \to \mathbb{R}_+$ on the bids of the opponents we can derive a least upper bound on the interim best replies of player i. We denote by $\phi^{\mathbf{B}}$ the new upper bound, that is,

$$\forall s_i \in [0,1], \quad \phi^{\mathbf{B}}(s_i) = \mathrm{E}[\mathbf{v}_i \mid s_i] - \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})} \pi^*(s_i;\mu),$$

where $\mathbf{B}_{-i} = \{\mathbf{B}, \mathbf{B}, \ldots\}$. The following proposition lists some useful properties of the map ϕ, under additional restrictions on the bound $\mathbf{B}$ that will be satisfied in our application to rationalizability.

Proposition 8. *For every continuous, increasing function* $\mathbf{B}$ *such that* $\mathbf{B} > 0$, *the function* $\phi^{\mathbf{B}}$ *satisfies the following properties*:

(1) $0 < \phi^{\mathbf{B}}(s_i) \leqslant \mathrm{E}[\mathbf{v}_i \mid s_i]$ *for all* $s_i \in (0,1]$. *If* $\mathbf{B}(0) \geqslant \mathrm{E}[\mathbf{v}_i \mid \mathbf{s}_i = 0]$, $\mathrm{E}[\mathbf{v}_i \mid \mathbf{s}_i = 0] = \phi^{\mathbf{B}}(0)$.

(2) *For every* $s_i > 0$ *and* $b^* \in \arg\max_{b \geqslant 0} \pi(b, s_i; \mathbf{B}_{-i})$, $b^* \leqslant \phi^{\mathbf{B}}(s_i) \leqslant \mathbf{B}(1)$ (*where* $\mathbf{B}_{-i} = \{\mathbf{B}, \mathbf{B}, \ldots\}$). *Furthermore, if* $\mathbf{B}(1) \notin \arg\max_{b \geqslant 0} \pi(b, s_i; \mathbf{B}_{-i})$, *both inequalities are strict.*
(3) $\phi^{\mathbf{B}}$ *is continuous.*
(4) *Either* $\phi^{\mathbf{B}}$ *is increasing or there exists a signal* $s^{\mathbf{B}}$ *such that* $\phi^{\mathbf{B}}$ *is increasing on* $[0, s^{\mathbf{B}})$ *and* $\phi^{\mathbf{B}}(s_i) = \mathbf{B}(1)$ *for all* $s_i \in [s^{\mathbf{B}}, 1]$.

Proof. (1) As noted above, if $\mathbf{B}$ is increasing, $\phi^{\mathbf{B}}(s_i) = \mathrm{E}[\mathbf{v}_i \mid s_i] - \pi^*(s_i; \mathbf{B}_{-i})$. This immediately implies that $\phi^{\mathbf{B}}(s_i) \leqslant \mathrm{E}[\mathbf{v}_i \mid s_i]$ for all s_i. Moreover, consider a signal $s_i > 0$ and the bid function $\mathbf{g}$ defined by

$$\forall s_j \in [0,1], \quad \mathbf{g}(s_j) = s_j \mathbf{B}(s_j).$$

Since $\mathbf{B}$ is continuous, so is $\mathbf{g}$, and $\mathbf{g}(0) = 0$. Thus, $\mathbf{g}_{-i} \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$. Moreover, by Remark 2, there exists $b^* \in \mathbb{R}_+$ such that $\pi(b^*, s_i; \mathbf{g}_{-i}) = \pi^*(s_i; \mathbf{g}_{-i})$; by Remark 3, $b^* > 0$; by (1) in Theorem 6, this implies that $\phi^{\mathbf{B}}(s_i) \geqslant b^* > 0$.

Finally, assume that $\mathbf{B}(0) \geqslant \mathrm{E}[\mathbf{v}_i \mid \mathbf{s}_i = 0]$. Note that, for all $b \in \mathbb{R}_+$, $\pi(b, 0; \mathbf{B}_{-i}) \leqslant (\mathrm{E}[\mathbf{v}_i \mid \mathbf{s}_i = 0] - b) \Pr[\mathbf{B}_{-i} \leqslant b \mid \mathbf{s}_i = 0]$ by Result 2. Now, either $\Pr[\mathbf{B}_{-i} \leqslant b \mid \mathbf{s}_i = 0] = 0$, or $b > \mathbf{B}(0) \geqslant \mathrm{E}[\mathbf{v}_i \mid \mathbf{s}_i = 0]$. Therefore $\pi(b, 0; \mathbf{B}_{-i}) \leqslant 0$ for all b, which implies $\phi^{\mathbf{B}}(0) = \mathrm{E}[\mathbf{v}_i \mid \mathbf{s}_i = 0] - \pi^*(0; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i \mid \mathbf{s}_i = 0]$.

(2) Let $b^* \in \arg\max_{b \geqslant 0} \pi(b, s_i; \mathbf{B}_{-i})$, where $s_i > 0$. If $b^* = \mathbf{B}(1)$, then $\pi^*(s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i \mid s_i] - \mathbf{B}(1)$ and the result is obvious. If $b^* < \mathbf{B}(1)$, then $\Pr[\mathbf{B}_{-i} \leqslant b^* \mid s_i] < 1$. Therefore (using Result 2 again)

$$\pi^*(s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i - b^* \mid s_i,\ \mathbf{B}_{-i} \leqslant b^*] \Pr[\mathbf{B}_{-i} \leqslant b^* \mid s_i] < \mathrm{E}[\mathbf{v}_i \mid s_i] - b^*.$$

Hence $b^* < \mathrm{E}[\mathbf{v}_i \mid s_i] - \pi^*(s_i; \mathbf{B}_{-i}) = \phi^{\mathbf{B}}(s_i)$.

Finally, note that if $\mathbf{B}(1) \notin \arg\max_{b \geqslant 0} \pi(b, s_i; \mathbf{B}_{-i})$ then $\pi^*(s_i; \mathbf{B}_{-i}) > \pi(\mathbf{B}(1), s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i \mid s_i] - \mathbf{B}(1)$. Thus $\mathbf{B}(1) > \mathrm{E}[\mathbf{v}_i \mid s_i] - \pi^*(s_i; \mathbf{B}_{-i}) = \phi^{\mathbf{B}}(s_i)$.

The proof of (3) and (4) may be found in Appendix A. □

4. Rationalizable bids

The standard definition of (interim) rationalizability captures the implications of the assumption that bidders are (interim) rational, and there is common certainty of this fact. We analyze a strengthening of this definition because we also assume that bidders' beliefs satisfy some restrictions, and there is common certainty of this fact too (see, e.g., Battigalli, 1999).

4.1. Definitions

Let $\Delta_i \subseteq \Delta(\mathcal{B}_{-i})$ be a restricted set of beliefs and let $\Delta = (\Delta_1, \ldots, \Delta_n)$. In particular, we shall consider the case $\Delta_i = \Delta(\mathcal{M}_{-i}) \cap \Delta^+(\mathcal{B}_{-i})$ in the next subsection, and $\Delta_i = \Delta^+(\mathcal{B}_{-i})$ in the following one. We provide a definition of interim Δ-rationalizability that captures the implications of the assumption that:

(a) the bidders are expected payoff maximizers,
(b) for each bidder $i = 1, \ldots, n$, i's beliefs belong to Δ_i (but different bidders and different types of the same bidder may have different beliefs), and
(c) there is common certainty of (a) and (b).

Some additional notation is required. First, fix a player i, a set of beliefs $\Delta_i^* \subseteq \Delta(\mathcal{B}_{-i})$ and a type s_i. We let

$$\rho_i(s_i, \Delta_i^*) = \bigl\{b \geqslant 0\colon \exists \mu \in \Delta_i^*,\ \pi(b, s_i; \mu) = \pi^*(s_i; \mu)\bigr\}$$

denote the set of bids rationalized for type s_i by beliefs in Δ_i^*. Observe that ρ_i is monotone in its second argument: that is, $\Delta_i' \subseteq \Delta_i''$ implies $\rho_i(s_i, \Delta_i') \subseteq \rho_i(s_i, \Delta_i'')$. Next, fix a $(n-1)$-tuple $\mathcal{C}_{-i} = (\mathcal{C}_j)_{j \neq i}$, where each $\mathcal{C}_j$ is a correspondence (multi-valued function) from S_i to $\mathbb{R}_+$. Since $\mathcal{C}_{-i}$ may be interpreted as a subset of $\mathcal{B}_{-i}$ it makes sense to write

$$\Delta(\mathcal{C}_{-i}) = \bigl\{\mu \in \Delta(\mathcal{B}_{-i})\colon\ \mu\bigl(\bigl\{\mathbf{b}_{-i}\colon \forall j \neq i,\ \forall s_j \in [0,1],\ \mathbf{b}_j(s_j) \in \mathcal{C}_j(s_j)\bigr\}\bigr) = 1\bigr\}.$$

The main definitions can now be provided.

Definition 9. An n-tuple of correspondences $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ has the *Δ-best response property* if $\mathcal{C}_j(s_j) \subseteq \rho_j(s_j, \Delta_j \cap \Delta(\mathcal{C}_{-i}))$ for all $j = 1, \ldots, n$, $s_j \in S_j$.

Definition 10. For all $i = 1, \ldots, n$, $s_i \in [0,1]$ and $k = 1, 2, \ldots$ let $\mathcal{R}_i^\Delta(s_i; 0) = \mathbb{R}_+$,

$$\mathcal{R}_i^\Delta(s_i; k) = \rho_i\bigl(s_i, \Delta_i \cap \Delta\bigl(\mathcal{R}_{-i}^\Delta(\cdot, k-1)\bigr)\bigr).$$

A bid b^* is *interim (Δ, k)-rationalizable* for type s_i^* of bidder i if $b^* \in \mathcal{R}_i^\Delta(s_i^*, k)$. A bid b^* is *interim Δ-rationalizable* for type s_i^* of bidder i if there is an n-tuple of correspondences $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ with the Δ-best response property such that $b^* \in \mathcal{C}_i(s_i^*)$.

Observe that interim Δ-rationalizability is defined via a best-response property, independently of the sets of (Δ, k) -rationalizable bids (cf. Osborne and Rubinstein, 1994, Definition 55.1). However, we will show below that the set of interim Δ-rationalizable bids for type s_i can indeed be obtained as the limit of the sets of (Δ, k)-rationalizable bids as $k \to \infty$.[15]

The following remarks are easily derived from Definition 10.

Remark 4. The set of Δ-rationalizable bids for type s_i is included in $\mathcal{R}_i^\Delta(s_i, k)$ for all $k = 1, 2, \ldots$.

Proof. Let $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ be an n-tuple of correspondences with the Δ-best-response property. Trivially, $\mathcal{C}_j(s_j) \subseteq \rho_j(s_j, \Delta_i \cap \Delta(\mathcal{C}_{-i})) \subseteq \mathbb{R}_+ = \mathcal{R}_i^\Delta(s_i; 0)$ for all j and s_j. Suppose that $\mathcal{C}_j(s_j) \subseteq \mathcal{R}_j^\Delta(s_j; k-1)$ for all j and s_j. Then $\Delta(\mathcal{C}_{-i}) \subseteq \Delta(\mathcal{R}_{-i}^\Delta(\cdot\,; k-1))$.

[15] In games with compact action spaces and continuous payoffs, this result follows from standard arguments: see Bernheim (1984, Proposition 3.2) and the generalization in Battigalli (1999, Proposition 3). But Lipman (1994) shows by example that the result does not hold in general discontinuous games. Hence, a direct proof is required in the present context. We thank an anonymous referee for pointing this out.

By the best-response property and monotonicity of $\rho_i(s_i, \cdot)$ we obtain $\mathcal{C}_i(s_i) \subseteq \rho_i(s_i, \Delta_i \cap \Delta(\mathcal{C}_{-i})) \subseteq \rho_i(s_i, \Delta_i \cap \Delta(\mathcal{R}_{-i}(\cdot, k-1))) = \mathcal{R}_i(s_i, k)$.

Therefore $\mathcal{C}_i(s_i) \subseteq \mathcal{R}_i(s_i, k)$ for all i, s_i and k, which implies the thesis. □

Remark 5. Let $(\mathbf{b}_1^{\text{eq}}, \ldots, \mathbf{b}_n^{\text{eq}})$ be a Bayesian Nash equilibrium such that $\mathbf{b}_{-i}^{\text{eq}} \in \Delta_i$, $i = 1, \ldots, n$. Then, for all i and s_i, the equilibrium bid $\mathbf{b}_i^{\text{eq}}(s_i)$ is interim Δ-rationalizable for type s_i.

Proof. Since $\mathbf{b}_{-i}^{\text{eq}} \in \Delta_i$ and $(\mathbf{b}_1^{\text{eq}}, \ldots, \mathbf{b}_n^{\text{eq}})$ is an equilibrium, $\mathbf{b}_i^{\text{eq}}(s_i) \in \rho_i(s_i, \Delta_i \cap \Delta(\{\mathbf{b}_{-i}^{\text{eq}}\}))$. Therefore $(\mathbf{b}_1^{\text{eq}}, \ldots, \mathbf{b}_n^{\text{eq}})$ (regarded as an n-tuple of correspondences) has the Δ-best-response property. □

4.2. Rationalizable bidding with monotonic beliefs

We now return to the auction setting of Section 3 and let $\Delta_i = \Delta^+(\mathcal{B}_{-i}) \cap \Delta(\mathcal{M}_{-i})$ $(i = 1, \ldots, n)$. To simplify the notation, we omit the superscript Δ from the set of interim Δ-rationalizable strategies.

We shall provide a full characterization of the set of interim (Δ, k) and Δ-rationalizable bids momentarily. However, a direct application of Theorem 6 and Definition 10 is sufficient to compare the predictions of equilibrium analysis and interim Δ-rationalizability.

From now on, we let $\mathbf{b}^{\text{eq}}$ denote the symmetric equilibrium bidding function of Milgrom and Weber (1982, Theorem 14). Our assumptions are sufficient to guarantee its existence; moreover, $\mathbf{b}^{\text{eq}}$ is increasing and satisfies $\mathbf{b}^{\text{eq}}(0) = 0$. Therefore $\mathbf{b}_{-i}^{\text{eq}} \in \Delta_i$. By Remark 5, $\mathbf{b}^{\text{eq}}(s_i)$ is Δ-rationalizable for type s_i.

Our first result shows that, for every type $s_i \in (0, 1)$, all bids below the equilibrium, as well as a nonempty interval of bids above it, are interim Δ-rationalizable.

Recall that, for any increasing bound $\mathbf{B}$, $\phi^{\mathbf{B}}(s_i) = \mathrm{E}[\mathbf{v}_i \mid s_i] - \pi^*(s_i; \mathbf{B}_{-i})$ is the new bound on best replies obtained by a given upper bound $\mathbf{B}$ on beliefs.

Proposition 11. *The function $\phi^{\mathbf{b}^{\text{eq}}}$ is increasing, and satisfies*:

(1) $\phi^{\mathbf{b}^{\text{eq}}}(1) = \mathbf{b}^{\text{eq}}(1)$.
(2) $\mathbf{b}^{\text{eq}}(s_i) < \phi^{\mathbf{b}^{\text{eq}}}(s_i) < \mathbf{b}^{\text{eq}}(1)$ *for all* $s_i \in (0, 1)$.
(3) *Every bid* $b \in (0, \phi^{\mathbf{b}^{\text{eq}}}(s_i))$ *is interim Δ-rationalizable for* s_i, *for all* $s_i \in (0, 1]$.

Proof. By the equilibrium condition $\mathbf{b}^{\text{eq}}(s_i) \in \arg\max_{b \geqslant 0} \pi(b, s_i; \mathbf{b}^{\text{eq}})$. Therefore, Proposition 8, (2) implies that $\mathbf{b}^{\text{eq}}(s_i) \leqslant \phi^{\mathbf{b}^{\text{eq}}}(s_i) \leqslant \mathbf{b}^{\text{eq}}(1)$, where the inequalities are strict for $s_i \in (0, 1)$, and hold as equalities if $s_i = 1$. This proves (1) and (2). Proposition 8 (4) implies that $\phi^{\mathbf{b}^{\text{eq}}}$ is increasing.

(3) To prove that every bid in the interval $(0, \phi^{\mathbf{b}^{\text{eq}}}(s_i))$ is interim Δ-rationalizable for s_i, we show that the n-tuple of correspondences $(s_j \mapsto (0, \phi^{\mathbf{b}^{\text{eq}}}(s_j)))_{j=1}^n$ (slightly modified for the extreme values $s_j = 0, 1$) has the best-response property:

- for $s_j \in (0,1)$, let $\mathcal{C}_j(s_j) = (0, \phi^{\mathbf{b}^{\mathrm{eq}}}(s_j))$, and $\widehat{\mathcal{C}}_j(s_j) = (0, \mathbf{b}^{\mathrm{eq}}(s_j)]$;
- for $s_j = 0$, let $\mathcal{C}_j(0) = \{0\} \cup (0, \phi^{\mathbf{b}^{\mathrm{eq}}}(0))$ and $\widehat{\mathcal{C}}_j(0) = \{0\}$;
- for $s_j = 1$, let $\mathcal{C}_j(1) = \widehat{\mathcal{C}}_j(1) = (0, \phi^{\mathbf{b}^{\mathrm{eq}}}(1)] = (0, \mathbf{b}^{\mathrm{eq}}(1)]$.

By (1) and (2), $\widehat{\mathcal{C}}_{-j}(s_j) \subseteq \mathcal{C}_{-j}(s_j)$ for all j and s_j. Thus, $\Delta(\widehat{\mathcal{C}}_{-j}) \subseteq \Delta(\mathcal{C}_{-j})$ for all j, so that $\rho_j(s_j; \Delta_j \cap \Delta(\widehat{\mathcal{C}}_{-j})) \subseteq \rho_j(s_j; \Delta_j \cap \Delta(\mathcal{C}_{-j}))$.

Now Theorem 6 implies that, for all j and $s_j \in (0,1)$, $\mathcal{C}_j(s_j) \subseteq \rho_j(s_j, \Delta_j \cap \Delta(\widehat{\mathcal{C}}_{-j}))$, and also that $(0, \phi^{\mathbf{b}^{\mathrm{eq}}}(s_j)) \subseteq \rho_j(s_j; \Delta_j \cap \Delta(\widehat{\mathcal{C}}_{-j}))$ for $s_j = 0, 1$.[16] Moreover, since $\mathbf{b}^{\mathrm{eq}}_{-j} \in \Delta_j$, by the equilibrium condition, also $0 \in \rho_j(0; \Delta_j \cap \Delta(\widehat{\mathcal{C}}_{-j}))$ and $\phi^{\mathbf{b}^{\mathrm{eq}}}(1) = \mathbf{b}^{\mathrm{eq}}(1) \in \rho_j(1; \Delta_j \cap \Delta(\widehat{\mathcal{C}}_{-j}))$. Thus, $\mathcal{C}_j(s_j) \subseteq \rho_j(s_j; \Delta_j \cap \Delta(\widehat{\mathcal{C}}_{-j}))$ for $s_j = 0, 1$ as well. Since $\rho_j(s_j; \Delta_j \cap \Delta(\widehat{\mathcal{C}}_{-j})) \subseteq \rho_j(s_j; \Delta_j \cap \Delta(\mathcal{C}_{-j}))$, the collection $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ has the best-response property. $\square$

We now turn to the characterization of interim Δ-rationalizability. For every signal $s_i \in [0,1]$, let

$$\mathbf{B}(s_i; 1) = \mathrm{E}[\mathbf{v}_i \mid s_i],$$
$$\mathbf{B}(s_i; k+1) = \mathrm{E}[\mathbf{v}_i \mid s_i] - \inf_{\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i}(\cdot\,; k))} \pi^*(s_i; \mu), \quad k \geqslant 1.$$

Since $\inf_{\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i}(\cdot\,; k))} \pi^*(s_i; \mu) \geqslant 0$, we have $\mathbf{B}(\cdot\,; 1) \geqslant \mathbf{B}(\cdot\,; 2)$; by induction, if $\mathbf{B}(\cdot\,; k-1) \geqslant \mathbf{B}(\cdot\,; k)$ for some $k > 1$,

$$\Delta^+\big(\mathcal{M}_{-i}; \mathbf{B}_{-i}(\cdot\,; k-1)\big) \supseteq \Delta^+\big(\mathcal{M}_{-i}; \mathbf{B}_{-i}(\cdot\,; k)\big),$$

so that, for every $s_i \in [0,1]$,

$$\inf_{\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i}(\cdot\,; k-1))} \pi^*(s_i; \mu) \leqslant \inf_{\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i}(\cdot\,; k))} \pi^*(s_i; \mu),$$

and hence $\mathbf{B}(\cdot\,; k) \geqslant \mathbf{B}(\cdot\,; k+1)$ for all k. Thus, the sequence $\{\mathbf{B}(\cdot\,; k)\}_{k \geqslant 1}$ is weakly decreasing pointwise, and we can define $\mathbf{B}(s_i; \infty) = \lim_{k \to \infty} \mathbf{B}(s_i; k)$ for all s_i. The main result of this section can now be stated.

Theorem 12. (1) *For all $k = 1, 2, \ldots$ and $s_i \in (0,1]$, $\mathcal{R}_i(s_i; k)$ is an interval with interior $(0, \mathbf{B}(s_i; k))$; the upper bound $\mathbf{B}(\cdot\,; k)$ is increasing, continuous, and satisfies $\mathbf{B}(\cdot\,; k) > 0$.*

(2) *For all $s_i \in (0,1]$, the set of interim Δ-rationalizable bids is an interval with interior $(0, \mathbf{B}(s_i; \infty))$; the upper bound $\mathbf{B}(\cdot\,; \infty))$ is nondecreasing and satisfies $\mathbf{B}(\cdot\,; \infty) > 0$.*

Proof. (1) The statement is true for $k = 1$ by Remark 3 and Assumption 4.[17] Suppose it is true for some $k \geqslant 1$. Then Theorem 6 implies that $\mathcal{R}(s_i; k+1)$ is an interval with interior $(0, \mathrm{E}[\mathbf{v}_i \mid s_i] - \pi^*(s_i; \mathbf{B}_{-i}(\cdot\,; k)) = (0, \mathbf{B}(s_i; k+1))$ for all $s_i \in (0,1]$.

By the inductive hypothesis, $\mathbf{B}(\cdot\,; k)$ is a continuous, increasing function such that $\mathbf{B}(\cdot\,; k) > 0$. Thus, Proposition 8, (1) and (3), immediately implies that $\mathbf{B}(\cdot\,; k+1) = \phi^{\mathbf{B}(\cdot\,; k)}$

[16] The latter inclusion may hold vacuously for $s_j = 0$, if $\phi^{\mathbf{b}^{\mathrm{eq}}}(0) = 0$.

[17] Continuity of the map $s_i \mapsto \mathrm{E}[\mathbf{v}_i \mid s_i]$ follows by Dominated Convergence.

is continuous and such that $\mathbf{B}(\cdot\,; k+1) > 0$. Moreover, Proposition 8(4) implies that either $\mathbf{B}(\cdot\,; k+1)$ is increasing, or there is some $s^k < 1$ such that $\mathbf{B}(s; k+1) = \mathbf{B}(1; k)$ for all $s \in [s^k, 1]$. By way of contradiction, suppose that $\mathbf{B}(\cdot\,; k+1)$ is not increasing. Then, $\mathbf{B}(s; k+1) = \mathbf{B}(1, k) > \mathbf{B}(s; k)$ for some $s \in (s^k, 1)$, which contradicts $\mathbf{B}(s; k+1) \leqslant \mathbf{B}(s; k)$. Therefore, $\mathbf{B}(\cdot\,; k+1)$ must also be increasing.

(2) As noted above, the sequence $\{\mathbf{B}(\cdot\,; k)\}_{k\geqslant 1}$ is weakly pointwise decreasing, and its elements are increasing and positive by Part (1). Hence, its pointwise limit $\mathbf{B}(\cdot\,; \infty)$ is nondecreasing. We first show that it satisfies $\mathbf{B}(\cdot\,; \infty) > 0$.

As noted in the text, for any s_i, the Milgrom–Weber equilibrium bid $\mathbf{b}^{\mathrm{eq}}(s_i)$ is interim Δ-rationalizable, and hence interim (Δ, k)-rationalizable for all $k \geqslant 1$. This implies that $\mathbf{B}(s_i; k) \geqslant \mathbf{b}^{\mathrm{eq}}(s_i)$ for all s_i and $k \geqslant 1$; hence, $\mathbf{B}(\cdot\,; \infty) \geqslant \mathbf{b}^{\mathrm{eq}} > 0$, as required.

Next we show that every bid in the open interval $(0, \mathbf{B}(s_i; \infty))$ is interim Δ-rationalizable for type s_i. Lemma 19 in Appendix A, applied to the sequence $\{\mathbf{B}(\cdot\,; k)\}_{k\geqslant 1}$ and to its limit $\mathbf{B}(\cdot\,; \infty)$, shows that

$$\lim_{k\to\infty} \pi^*\bigl(s_i; \mathbf{B}_{-i}(\cdot\,; k)\bigr) = \inf_{\mu\in\Delta^+(\mathcal{M}_{-i}, \mathbf{B}_{-i}(\cdot\,;\infty))} \pi^*(s_i; \mu).$$

Therefore

$$\begin{aligned} \phi^{\mathbf{B}(\cdot\,;\infty)}(s_i) &= \mathrm{E}[\mathbf{v}_i \mid s_i] - \inf_{\mu\in\Delta^+(\mathcal{M}_{-i}, \mathbf{B}_{-i}(\cdot\,;\infty))} \pi^*(s_i; \mu) \\ &= \lim_{k\to\infty} \Bigl(\mathrm{E}[\mathbf{v}_i \mid s_i] - \pi^*\bigl(s_i; \mathbf{B}_{-i}(\cdot\,; k)\bigr)\Bigr) = \lim_{k\to\infty} \mathbf{B}(s_i; k+1) = \mathbf{B}(s_i; \infty). \end{aligned}$$

Theorem 6 then implies that every bid $b^* \in (0, \mathbf{B}(s_i; \infty))$ is a best reply to some belief $\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i}(\cdot\,; \infty))$.

We prove that the collection of correspondences $(s_j \mapsto (0, \mathbf{B}(s_j; \infty)))_{j=1}^n$(slightly modified for the extreme values $s_j = 0, 1$) has the best-response property:

- for $s_j \in (0, 1)$, let $\mathcal{C}_j(s_j) = (0, \mathbf{B}(s_j; \infty))$;
- for $s_j = 0$, let $\mathcal{C}_j(0) = \{0\} \cup (0, \mathbf{B}(0; \infty))$;
- for $s_j = 1$, let $\mathcal{C}_j(1) = (0, \mathbf{B}(1; \infty)) \cup \{\mathbf{b}^{\mathrm{eq}}(1)\}$.

The zero bid must be included for type $s_j = 0$ to ensure that $\mathcal{C}_j(0) \neq \emptyset$ and $\Delta_j \cap \Delta(\mathcal{C}_{-j}) \neq \emptyset$. As shown below, including bid $b = \mathbf{b}^{\mathrm{eq}}(1)$ for type $s_j = 1$ allows to rationalize the zero bid for $s_j = 0$. Also, since $\mathbf{b}^{\mathrm{eq}}(1)$ is rationalizable for type $s_j = 1$, $\mathbf{B}(1; \infty) \geqslant \mathbf{b}^{\mathrm{eq}}(1)$; hence, $\mathcal{C}_j(1)$ is an interval, and $\mathbf{b}^{\mathrm{eq}}(1)$ is either one of its interior points, or its right endpoint.

$\phi^{\mathbf{B}(\cdot\,;\infty)} = \mathbf{B}(\cdot\,; \infty)$ implies that $\mathcal{C}_j(s_j) = (0, \mathbf{B}(s_j; \infty)) \subseteq \rho_j(s_j; \Delta_j \cap \Delta(\mathcal{C}_{-j}))$ for all $s_j > 0$; moreover, it implies that $(0, \mathbf{B}(s_j; \infty)) \subseteq \rho_j(s_j; \Delta_j \cap \Delta(\mathcal{C}_{-j})$ for $s_j = 0, 1$.

Furthermore, note that Proposition 11 implies that $\mathbf{b}^{\mathrm{eq}}(s_k) \in \mathcal{C}_k(s_k)$ for $s_k \in (0, 1)$; the definition of $\mathcal{C}_k(s_k)$ for $s_k = 0, 1$ ensures that $0 = \mathbf{b}^{\mathrm{eq}}(0) \in \mathcal{C}_k(0)$, and $\mathbf{b}^{\mathrm{eq}}(1) \in \mathcal{C}_k(1)$; thus, by the equilibrium condition, $s_j \in \rho_j(s_j; \Delta_j \cap \Delta(\mathcal{C}_{-j})$ for $s_j = 0, 1$ as well, and we conclude that $\mathcal{C}_j(s_j) \subseteq \rho_j(s_j; \Delta_j \cap \Delta(\mathcal{C}_{-j})$ for all s_j. Therefore, the collection $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ has the best-reply property. This proves that every bid in the interval $(0, \mathbf{B}(s_i; \infty))$ is Δ-rationalizable for type s_i.

On the other hand, for every bid $b > \mathbf{B}(s_i; \infty)$ there is some k such that $b > \mathbf{B}(s_i; k)$ and this implies that b cannot be Δ-rationalizable for s_i (see Remark 4). □

There are interesting examples of (symmetric) interdependent-values models where (a) signals are independent, and (b) valuations functions are *quasi-linear*: that is, the valuation function has the form $\mathbf{v}(s_i, s_{-i}) = \boldsymbol{\upsilon}(s_{-i})s_i + \boldsymbol{\kappa}(s_{-i})$, where $\boldsymbol{\upsilon}$ and $\boldsymbol{\kappa}$ are nondecreasing and $\boldsymbol{\upsilon} > 0$. Auctions with independent private values and the "wallet game" (see (Klemperer, 1998)) belong to this class of models. For such models we find stronger results about rationalizable bids. In particular, the following proposition shows that interim Δ-rationalizability *implies* substantial shading for high conditional valuations (signals), but is consistent with negligible shading for low conditional valuations (of course, interim Δ-rationalizability is also consistent with extreme shading for all signals, because every positive bid below the equilibrium is interim Δ-rationalizable for bidders with positive signals).

Proposition 13. *Suppose that the signals are independent and the valuation functions quasi-linear. Then, for all $k = 2, 3, \ldots, \infty$,*

(1) *the upper bound $\mathbf{B}(\cdot\,; k)$ is concave*;
(2) *the "minimum-shading" function $\mathbf{S}(s_i; k) = \mathrm{E}[\mathbf{v}_i \mid s_i] - \mathbf{B}(s_i; k)$ is increasing* (*nondecreasing for $k = \infty$*) *and convex, with $\mathbf{S}(0; k) = 0$ and $\mathbf{S}(s_i; k) > 0$ for all $s_i \in (0, 1]$*;
(3) *the "minimum-proportional-shading" function $\mathbf{S}(s_i, k)/\mathrm{E}[\mathbf{v}_i \mid s_i]$* (*defined for $s_i > 0$*) *is positive and nondecreasing*; *however,*

$$\lim_{s_i \downarrow 0} \frac{\mathbf{S}(s_i, k)}{\mathrm{E}[\mathbf{v}_i \mid s_i]} = 0.$$

Proof. Fix an increasing bound $\mathbf{B}$. We claim that the value function $\pi^*(s_i; \mathbf{B}_{-i})$ is increasing and convex in s_i.

To see this, note that, by quasi-linearity and independence,

$$\begin{aligned} \pi(b, s_i; \mathbf{B}_{-i}) &= \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{-i} \leqslant b] \Pr[\mathbf{B}_{-i} \leqslant b] \\ &= \big(\mathrm{E}[\boldsymbol{\upsilon} \mid \mathbf{B}_{-i} \leqslant b] s_i + \mathrm{E}[\boldsymbol{\kappa} \mid \mathbf{B}_{-i} \leqslant b] - b\big) \Pr[\mathbf{B}_{-i} \leqslant b], \end{aligned} \tag{6}$$

i.e., $\pi(b, s_i; \mathbf{B}_{-i})$ is linear in s_i, for any b.

To show that $\pi^*(\cdot\,; \mathbf{B}_{-i})$ is increasing in its first argument, pick $s_i' > s_i$. Suppose that $\pi^*(s_i; \mathbf{B}_{-i}) > 0$; choose $b^* \in \arg\max_{b \geqslant 0} \pi(b, s_i; \mathbf{B}_{-i})$, and note that $\Pr[\mathbf{B}_{-i} \leqslant b^*] > 0$ and $\boldsymbol{\upsilon} > 0$. Then (6) implies that $\pi^*(s_i'; \mathbf{B}_{-i}) \geqslant \pi(b^*, s_i'; \mathbf{B}_{-i}) > \pi(b^*, s_i; \mathbf{B}_{-i}) = \pi^*(s_i; \mathbf{B}_{-i})$. If instead $\pi^*(s_i; \mathbf{B}_{-i}) = 0$, Remark 3 shows that $\pi^*(s_i'; \mathbf{B}_{-i}) > \pi^*(s_i; \mathbf{B}_{-i}) = 0$.

To see that $\pi^*(\cdot\,; \mathbf{B}_{-i})$ is convex, choose $s_i, s_i', \lambda \in [0, 1]$ and let

$$b^* \in \arg\max_{b \geqslant 0} \pi\big(b, \lambda s_i + (1-\lambda)s_i'; \mathbf{B}_{-i}\big).$$

Then, by linearity,

$$\pi^*\big(\lambda s_i + (1-\lambda)s_i'; \mathbf{B}_{-i}\big) = \pi\big(b^*, \lambda s_i + (1-\lambda)s_i'; \mathbf{B}_{-i}\big)$$

$$= \lambda\pi(b^*, s_i; \mathbf{B}_{-i}) + (1-\lambda)\pi\big(b^*, s_i'; \mathbf{B}_{-i}\big)$$
$$\leqslant \lambda\pi^*(s_i; \mathbf{B}_{-i}) + (1-\lambda)\pi^*\big(s_i'; \mathbf{B}_{-i}\big). \tag{7}$$

This completes the proof of the preliminary claim.

We first prove (1)–(3) for $k < \infty$.

(1) By Theorem 12(1), the upper bound $\mathbf{B}(\cdot\,; k-1)$ is indeed increasing for $k-1 > 0$. Hence the argument above implies that $\mathbf{B}(\cdot\,; k) = \phi^{\mathbf{B}(\cdot\,;k-1)}$ is concave for each $k > 1$. Concavity is preserved by taking the pointwise limit. Therefore $\mathbf{B}(\cdot\,; \infty)$ is also concave.

(2) The argument given above also implies that $\mathbf{S}(s_i; k) = \pi^*(s_i; \mathbf{B}_{-i}(\cdot\,; k-1))$ is increasing and convex. Proposition 8(1) implies that $\mathbf{B}(0; k-1) = \mathbf{B}(0; k) = \mathrm{E}[\mathbf{v}_i \mid \mathbf{s}_i = 0]$. Therefore $\mathbf{S}(0; k) = 0$ and $\mathbf{S}(s_i; k) > 0$ for any $s_i > 0$.

(3) Parts (1) and (2) imply that $\mathbf{S}(s_i, k)/\mathrm{E}[\mathbf{v}_i \mid s_i]$ is positive and nondecreasing. Furthermore, we know that $\mathbf{b}^{\mathrm{eq}} \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i}(\cdot\,; k-1))$, where $\mathbf{b}^{\mathrm{eq}}$ is the Milgrom–Weber equilibrium bidding function. Then, Theorem 6 and Result 2 yield

$$\mathbf{S}(s_i; k) = \inf_{\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i}(\cdot\,;k-1))} \pi^*(s_i; \mu) \leqslant \pi^*\big(s_i; \mathbf{b}^{\mathrm{eq}}_{-i}\big) \leqslant \mathrm{E}[\mathbf{v}_i \mid s_i]\Pr[\mathbf{B}_{-i} \leqslant b]$$
$$= \mathrm{E}[\mathbf{v}_i \mid s_i]\big[G\big(\mathbf{B}^{-1}(s_i)\big)\big]^{n-1},$$

where G denotes the common marginal c.d.f. of the signals. Thus

$$0 \leqslant \lim_{s_i \downarrow 0} \frac{\mathbf{S}(s_i; k)}{\mathrm{E}[\mathbf{v}_i \mid s_i]} \leqslant \lim_{s_i \downarrow 0}\big[G\big(\mathbf{B}^{-1}(s_i)\big)\big]^{n-1} = 0.$$

Results (1)–(3) also hold for $k = \infty$. In particular, $\mathbf{S}(s_i; \infty) \geqslant \mathbf{S}(s_i, 2) > 0$ for $s_i > 0$. Concavity/convexity and weak monotonicity are preserved by taking the limit $k \to \infty$. As for proportional shading by low types, Lemma 19 in Appendix A and the argument above imply

$$\mathbf{S}(s_i; \infty) = \lim_{k \to \infty} \inf_{\mu \in \Delta^+(\mathcal{M}_{-i}, \mathbf{B}_{-i}(\cdot\,;k))} \pi^*(s_i; \mu) \leqslant \mathrm{E}[\mathbf{v}_i \mid s_i]\big[G\big(\mathbf{B}^{-1}(s_i)\big)\big]^{n-1}.$$

Therefore

$$0 \leqslant \lim_{s_i \downarrow 0} \frac{\mathbf{S}(s_i; \infty)}{\mathrm{E}[\mathbf{v}_i \mid s_i]} \leqslant \lim_{s_i \downarrow 0}\big[G\big(\mathbf{B}^{-1}(s_i)\big)\big]^{n-1} = 0. \qquad \square$$

4.2.1. Private values

In the previous subsection we analyzed (interim) Δ-rationalizability with $\Delta_i = \Delta^+(\mathcal{B}_{-i}) \cap \Delta(\mathcal{M}_{-i})$. But we remarked in Section 3 that the restriction to monotonic beliefs is immaterial if the auction game has private values, i.e., if signals and valuations coincide. In particular, Theorem 6(4) implies that all the results about Δ-rationalizability of the previous subsection also hold for $\Delta_i = \Delta^+(\mathcal{B}_{-i})$.

In the case of private values it makes sense to denote a generic signal by v_i. The upper bound on best responses to beliefs in $\Delta^+(\mathcal{B}_{-i})$ is $\mathbf{B}(v_i; 1) = v_i$. Therefore we obtain

Remark 6. $\mathcal{R}(v_i; 1) = (0, v_i)$ for all $v_i \in (0, 1]$ and $\mathcal{R}(0; k) = \{0\}$ for all k. This implies that for all μ and all $k = 1, 2, \ldots$, $\mu \in \Delta_i \cap \Delta(\mathcal{R}_{-i}(\cdot\,, k))$ if and only if $\mu \in \Delta(\mathcal{R}_{-i}(\cdot\,, k))$, that is, the condition $\forall b > 0$, $\int_{\mathcal{B}_{-i}} \Pr[\mathbf{b}_{-i} < b \mid v_i]\mu(d\mathbf{b}_{-i}) > 0$ is superfluous because it is implied by $\mu \in \Delta(\mathcal{R}_{-i}(\cdot\,, k))$.

Using Theorem 6(4), one can easily prove that for every $v_i > 0$ a bid b is weakly dominated for valuation/type v_i if and only if $b \notin (0, v_i)$.[18] We can conclude that the procedure given by Definition 10 with $\Delta_i = \Delta^+(\mathcal{B}_{-i})$ is equivalent to performing one round of elimination of all weakly dominated bids for each type, followed by the iterated elimination, for each type, of bids which are never best responses.[19] Therefore, in this case Δ-rationalizability captures the implications of the following assumptions:

(a) every bidder is rational,
(b) every bidder is *cautious* (i.e., she would never choose a weakly dominated bid),
(c) there is common certainty of (a) and (b).

5. Discussion

This section discusses the relationship between our findings and the experimental evidence on first-price auctions, and indicates a number of extensions. To fix ideas, we begin with an overview of our results.

Our key analytical tool is a characterization of the best responses for a type to beliefs satisfying certain restrictions. Specifically, if a (risk neutral) rational player i of type s_i believes that his opponents are not going to bid above a given, type-dependent least upper bound $\mathbf{B}$, then she can choose any bid in the interval $(0, \phi^{\mathbf{B}}(s_i))$; $\phi^{\mathbf{B}}$ is thus the new least upper bound derived from $\mathbf{B}$.

We then use this result to obtain an iterative characterization of interim rationalizable bids. The upper bound obtained in the first step of the algorithm is $\mathbf{B}(s_i, 1) = \mathrm{E}(\mathbf{v}_i \mid s_i)$, where $\mathrm{E}(\mathbf{v}_i \mid s_i)$ is the conditional valuation of the good for type s_i. In subsequent steps $k = 2, 3, \ldots,$ we apply our characterization result and derive an upper bound $\mathbf{B}(s_i; k) = \phi^{\mathbf{B}(\cdot\,; k-1)}(s_i)$ from the previous upper bound $\mathbf{B}(\cdot\,; k-1)$. (Of course, $\mathbf{B}(s_i; k) \leqslant \mathbf{B}(s_i; k-1)$.) The set of interim rationalizable bids for type s_i is an interval with interior $(0, \lim_{k\to\infty} \mathbf{B}(s_i; k))$. This provides a relatively simple implementation of interim rationalizability.

We show that the upper bound on interim rationalizable bids is nondecreasing and *strictly above the Nash equilibrium*. Furthermore, if signals are independent and valuation functions quasi-linear (e.g., in auctions with independent private values), this upper bound is *concave*, implying substantial proportional shading for high types.

Our analysis has implications for revenue comparisons in auctions with independent private values and risk-neutral bidders. We rely on a form of "cautiousness" that rules out weakly dominated bids. In *second*-price auctions, this implies that players will bid their valuation, so that rationalizable bids coincide with equilibrium bids. On the other hand, rationalizable expected revenues in a first-price auction can be lower or

[18] A bid b is *weakly dominated for valuation* v_i if there is another bid b' such that $\pi(b, v_i; \mathbf{b}_{-i}) \leqslant \pi(b', v_i; \mathbf{b}_{-i})$ for all $\mathbf{b}_{-i} \in \mathcal{B}_{-i}$ and the inequality is strict for at least one $\mathbf{b}_{-i}$.

[19] The "ex-ante version" of this procedure was first put forward and motivated by Dekel and Fudenberg (1990). Since then, several papers provided other epistemic characterizations. See Section 6 in the survey by Dekel and Gul (1997) and the references therein.

higher than equilibrium expected revenues. By the equilibrium revenue equivalence result, rationalizable expected revenues in a first-price auction can be lower or higher than in a second-price auction.

Perhaps more interestingly, rationalizable expected revenues in a first-price auction can be arbitrarily close to zero, and will always be lower than they would be if players were to bid their valuation. Observe that, with independent private values, as the number of bidders grows, the first-price equilibrium bid function approaches the identity function, implying that the same must hold for the upper bound on rationalizable bids. Thus, in rough, intuitive terms, in the limit, revenue equivalence obtains only in the "best scenario" (from the point of view of the seller) where players bid close to their upper bound.

5.1. Experiments and deviations from the risk-neutral Nash equilibrium

There are (at least) three "stylized facts" emerging from the experimental studies on first-price auctions with independent private values, which we find relevant in relation to our theoretical analysis:[20]

Overbidding. A large majority of subjects show a persistent tendency to bid above the risk-neutral Nash equilibrium (RNNE).

Decreasing proportional deviations. Deviations from RNNE are proportionally larger for subjects with smaller valuations; in other words, the ratio

$$\frac{|\text{actual bid-RNNE bid}|}{\text{valuation}}$$

is negatively correlated with subjects' private valuations.[21]

Heterogeneity. Bidding behavior is heterogeneous across subjects. [22]

In a series of papers, Cox, Smith, and Walker try to explain the data with a family of models featuring bidders with heterogeneous degrees of (constant relative) risk aversion. In such models, equilibrium bidding functions are linear (like the RNNE function) except for the largest valuations, but have heterogenous slopes and are steeper than the RNNE (e.g., Cox et al., 1988, 1992). The risk-aversion explanation of Overbidding is controversial. In particular, it leaves Decreasing Proportional Deviations largely unexplained. Furthermore, it is at odds with experimental findings concerning different auction settings.[23]

[20] See, for example, Kagel (1995) for a survey about experiments on auctions.

[21] See, e.g., Kagel and Roth (1992, p. 1381).

[22] For example, Cox et al. (1988) reject the null hypothesis of a common bidding function in symmetric IPV auctions.

[23] In *third-price* auctions, risk aversion implies bidding below the RNNE, whereas experiments show significant bidding above the RNNE (Kagel and Levin, 1993). Other partial explanations of Overbidding involve (i) psychological biases related to frame effects and the complexity of the decision problem (e.g., Kagel, 1995, Section I.B), and (ii) lack of experimental control on subjects incentives due to a small expected cost of deviations from the optimal bid (Harrison, 1989). Section I.G in Kagel (1995) provides a discussion of the debate about the risk-aversion explanation. Kagel and Roth (1992) presents Decreasing Proportional Deviations as one of the empirical findings at odds with the constant relative risk aversion model. A recent experimental paper by Goeree et al. (2000) provides support for the risk-aversion explanation.

We believe that our paper sheds light on a different explanation of these experimental findings: different subjects have different beliefs about the bidding behavior of their competitors, and the limited feedback they get from the outcomes of previous auctions prevents them from approaching the equilibrium sufficiently fast (e.g., Friedman, 1992). But even if subjects do not hold equilibrium beliefs, they may be sophisticated enough to take into account that their competitors' behavior satisfies some rationality restrictions and, possibly, that also their opponents' beliefs conform to analogous assumptions. Our paper identifies the least upper bound on bids of strategically sophisticated, risk-neutral bidders with heterogenous beliefs. Since the upper bound is above the (linear) RNNE and concave, Overbidding and Decreasing Proportional Deviations are qualitatively consistent with risk-neutrality and (a degree of) strategic sophistication.

We regard nonequilibrium (but strategically sophisticated) bidding as a complementary explanation of experimental findings, which can be integrated with risk-aversion. In (Battigalli and Siniscalchi, 2000) we show how risk-aversion can be incorporated into our analysis.

Experimental evidence suggests a number of extensions to our results. First, our analysis so far does not offer an explanation of the asymmetry in subjects' deviations from RNNE (i.e., the tendency to bid *above* the RNNE), nor does it explain why very small bids are so rare for subjects with intermediate or high valuations. Second, it may be argued that rational bidders should form their beliefs about the competitors and make plans before they are told their valuation, and therefore *ex-ante* rationalizability is a more appropriate solution concept in this context.[24] Third, in most experimental settings there is an (explicit or implicit) minimum bid increment. We discuss these issues in the following subsection.

5.2. Some extensions and related results

Unknown distribution of signals. We have derived our results under the assumption that the distribution of signals F is common knowledge. However, similar results can be obtained under more general assumptions. For the sake of simplicity, we discuss here the (symmetric) IPV case with two bidders.

Suppose that the true c.d.f. F is not known, but it is common knowledge that the valuation of a generic bidder is distributed according to some continuous density f bounded below by a strictly positive continuous function $g:[0,1] \to \mathbb{R}$ such that $\int_0^1 g(v)\,\mathrm{d}v < 1$. Suppose that bidder i believes that her competitor $-i$ will not bid above a given increasing upper bound $\mathbf{B} > 0$ (e.g., $\mathbf{B}(v) = v$). Adapting the proof of Theorem 6, we can show that the new least upper bound on bids derived from $\mathbf{B}$ is

$$\phi^{\mathbf{B}}(v) = v - \max_{0 \leqslant b \leqslant v} (v - b) G^*\bigl(\mathbf{B}^{-1}(b)\bigr),$$

[24] Which solution concept (interim or ex-ante) is more appropriate depends on our interpretation of the formal asymmetric-information model. If it represents a situation with genuine incomplete information without an ex-ante stage, then interim rationalizability is appropriate. If it represents a situation where the bidders really obtain information about the outcome of a chance move, then ex-ante rationalizability may be more appropriate. Experimental games fall in the second category.

where G^* is the "pessimistic" c.d.f. that assigns probability mass $1 - \int_0^1 g(w)\,\mathrm{d}w$ to the highest type $v_{-i} = 1$ and probability $\int_{v'}^{v''} g(w)\,\mathrm{d}w$ to any interval $[v', v'']$ with $v'' < 1$ (G^* can be obtained as the limit of a sequence of continuously differentiable c.d.f.'s F_n such that $F_n' = f_n \geqslant g$).

Our qualitative results on bounds continue to hold in this setting. Of course, the upper bounds we find will be *higher* than in the model where a density f (with $f(v) \geqslant g(v)$) is common knowledge: since more beliefs are allowed, more bids are rationalizable. For example, if $g(v) = c < 1$, then second-step upper bound is $\mathbf{B}(v; 2) = v - cv^2/4$ which is above the upper bound obtained with a common knowledge uniform distribution, i.e., $v - v^2/4$ (see Section 2).[25]

Lower bounds. Theorem 12 shows that imposing successively higher-order mutual belief in the assumption that players are rational (and that positive bids win with positive probability) yields a decreasing sequence of least upper bounds $\mathbf{B}(\cdot\,; k)$ on the set of best replies for every bidder type. However, arbitrarily small but positive bids are interim rationalizable.

In view of the experimental findings mentioned above, it may be interesting to *exogenously* specify a lower bound $\mathbf{L} \in \mathcal{B}_{-i}$ to players' bids, and investigate the consequences of the further assumption that (it is mutual belief that) players do not expect their opponents to bid below $\mathbf{L}$. A preliminary analysis may be found in Appendix 6.2.2 of (Battigalli and Siniscalchi, 2000). It is shown that the upper bounds obtained with this modified solution procedure are similar to those found here.

Ex-ante rationalizability. Our analysis of *interim* rationalizability provides an upper bound on *ex-ante* rationalizable bidding functions, but we are not able to show the upper bound is tight. However, some of our qualitative results also hold for ex-ante rationalizability. In particular, we can show that the analog of Proposition 11 continues to hold: the upper bound on rationalizable bids is strictly above the RNNE and every positive bid below this upper bound is rationalizable (see Battigalli and Siniscalchi, 2000, p. 24).

Discrete bids. In most experimental settings, bids are discrete because there is a (possibly only implicit) minimum increment δ (e.g., a cent), so that the set of bids is $\{0, \delta, 2\delta, \ldots, k\delta, (k+1)\delta, \ldots\}$. Our analysis of rationalizable bids provides an acceptable approximation if the number of players is not very large and δ is small. This appears to be the case in most experiments, as well as in many real-life situations.

Dekel and Wolinski (2000) analyze the opposite case (a large population of players and a nonnegligible minimum increment) in an IPV setting. They identify a nondecreasing lower bound to the set of rationalizable bids and show that, for any *fixed* minimum increment δ, as the number of bidders n gets large, this lower bound approaches the equilibrium bidding function. Thus, rationalizability and equilibrium roughly coincide in large IPV auctions with a nonnegligible minimum increment on bids.

[25] In an interdependent-values setting with unknown distribution of signals, Chung and Ely (2000) analyze the efficiency properties of a generalized Vickrey–Clarke–Groves mechanism using a notion of "iterated ex-post weak dominance."

Asymmetries. The approach presented in this paper may be extended to environments where players are not symmetric, i.e., Assumptions 1 and 2 are violated. This is carried out in (Battigalli and Siniscalchi, 2000). Theorem 6 and Proposition 8 are easily extended, but the upper bounds on k-rationalizable bids may be flat at the top.

Acknowledgments

We thank two anonymous referees for helpful comments, and Filippo Vergara Caffarelli for excellent research assistance.

Appendix A

A.1. Derivation and properties of bounds

We begin by introducing additional notation used in the proofs.

A.1.1. Notation

Sets of bidders and signals. Let $N = \{1, \dots, n\}$ denote the set of players. For any subset of players $J \subseteq N$, let $S_J = \prod_{j \in J} S_j$, and for simplicity define $S_{N \setminus \{i\}} = S_{-i}$. A generic element of S_J is denoted s_J. For any partition $\{K, L\}$ of the set of players J and any $s_K \in S_K$, $s_L \in S_L$, (s_K, s_L) is the element of S_J obtained from s_K and s_L.

For any nonempty subset $J \subseteq N$ of players, denote by F_J the marginal of c.d.f. F on S_J, and by F_{-J} its marginal on $S_{N \setminus J}$; finally, given a subset K of players such that $K \cap J = \emptyset$, denote by $F_{K|J}$ the conditional c.d.f on S_K given the signals of players $j \in J$.

Private values. Recall that, by Assumption 4, the valuation function $\mathbf{v}$ is continuous and increasing in its first argument. Thus, if the game has private values, it may be assumed without loss of generality that $\mathbf{v}(s_i, s_{-i}) = s_i$ for all s_i and s_{-i}.

Random vectors and events. We denote m-dimensional random vectors, defined as (Borel measurable) functions with domain S_J and range $\mathbb{R}^m$, with boldface letters. In particular, for any set of players J and random variables $\mathbf{b}_j$, $j \in J$, we let $\mathbf{b}_J$ denote the joint function (random vector) defined by $\mathbf{b}_J(s_J) = \{\mathbf{b}_j(s_j)\}_{j \in J}$. For any set of players J and tuple of random variables $\mathbf{b}_J$, we denote by $\mathbf{b}_J^{\max}$ the scalar-valued random variable defined by the map $s_J \mapsto \max_{j \in J} \mathbf{b}_j(s_j)$.

Events related to the signals of players in set J are represented by means of square brackets with an index J specifying that we refer to a subset of S_J. For example, let $K \subseteq J$; then $[\mathbf{b}_K < b]_J \subseteq S_J$ is the set of vectors of signals s_J such that $\mathbf{b}_j(s_j) < b$ for all $j \in K$. When $J = N \setminus \{i\}$ we suppress the index, as S_{-i} is the basic space of uncertainty from the point of view of bidder i. Thus, for example, $[\forall j \in K, \mathbf{b}_j < b_j] \subseteq S_{-i}$.

Expected payoff. Taking the possibility of ties into account, the expected payoff of bidding b given signal s_i and belief $\mu \in \Delta(\mathcal{B}_{-i})$ is

$$\pi(b, s_i; \mu) = \int_{\mathcal{B}_{-i}} \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{b}_{-i} < b] \Pr[\mathbf{b}_{-i} < b \mid s_i] \mu(\mathrm{d}\mathbf{b}_{-i})$$
$$+ \sum_{\emptyset \neq J \subseteq N \setminus \{i\}} \frac{1}{1 + |J|} \int_{\mathcal{B}_{-i}} \mathrm{E}\big[\mathbf{v}_i - b \,\big|\, s_i, T(J, b; \mathbf{b}_{-i})\big] \Pr\big[T(J, b, \mathbf{b}_{-i}) \,\big|\, s_i\big] \mu(\mathrm{d}\mathbf{b}_{-i}), \tag{A.1}$$

where $T(J, b; \mathbf{b}_{-i}) = [\mathbf{b}_{-J \cup \{i\}} < b] \cap [\forall j \in J, \mathbf{b}_j = b]$.

It is convenient to define a modified version of the expected payoff function for a bidder with nondecreasing conjectures, which may assign positive probability to ties. For any tuple $\{\mathbf{B}_j\}_{j\neq i}$ of (nondecreasing) real-valued functions on $[0,1]$, define

$$\bar{\pi}(b,s_i;\mathbf{B}_{-i}) = \big(\mathrm{E}[\mathbf{v}_i \mid s_i,\mathbf{B}_{-i}<b]-b\big)\Pr[\mathbf{B}_{-i}<b\mid s_i] + \max\big\{0,\big(\mathrm{E}\big[\mathbf{v}_i \bigm| s_i,\mathbf{B}_{-i}^{\max}=b\big]-b\big)\Pr\big[\mathbf{B}_{-i}^{\max}=b \bigm| s_i\big]\big\}. \tag{A.2}$$

To avoid repeating tedious qualifications, for every random variable $\mathbf{h}$ and event F, if $\Pr[F]=0$, we assume that $\mathrm{E}[\mathbf{h}\mid F]$ is such that $\mathrm{E}[\mathbf{h}\mid F]\Pr[F]=0$.

If the game has private values, the above function takes up a particularly simple form. We define modified payoffs for arbitrary beliefs $\mu\in\Delta(\mathcal{B}_{-i})$.

$$\bar{\pi}(b,s_i;\mu) = \begin{cases} (s_i-b)\displaystyle\int_{\mathcal{B}_{-i}}\Pr[\mathbf{B}_i\leqslant b\mid s_i]\mu(\mathrm{d}\mathbf{b}_{-i}), & b\in[0,s_i],\\ (s_i-b)\displaystyle\int_{\mathcal{B}_{-i}}\Pr[\mathbf{B}_i< b\mid s_i]\mu(\mathrm{d}\mathbf{b}_{-i}), & b>s_i. \end{cases} \tag{A.3}$$

A.1.2. Preliminary results

Lemma 14. *Consider a nondecreasing bid function* $\mathbf{B}$ *and let* $\mathbf{B}_{-i}=(\mathbf{B},\mathbf{B},\ldots)$. *For every signal* $s_i\in[0,1]$, *the function* $\bar{\pi}(\cdot,s_i;\mathbf{B}_{-i}):\mathbb{R}_+\to\mathbb{R}$ *is upper semicontinuous. Therefore,* $\sup_{b\geqslant 0}\bar{\pi}(b,s_i;\mathbf{B}_{-i})$ *is attained. Finally, if* $\mathbf{B}$ *is increasing (i.e.,* $\mathbf{B}_{-i}\in\mathcal{M}_{-i}$*), then* $\bar{\pi}(\cdot,s_i;\mathbf{B}_{-i})=\pi(\cdot,s_i;\mathbf{B}_{-i})$*, and both functions are continuous.*

Proof. Fix $b\geqslant 0$ and consider a sequence $b^k\to b$ such that $\lim_{k\to\infty}\bar{\pi}(b^k,s_i;\mathbf{B}_{-i})$ exists; denote the latter by L. We must show that $L\leqslant\bar{\pi}(b,s_i;\mathbf{B}_{-i})$. Observe that, since $\sum_{k\geqslant 1}\Pr[\mathbf{B}_{-i}^{\max}=b^k\mid s_i]\leqslant 1$, it must be the case that $\Pr[\mathbf{B}_{-i}^{\max}=b^k\mid s_i]\to 0$.

Assume first that $b^k\uparrow b$. Note that the indicator functions of the events $[\mathbf{B}_{-i}<b^k]$ converge pointwise to the indicator function of $[\mathbf{B}_{-i}<b]$. Therefore, since $\mathbf{v}_i$ is bounded and so is the convergent sequence $\{b^k\}$, by Dominated Convergence (cf. Aliprantis and Border, 1994, p. 323) and the observation that the probability of ties vanishes as $k\to\infty$ we conclude that $L=(\mathrm{E}[\mathbf{v}_i\mid s_i,\mathbf{B}_{-i}<b]-b)\Pr[\mathbf{B}_{-i}<b\mid s_i]$, so $L\leqslant\bar{\pi}(b,s_i;\mathbf{B}_{-i})$. Observe that the inequality can only be strict if $\Pr[\mathbf{B}_{-i}^{\max}=b\mid s_i]>0$.

Assume next that $b^k\downarrow b$. Now the indicator functions of the events $[\mathbf{B}_{-i}<b^k]$ converge pointwise to the indicator function of $[\mathbf{B}_{-i}\leqslant b]$, so $L=(\mathrm{E}[\mathbf{v}_i\mid s_i,\mathbf{B}_{-i}\leqslant b]-b)\Pr[\mathbf{B}_{-i}\leqslant b\mid s_i]\leqslant\bar{\pi}(b,s_i;\mathbf{B}_{-i})$. Again, the inequality can only be strict if $\Pr[\mathbf{B}_{-i}^{\max}=b\mid s_i]>0$.

To complete the proof, consider an arbitrary convergent sequence $b^k\to b$ and an arbitrary subsequence $\{b^{k_n}\}$ such that $\lim_{n\to\infty}\bar{\pi}(b^{k_n},s_i;\mathbf{B}_{-i})\equiv L$ exists. If the subsequence is itself monotonic, then $L\leqslant\bar{\pi}(b,s_i;\mathbf{B}_{-i})$ by the above arguments. Otherwise, it must contain a monotonic sub-subsequence $\{b^{k_{n_m}}\}$ such that $b^{k_{n_m}}\to b$ and $\bar{\pi}(b,s_i;\mathbf{B}_{-i})\geqslant\lim_{m\to\infty}\bar{\pi}(b^{k_{n_m}},s_i;\mathbf{B}_{-i})=L$.

Finally, to prove the last claim note that if $\mathbf{B}_{-i}\in\mathcal{M}_{-i}$, then $\Pr[\mathbf{B}_{-i}^{\max}=b\mid s_i]=0$ for all b. □

In a private-values setting, the relationship between the payoff function π and the modified payoff function $\bar{\pi}$ is even closer.

Lemma 15. *Assume the game has private values. Then, for every signal* s_i*, bid* $b\in[0,s_i]$ *and belief* $\mu\in\Delta(\mathcal{B}_{-i})$, $\bar{\pi}(b,s_i;\mu)\geqslant\pi(b,s_i;\mu)$. *Moreover,* $\pi^*(s_i;\mu)=\max_{b\geqslant 0}\bar{\pi}(b,s_i;\mu)$.

Proof. The first claim is obvious upon inspecting Eqs. (A.1) and (A.3). This implies that $\pi^*(s_i;\mu)\leqslant\max_{b\geqslant 0}\bar{\pi}(b,s_i;\mu)$. Now choose $b^*\in\arg\max_{b\geqslant 0}\bar{\pi}(b,s_i;\mu)$; notice that $b^*\in[0,s_i]$ for at least one maximizer b^*. Since there can be at most countably many bids b such that $\int_{\mathcal{B}_{-i}}\Pr[\mathbf{b}_{-i}^{\max}=b\mid s_i]\mu(\mathrm{d}\mathbf{b}_{-i})>0$, there exists a sequence $b^k\downarrow b^*$ such that $\int_{\mathcal{B}_{-i}}\Pr[\mathbf{b}_{-i}^{\max}=b^k\mid s_i]\mu(\mathrm{d}\mathbf{b}_{-i})=0$ for all k. For each $\mathbf{b}_{-i}\in\mathcal{B}_{-i}$, $\bigcap_{k\geqslant 1}[\mathbf{b}_{-i}\leqslant b^k]=[\mathbf{b}_{-i}\leqslant b^*]$; thus, by continuity of the measure $\Pr[\cdot\mid s_i]$, $\Pr[\mathbf{B}_{-i}\leqslant b^k\mid s_i]\to\Pr[\mathbf{B}_{-i}\leqslant b\mid s_i]$ for all $k\geqslant 1$. Hence, by Dominated Convergence, $\int_{\mathcal{B}_{-i}}\Pr[\mathbf{b}_{-i}\leqslant b^k\mid s_i]\mu(\mathrm{d}\mathbf{b}_{-i})\to$

$\int_{\mathcal{B}_{-i}} \Pr[\mathbf{b}_{-i} \leqslant b^* \mid s_i]\mu(\mathrm{d}\mathbf{b}_{-i})$, and therefore $\pi(b^k, s_i; \mu) \to \bar{\pi}(b^*, s_i; \mu)$. Since $\pi^*(s_i; \mu) \geqslant \pi(b^k, s_i; \mu)$ for all k, $\pi^*(s_i; \mu) \geqslant \max_{b\geqslant 0} \bar{\pi}(b, s_i; \mu)$. This establishes the second claim. □

We state another preliminary lemma, which verifies that the conjecture $\mathbf{B}_{-i}$ is "more pessimistic" than any belief $\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$.

Lemma 16. *Consider a nondecreasing bid function* $\mathbf{B}$ *and let* $\mathbf{B}_{-i} = (\mathbf{B}, \mathbf{B}, \ldots)$. *For every signal* $s_i \in [0, 1]$ *and bid* $b \geqslant 0$:

(1) *if* $\bar{\pi}(b, s_i; \mathbf{B}_{-i}) > 0$ *and* $\Pr[\mathbf{B}^{\max}_{-i} = b \mid s_i] = 0$, *then* $\pi(b, s_i; \mu) \geqslant \bar{\pi}(b, s_i; \mathbf{B}_{-i})$ *for* $\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$;
(1-PV) *if the game has private values and* $b \in [0, s_i]$, *then* $\bar{\pi}(b, s_i; \mu) \geqslant \bar{\pi}(b, s_i; \mathbf{B}_{-i})$ *for any* $\mu \in \Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})$;
(2) *if* $\bar{\pi}(b, s_i; \mathbf{B}_{-i}) \geqslant 0$, $b > 0$ *and* $\Pr[\mathbf{B}^{\max}_{-i} = b] > 0$, *then* $\mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}^{\max}_{-i} = b] \geqslant 0$.

Therefore:

(3) $\inf_{\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})} \pi^*(s_i; \mu) \geqslant \max_{b\geqslant 0} \bar{\pi}(b, s_i; \mathbf{B}_{-i})$;
(3-PV) *if the game has private values, then* $\inf_{\mu \in \Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})} \pi^*(s_i; \mu) \geqslant \max_{b\geqslant 0} \bar{\pi}(b, s_i; \mathbf{B}_{-i})$.

Proof. The claims pertaining to the private-values case are simple to establish, and illustrate the basic ideas.

(1-PV) For any bid b and conjecture $\mathbf{b}_{-i} \in \mathcal{B}_{-i}$ such that $\mathbf{b}_{-i} < \mathbf{B}_{-i}$, $\Pr[\mathbf{b}_{-i} \leqslant b \mid s_i] \geqslant \Pr[\mathbf{B}_{-i} \leqslant b \mid s_i]$. Therefore, for any bid $b \in [0, s_i]$,

$$\bar{\pi}(b, s_i; \mu) = (s_i - b) \int_{\mathcal{B}_{-i}} \Pr[\mathbf{b}_{-i} \leqslant b \mid s_i]\mu(\mathrm{d}\mathbf{b}_{-i}) \geqslant (s_i - b)\Pr[\mathbf{B}_{-i} \leqslant b \mid s_i] = \bar{\pi}(b, s_i; \mathbf{B}_{-i}).$$

(3-PV) Clearly, for any belief $\mu \in \Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})$, no bid $b > s_i$ can maximize $\bar{\pi}(\cdot, s_i; \mu)$. Thus, (1-PV) implies that, for any such belief, $\max_{b\geqslant 0} \bar{\pi}(b, s_i; \mu) \geqslant \max_{b\geqslant 0} \bar{\pi}(b, s_i; \mathbf{B}_{-i})$. Hence, by Lemma 15,

$$\inf_{\mu \in \Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})} \pi^*(s_i; \mu) = \inf_{\mu \in \Delta^+(\mathcal{B}_{-i}; \mathbf{B}_{-i})} \max_{b\geqslant 0} \bar{\pi}(b, s_i; \mu) \geqslant \max_{b\geqslant 0} \bar{\pi}(b, s_i; \mathbf{B}_{-i}).$$

Under our monotonicity and affiliation assumptions, the same basic ideas generalize to the interdependent-values setting. We first illustrate the argument by proving claim (1) for the case of a single competitor $-i = j$ and continuous, increasing conjectures $\mathbf{B}_j, \mathbf{b}_j$; thus, there are no positive-probability ties: $\bar{\pi}(\cdot, s_i, \mathbf{b}_j) = \pi(\cdot, s_i, \mathbf{b}_j)$, and similarly for $\mathbf{B}_j$. Note that

$$\pi(b, s_i; \mathbf{b}_j) = \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B} < b]\Pr[\mathbf{B} < b \mid s_i] + \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{b}_j < b \leqslant \mathbf{B}] \cdot \Pr[\mathbf{b}_j < b \leqslant \mathbf{B} \mid s_i].$$

Since $\Pr[\mathbf{B} = b \mid s_i] = 0$, the first term in the r.h.s. equals $\pi(b, s_i; \mathbf{B})$. If $\Pr[\mathbf{b}_j < b \leqslant \mathbf{B} \mid s_i] = 0$ we are done. If $\Pr[\mathbf{b}_j < b \leqslant \mathbf{B} \mid s_i] > 0$, we must show that $\mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{b}_j < b \leqslant \mathbf{B}] \geqslant 0$. Since $\pi(b, s_i; \mathbf{B}_{-i}) > 0$, $b > \mathbf{B}^{-1}(0)$. Therefore $[\mathbf{B} < b]_j = [0, \mathbf{B}^{-1}(b))$ and $[\mathbf{b}_j < b \leqslant \mathbf{B}]_j = [\mathbf{B}^{-1}(b), \mathbf{b}_j^{-1}(b))$ (let $\mathbf{b}_j^{-1}(b) = 1$ if $b > \mathbf{b}_j(1)$). Hence, by Result 2, $\mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{b}_j < b \leqslant \mathbf{B}] \geqslant \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B} < b] > 0$, where the latter inequality follows from $\pi(b, s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B} < b]\Pr[\mathbf{B} < b \mid s_i] > 0$. This establishes the claim in the simple case.

(1) It is enough to prove the claim for a belief μ concentrated on a single profile $\mathbf{b}_{-i} \in \mathcal{M}_{-i}$ such that $\mathbf{b}_{-i} < \mathbf{B}_{-i}$. Note that

$$\begin{aligned}\pi(b, s_i; \mathbf{b}_{-i}) = {} & \mathrm{E}[\mathbf{v}_i - b \mid s_i,\ \mathbf{B}_{-i} < b]\Pr[\mathbf{B}_{-i} < b \mid s_i] \\ & + \sum_{\emptyset \neq J \subseteq N \setminus \{i\}} \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{N\setminus(J\cup i)} < b,\ \mathbf{b}_J < b \leqslant \mathbf{B}_J] \\ & \qquad \times \Pr[\mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{b}_J < b \leqslant \mathbf{B}_J \mid s_i].\end{aligned}$$

Since $\Pr[\mathbf{B}^{\max}_{-i} = b \mid s_i] = 0$, the first term in the r.h.s. equals $\bar{\pi}(b, s_i; \mathbf{B}_{-i})$. If $\Pr[\mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{b}_J < b \leqslant \mathbf{B}_J \mid s_i] = 0$ for all nonempty $J \subseteq N \setminus \{i\}$, the proof of the claim is complete. Otherwise, for any nonempty $J \subseteq N \setminus \{i\}$ such that $\Pr[\mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{b}_J < b \leqslant \mathbf{B}_J \mid s_i] > 0$, we must show that $\mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{b}_J < b \leqslant$

$\mathbf{B}_J] \geqslant 0$. Fix one such J and note that, for every $j \neq i$, the greatest lower bound of the set $[\mathbf{B}_j < b]_j$ is 0 (recall that $\bar{\pi}(b, s_i; \mathbf{B}_{-i}) > 0$), and its least upper bound is $\sup\{s_j\colon \mathbf{B}_j(s_j) < b\}$; also, the g.l.b of $[\mathbf{b}_j < b \leqslant \mathbf{B}_j]_j$ is $\inf\{s_j\colon \mathbf{B}_j(s_j) \geqslant b\} \geqslant 0$ and its l.u.b. is $\sup\{s_j\colon \mathbf{b}_j(s_j) < b\} \geqslant \sup\{s_j\colon \mathbf{B}_j(s_j) < b\}$. [26]

Hence, by Remark 2, $\mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{b}_J < b \leqslant \mathbf{B}_J] \geqslant \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{-i} < b]$; since $\bar{\pi}(b, s_i; \mathbf{B}_{-i}) > 0$ and $\Pr[\mathbf{B}_{-i}^{\max} = b \mid s_i] = 0$ by assumption, we must have $\Pr[\mathbf{B}_{-i} < b \mid s_i] > 0$ and $\mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{-i} < b] > 0$, which establishes the claim.

(2) If $\mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{-i} < b] < 0$ the claim follows immediately because $\bar{\pi}(b, s_i; \mathbf{B}_{-i}) \geqslant 0$. If instead $\mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{-i} < b] \geqslant 0$, note that

$$\begin{aligned} &\mathrm{E}\bigl[\mathbf{v}_i - b \bigm| s_i,\ \mathbf{B}_{-i}^{\max} = b\bigr] \\ &\quad = \sum_{\emptyset \neq J \subseteq N\setminus\{i\}} \mathrm{E}[\mathbf{v}_i - b \mid s_i,\ \mathbf{B}_{N\setminus(J\cup i)} < b,\ \mathbf{B}_J = b] \Pr\bigl[\mathbf{B}_{N\setminus(J\cup i)} < b,\ \mathbf{B}_J = b \bigm| s_i, \mathbf{B}_{-i}^{\max} = b\bigr]. \end{aligned}$$

As above, Remark 2 implies that, for any nonempty $J \subseteq N \setminus \{i\}$ such that $\Pr[\mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{B}_J = b \mid s_i, \mathbf{B}_{-i}^{\max} = b] > 0$, $\mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{N\setminus(J\cup i)} < b, \mathbf{B}_J = b] \geqslant \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{-i} < b] \geqslant 0$, which proves the claim.[27]

(3) Choose $b^* \in \arg\max_{b\geqslant 0} \bar{\pi}(b, s_i; \mathbf{B}_{-i})$. The required inequality holds trivially if $\bar{\pi}(b^*, s_i; \mathbf{B}_{-i}) = 0$, so assume $\bar{\pi}(b^*, s_i; \mathbf{B}_{-i}) > 0$. In particular, $\mathbf{B}_{-i} > 0$ implies that $\Pr[\mathbf{B}_{-i}^{\max} = 0 \mid s_i] = 0$, so we can assume that $b^* > 0$. Choose $\mu \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$. Assume that $\Pr[\mathbf{B}_{-i}^{\max} = b^* \mid s_i] = 0$; then, by claim (1), $\pi^*(s_i; \mu) \geqslant \pi(b^*, s_i; \mu) \geqslant \bar{\pi}(b^*, s_i; \mathbf{B}_{-i})$, and we are done. If instead $\Pr[\mathbf{B}_{-i}^{\max} = b^* \mid s_i] > 0$, consider a sequence $b^k \downarrow b^*$ such that $\Pr[\mathbf{B}_{-i}^{\max} = b^k \mid s_i] = 0$ for all k: this is possible because there can be at most countably many positive-probability ties. Since, for any $\mathbf{b}_{-i} \in \mathcal{M}_{-i}$, $\pi(\cdot, s_i; \mathbf{b}_{-i})$ is continuous, $\pi(b^*, s_i; \mathbf{b}_{-i}) = \lim_{k\to\infty} \pi(b^k, s_i; \mathbf{b}_{-i}) \geqslant \lim_{k\to\infty} \bar{\pi}(b^k, s_i; \mathbf{B}_{-i}) = \bar{\pi}_i(b^*, s_i; \mathbf{B}_{-i})$, where the inequality follows from claim (1), and the second equality follows from claim (2). Integrating with respect to μ, this implies that $\pi^*(s_i; \mu) \geqslant \pi(b^*, s_i; \mu) \geqslant \bar{\pi}(b^*, s_i; \mathbf{B}_{-i})$. □

We next develop the machinery required to approximate $\mathbf{B}_{-i}$ with beliefs belonging to $\Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$. Define an increasing and continuous map $\sigma : [0,1] \times (0,1] \to [0,1]$ by

$$\forall (x,\alpha) \in [0,1] \times (0,1], \quad \sigma(x,\alpha) = \begin{cases} \dfrac{1-\alpha}{\alpha} x & x \in [0,\alpha], \\ \left(1 - \dfrac{\alpha}{1-\alpha}\right) + \dfrac{\alpha}{1-\alpha} x & x \in (\alpha, 1], \end{cases} \tag{A.4}$$

That is, for every $\alpha \in (0,1]$, the graph of $\sigma(\cdot, \alpha)$ is the piecewise linear function joining the origin with the point $(\alpha, 1-\alpha)$, and the latter with the point $(1,1)$. Each function $\sigma(\cdot, \alpha)$ is continuous and differentiable everywhere except at $x = \alpha$. Its inverse $\tau : [0,1] \times (0,1] \to [0,1]$ is given by

$$\forall (y,\alpha) \in [0,1] \times (0,1], \quad \tau(y,\alpha) = \begin{cases} \dfrac{\alpha}{1-\alpha} y & y \in [0, 1-\alpha], \\ \left(1 - \dfrac{1-\alpha}{\alpha}\right) + \dfrac{1-\alpha}{\alpha} y & y \in (1-\alpha, 1]. \end{cases} \tag{A.5}$$

i.e., the piecewise linear function joining the origin with the point $(1-\alpha, \alpha)$, and the latter with the point $(1,1)$. Each function τ is continuous and differentiable everywhere except at $y = 1-\alpha$.

Note that, as $\alpha \downarrow 0$, $\sigma(\cdot, \alpha)$ converges pointwise on $(0,1]$ to the constant function 1; for notational convenience, we let $\sigma(x, 0) = 1$ for all $x \in [0,1]$. Similarly, τ converges pointwise to the constant function $0 \equiv \tau(y, 0)$.

Now, for any nonnegative real number $\alpha \geqslant 0$ and bid function $\mathbf{b} : [0,1] \to \mathbb{R}_+$, define the function $\mathbf{b}^{(\alpha)} : [0,1] \to \mathbb{R}_+$ by

$$\forall s_j \in [0,1], \quad \mathbf{b}^{(\alpha)}(s_j) = \sigma(s_j, \alpha)\mathbf{b}(s_j). \tag{A.6}$$

[26] The above, detailed argument will be omitted henceforth.

[27] The claim may be false for $b = 0$, if $\Pr[\mathbf{B}_{-i}^{\max} = 0 \mid s_i] > 0$.

In conjunction with Lemma 16, the following lemma implies that $\sup_{b\geqslant 0}\bar{\pi}(b, s_i; \mathbf{B}_{-i})$ is the least upper bound bidder i may obtain by best-responding to beliefs in $\Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$.

Lemma 17. *Let $\mathbf{B}$ be a nondecreasing bid function, and define $\mathbf{B}_{-i} = (\mathbf{B}, \mathbf{B}, \ldots)$. Fix arbitrarily a signal $s_i \in [0, 1]$.*

(1) *For all $\alpha > 0$, $\mathbf{B}_{-i}^{(\alpha)} \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$.*

Moreover, for every sequence $\alpha_k \downarrow 0$:

(2) *$\mathbf{B}^{(\alpha_k)} \to \mathbf{B}$ a.s. pointwise;*
(3) *for every $b \geqslant 0$ and every sequence $b^k \to b$, $\lim_{k\to\infty} \pi(b^k, s_i; \mathbf{B}_{-i}^{(\alpha_k)}) \leqslant \bar{\pi}(b, s_i; \mathbf{B}_{-i})$;*
(4) *$\pi^*(s_i; \mathbf{B}_{-i}^{(\alpha_k)}) \to \max_{b\geqslant 0} \bar{\pi}(b, s_i; \mathbf{B}_{-i})$.*

Proof. (1) Denote an arbitrary opponent of player i by j throughout this proof. Note that, for any $\alpha > 0$, $\mathbf{B}^{(\alpha)} < \mathbf{B}$ (with equality for $s_j = 1$). Moreover, $s_j' > s_j$ implies $\sigma(s_j, \alpha)\mathbf{B}(s_j) < \sigma(s_j', \alpha)\mathbf{B}(s_j) \leqslant \sigma(s_j', \alpha)\mathbf{B}(s_j')$, so $\mathbf{B}^{(\alpha)}$ is increasing. Finally, for any $b > 0$ there exists $s_j > 0$ such that $\sigma(s_j, \alpha)\mathbf{B}(1) < b$, and thus $\mathbf{B}^{(\alpha)}(s_j) < b$. Therefore, $\Pr[\mathbf{B}_{-i}^{(\alpha)} < b \mid s_i] > 0$, and the first assertion of the lemma is proved.

Now fix $b \geqslant 0$ and consider sequences $\alpha_k \downarrow 0$ and $b^k \to b$.

(2) Pointwise convergence on $(0, 1]$ follows from the properties of the function σ.

(3) We begin by computing the a.s. pointwise limit of the sequence of indicator functions corresponding to the events $[\mathbf{B}_{-i}^{(\alpha_k)} \leqslant b]$.

Choose any $s_j > 0$. If $\mathbf{B}(s_j) < b$, then for k large $\mathbf{B}^{(\alpha_k)}(s_j) < b^k$; similarly, if $\mathbf{B}(s_j) > b$, then for k large $\mathbf{B}^{(\alpha_k)}(s_i) > b^k$. Also note that, if $\mathbf{B}(0) > b$, then, for every $\epsilon > 0$, there exists K such that $k \geqslant K$ implies $\mathbf{B}^{(\alpha_k)}(\epsilon) > b^k$, so $\bigcap_{k\geqslant 1}[\mathbf{B}^{(\alpha_k)} \leqslant b^k]_j = \{0\}$.

Finally, suppose $\mathbf{B}(s_j) = b$. Since $\mathbf{B}^{(\alpha_k)}$ is increasing for every k, it follows that, for every k, the set $L(k) = \{s_j \in [0, 1]\colon \mathbf{B}(s_j) = b, \mathbf{B}^{(\alpha_k)}(s_j) > b^k\}$ satisfies

$$s_j \in L(k), \quad \mathbf{B}(s_j') = b, \quad s_j' > s_j \quad \Rightarrow \quad s_j' \in L(k).$$

Now define $\ell(k) = \min(1, \inf L(k))$ (where $\inf\emptyset = \infty$). Thus, if $\mathbf{B}(s_j) = b$, then $s_j > \ell(k)$ implies $\mathbf{B}^{(\alpha_k)}(s_j) > b^k$ (otherwise s_j would be a greater lower bound to $L(k)$ than $\ell(k)$), and $s_j < \ell(k)$ implies $\mathbf{B}^{(\alpha_k)}(s_j) \leqslant b^k$ (either because $L(k) = \emptyset$, or because otherwise there would be some other $s_j' \in (s_j, \ell(k))$ such that $\mathbf{B}(s_j') = b$ and $\mathbf{B}^{(\alpha_k)}(s_j') > b^k$).

Next, w.l.o.g. assume that $\ell(k) \to \ell$. Then the indicator functions of the events $[\mathbf{B}_{-i}^{(\alpha_k)} \leqslant b^k]$ converge a.s. pointwise to the indicator function of the event

$$[\mathbf{B}_{-i} < b] \cup \left(\left[\mathbf{B}_{-i}^{\max} = b\right] \cap [\mathbf{s}_{-i} \leqslant \ell_{-i}]\right) \equiv [\mathbf{B}_{-i} < b] \cup T_{-i}(b).$$

By the Dominated Convergence theorem, this yields

$$\pi\left(b^k, s_i; \mathbf{B}_{-i}^{(\alpha_k)}\right) \to \int\limits_{[\mathbf{B}_{-i} < b]} (\mathbf{v}_i - b) F_{-i|i}(\mathrm{d}s_{-i} \mid s_i) + \int\limits_{T_{-i}(b)} (\mathbf{v}_i - b) F_{-i|i}(\mathrm{d}s_{-i} \mid s_i).$$

If $\Pr[T_{-i}(b) \mid s_i] = 0$, then $\lim_{k\to\infty} \pi_i(b^k, s_i; \mathbf{B}_{-i}^{(\alpha_k)}) \leqslant \bar{\pi}(b, s_i; \mathbf{B}_{-i})$ follows immediately. Otherwise, by Remark 2, $\mathrm{E}[\mathbf{v}_i - b \mid s_i, T_{-i}(b)] \leqslant \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{-i}^{\max} = b] \leqslant \max(0, \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{B}_{-i}^{\max} = b])$, and claim (3) follows.

(4) Consider a sequence $\alpha_k \downarrow 0$ and, for every k, choose $b^k \in \arg\max_{b\geqslant 0} \pi(b, s_i; \mathbf{B}_{-i}^{(\alpha_k)})$; note that the maximum is achieved because $\pi(\cdot, s_i; \mathbf{B}_{-i}^{(\alpha_k)})$ is continuous. Assume w.l.o.g. that the sequence of maximizers converges, and let $b^* = \lim_{k\to\infty} b^k$. Claim (3) implies that $\lim_{k\to\infty} \pi^*(s_i; \mathbf{B}_{-i}^{(\alpha_k)}) = \lim_{k\to\infty} \pi(b^k, s_i; \mathbf{B}_{-i}^{(\alpha_k)}) \leqslant \bar{\pi}(b^*, s_i; \mathbf{B}_{-i}) \leqslant \max_{b\geqslant 0} \bar{\pi}(b, s_i; \mathbf{B}_{-i})$. Since the reverse inequality follows from Lemma 16(3), the proof is complete. $\square$

A.1.3. Theorem 6

We are finally able to prove the main result of Section 3, Theorem 6. The key step is the proof of claim (2). We sketch the main argument here (see also the discussion in Section 2).

For any bid $b^* < \phi^{\mathbf{B}_{-i}}(s_i)$, the justifying belief $\mathbf{g}_{-i}^{(\alpha)}$ has the qualitative features illustrated in Fig. 1. Specifically, $\mathbf{g}^{(\alpha)}$ is increasing and lies below $\mathbf{B}$; moreover, it approximates the upper bound $\mathbf{B}$ up to the point $\bar{s}$ where the latter crosses the bid b^*, and approximates b^* thereafter.

To verify the optimality of b^* given $\mathbf{g}_{-i}^{(\alpha)}$, we proceed in two steps. First, we argue that bidder i's payoff function $\pi(b, s_i; \mathbf{g}_{-i}^{(\alpha)})$ is pointwise dominated by the "two-bidder, private-values" objective function $(\mathrm{E}[\mathbf{v}_i \mid s_i] - b)\Pr[\mathbf{g}_{-i}^{(\alpha)} \leqslant b \mid s_i]$. Moreover, the two functions share the same value $\mathrm{E}[\mathbf{v}_i \mid s_i] - b^*$ for $b = b^*$. We then prove that $b = b^*$ maximizes this auxiliary objective function among all bids b chosen by at least one opponent j with type $s_j \geqslant \bar{s}$—that is, a type for which $\mathbf{g}^{(\alpha)}(s_j)$ approximates b^*.

The second and concluding step entails verifying that, for all remaining bids b, bidder i's payoff given $\mathbf{g}_{-i}^{(\alpha)}$ is close to her payoff given $\mathbf{B}_{-i}$, her "pessimistic" conjecture. By the definition of $\phi^{\mathbf{B}_{-i}}(s_i)$, this implies that no such bids can be profitable deviations from b^*.

Proof of Theorem 6. Note first that Lemmata 16 and 17 imply that part (3) of the theorem is true. Moreover, if the game has private values, we have

$$\max_{b\geqslant 0}\bar{\pi}(b, s_i; \mathbf{B}_{-i}) = \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})}\pi^*(s_i;\mu) \geqslant \inf_{\mu\in\Delta^+(\mathcal{B}_{-i};\mathbf{B}_{-i})}\pi^*(s_i;\mu) \geqslant \max_{b\geqslant 0}\bar{\pi}(b, s_i; \mathbf{B}_{-i}),$$

the first equality appears in part (3) of the Theorem, the first inequality reflects the fact that $\mathcal{M}_{-i} \subset \mathcal{B}_{-i}$, and the second inequality is claim (3-PV) in Lemma 16. Also, Lemma 15 states that $\max_{b\geqslant 0}\bar{\pi}(b, s_i; \mathbf{B}_{-i}) = \pi^*(s_i; \mathbf{B}_{-i})$. Thus, (4) holds.

To see that (1) holds, observe that, by Remark 3, for any $\mathbf{b}_{-i} \in \mathcal{M}_{-i}$, $\mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{b}_{-i} < b^*] \leqslant \mathrm{E}[\mathbf{v}_i \mid s_i]$. Hence, if $b^* > \mathrm{E}[\mathbf{v}_i \mid s_i] - \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})}\pi^*(s_i;\mu)$ and $b^* \leqslant \mathrm{E}[\mathbf{v}_i \mid s_i]$, then, for any $\mu^* \in \Delta^+(\mathcal{M}_{-i}; \mathbf{B}_{-i})$,

$$\begin{aligned}\pi(b^*, s_i; \mu^*) &= \int\limits_{\mathcal{M}_{-i}} \big(\mathrm{E}[\mathbf{v}_i \mid s_i,\ \mathbf{b}_{-i} < b^*] - b^*\big)\Pr[\mathbf{b}_{-i} < b^* \mid s_i]\mu^*(\mathrm{d}\mathbf{b}_{-i}) \leqslant \mathrm{E}[\mathbf{v}_i \mid s_i] - b^* \\ &< \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})}\pi_i^*(s_i;\mu) \leqslant \pi^*(s_i;\mu^*),\end{aligned}$$

so b^* cannot be a best reply to $\mathbf{b}_{-i}$. On the other hand, if $b^* > \mathrm{E}[\mathbf{v}_i \mid s_i]$, then $\pi(b^*, s_i; \mu^*) < 0$ (recall that $b^* > 0$ and positive bids win with positive probability), so again b^* cannot be a best reply to μ^*.

We now prove (2).

Claim. $b^* < \min(\mathrm{E}[\mathbf{v}_i \mid s_i], \lim_{s_j\uparrow 1}\mathbf{B}(s_j))$.

To see this, note that $\mathrm{E}[\mathbf{v}_i \mid s_i] - b^* > \inf_{\mu\in\Delta^+(\mathcal{M}_{-i};\mathbf{B}_{-i})}\pi^*(s_i;\mu) = \sup_{b\geqslant 0}\bar{\pi}(b, s_i; \mathbf{B}_{-i})$ implies that $b^* < \mathrm{E}[\mathbf{v}_i \mid s_i]$. Moreover, suppose that $b^* \geqslant \lim_{s_j\uparrow 1}\mathbf{B}(s_j)$. If $\Pr[\mathbf{B}_{-i}^{\max} = b^* \mid s_i] = 0$, then $\bar{\pi}(b^*, s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i \mid s_i] - b^* > \sup_{b\geqslant 0}\bar{\pi}(b, s_i; \mathbf{B}_{-i})$, a contradiction. If instead $\Pr[\mathbf{B}_{-i}^{\max} = b^* \mid s_i] > 0$ (so $b^* = \lim_{s_j\uparrow 1}\mathbf{B}(s_j)$), note that, by Remark 2, $\mathrm{E}[\mathbf{v}_i - b^* \mid s_i, \mathbf{B}_{-i}^{\max} = b^*] \geqslant \mathrm{E}[\mathbf{v}_i - b^* \mid s_i, \mathbf{B}_{-i} \leqslant b^*] = \mathrm{E}[\mathbf{v}_i \mid s_i] - b^* > 0$; hence, we again obtain $\bar{\pi}(b^*, s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i \mid s_i] - b^*$, which yields the same contradiction.

We now construct the conjecture to which b^* is a unique best response. First, define a bounded, nondecreasing and measurable function $\mathbf{g}: [0, 1] \to \mathbb{R}_+$ by

$$\forall s_j \in [0, 1],\quad \mathbf{g}(s_j) = \min\big(\mathbf{B}(s_j), b^*\big). \tag{A.7}$$

Correspondingly, define the quantities

$$\bar{s} = \min\big(1, \inf\{s_j \in [0, 1]:\ \mathbf{B}(s_j) > b^*\}\big),\qquad \bar{b}^{(\alpha)} = \sigma(\bar{s}, \alpha)b^*. \tag{A.8}$$

Recall that $\mathbf{g}^{(\alpha)}(s_j) = \sigma(s_j, \alpha)\mathbf{g}(s_j)$ (Eq. (A.6)). For every $\alpha > 0$ and $b \geqslant 0$, by Remark 2 and the observation that $\mathbf{g}_{-i}^{(\alpha)} \in \mathcal{M}_{-i}$, $\pi(b, s_i; \mathbf{g}_{-i}^{(\alpha)}) \leqslant (\mathrm{E}[\mathbf{v}_i - b \mid s_i])\Pr[\mathbf{g}_{-i}^{(\alpha)} \leqslant b \mid s_i]$, with equality for $b = b^*$. Also note that $\bar{s} < 1$, for otherwise we would have $\mathbf{B}(s_j) \leqslant b^*$ for all $s_j \in [0, 1)$, and therefore $b^* \geqslant \lim_{s_j\uparrow 1}\mathbf{B}(s_j)$.

Letting $\bar{v} = \mathrm{E}[\mathbf{v}_i \mid s_i]$ for notational convenience, and fixing an arbitrary opponent j of player i,

$$\begin{aligned}\Pr\big[\mathbf{g}_{-i}^{(\alpha)} \leqslant b \,\big|\, s_i\big] &\leqslant \Pr\big[s_j\colon\ \mathbf{g}^{(\alpha)}(s_j) \leqslant b \,\big|\, s_i\big] \\ &= \Pr\big[s_j\colon\ s_j < \bar{s} \text{ and } \mathbf{B}^{(\alpha)}(s_j) < b \text{ or } s_j > \bar{s} \text{ and } \sigma(s_j,\alpha)b^* \leqslant b \,\big|\, s_i\big] \\ &\leqslant \Pr\big[s_j\colon\ s_j < \bar{s} \text{ or } s_j > \bar{s} \text{ and } \sigma(s_j,\alpha)b^* \leqslant b \,\big|\, s_i\big],\end{aligned}$$

with equality for $b = b^*$. Moreover, for $b \in [\bar{b}^{(\alpha)}, b^*]$, $s_j < \bar{s}$ implies $\sigma(s_j,\alpha)b^* < \sigma(\bar{s},\alpha)b^* = \bar{b}^{(\alpha)} \leqslant b$, so we also have

$$\Pr\big[\mathbf{g}_{-i}^{(\alpha)} \leqslant b \,\big|\, s_i\big] \leqslant \Pr\big[\sigma(s_j,\alpha)b^* \leqslant b \,\big|\, s_i\big] = F_{j|i}\left(\tau\left(\frac{b}{b^*},\alpha\right) \,\middle|\, s_i\right)$$

(recall that $\tau(\cdot,\alpha)$ is the inverse of $\sigma(\cdot,\alpha)$), and therefore $\pi(b, s_i; \mathbf{g}_{-i}^{(\alpha)}) \leqslant (\bar{v} - b)F_{j|i}(\tau(b/b^*,\alpha) \mid s_i)$, with equality for $b = b^*$. Hence, if b^* is the unique maximizer of the r.h.s. in the region $[\bar{b}^{(\alpha)}, b^*]$ of bids, then b^* is also the unique maximizer of player i's payoff in the same region. We now show that this is indeed the case.

It is expedient to represent bids as convex combinations of $\bar{b}^{(\alpha)} = \sigma(\bar{s},\alpha)b^*$ and b^*. For every $\lambda \in [0,1]$, define

$$b(\lambda,\alpha) = \big[(1-\lambda)\sigma(\bar{s},\alpha) + \lambda\big]b^*; \tag{A.9}$$

note that $\frac{\partial}{\partial\lambda}b(\lambda,\alpha) = [1 - \sigma(\bar{s},\alpha)]b^*$. Also, for every $\lambda \in [0,1]$, define the quantity

$$s(\lambda,\alpha) = \tau\left(\frac{b(\lambda,\alpha)}{b^*},\alpha\right) = \tau\big((1-\lambda)\sigma(\bar{s},\alpha) + \lambda,\alpha\big), \tag{A.10}$$

which yields the type (of player j) who bids $b(\lambda,\alpha)$, according to the bid function $\mathbf{g}^{(\alpha)}$, if $\mathbf{B}$ is right-continuous at $\bar{s}$.

Claim. *The function $H : [0,1] \times (0,1] \to \mathbb{R}_+$ defined by*

$$H(\lambda,\alpha) = \big(\bar{v} - b(\lambda)\big)F_{j|i}\big(s(\lambda,\alpha) \,\big|\, s_i\big) \tag{A.11}$$

has a unique maximum at $\lambda = 1$ for sufficiently small α.

Case 1 ($\bar{s} > 0$). Consider $\alpha \in (0,\bar{s})$. Then $\sigma(\bar{s},\alpha) = (1 - \alpha/(1-\alpha)) + \frac{\alpha}{1-\alpha}\bar{s}$, so $(1-\lambda)\sigma(\bar{s},\alpha) + \lambda = 1 - (1-\lambda)\frac{\alpha}{1-\alpha}(1-\bar{s}) > 1 - \alpha$, where the inequality follows from the choice of α. Therefore, $s(\lambda,\alpha) = 1 - (1-\lambda)(1-\bar{s}) \equiv s(\lambda)$, which is independent of α. Also, $\frac{\partial}{\partial\lambda}s(\lambda) = (1-\bar{s})$ for $\lambda \in (0,1)$.

We conclude that

$$\frac{\partial}{\partial\lambda}H(\lambda,\alpha) = -\big[1 - \sigma(\bar{s},\alpha)\big]b^*F_{j|i}\big(s(\lambda) \,\big|\, s_i\big) + \big(\bar{v} - b(\lambda,\alpha)\big)f_{j|i}\big(s(\lambda) \,\big|\, s_i\big)(1-\bar{s}).$$

Now let $f_{j|i}^{\min}(s_i) = \min_{x\in[0,1]} f_{j|i}(x \mid s_i) > 0$: then, since $\bar{s} < 1$, $\frac{\partial}{\partial\lambda}H(\lambda,\alpha) \geqslant -[1 - \sigma(\bar{s},\alpha)]b^* + (\bar{v} - b^*) \times f_{j|i}^{\min}(s_i)(1-\bar{s}) \equiv h(\alpha)$. Since $\bar{s} > 0$, $\lim_{\alpha\to 0}\sigma(\bar{s},\alpha) = 1$; this implies that, for α sufficiently small, $h(\alpha) > 0$, and therefore $\frac{\partial}{\partial\lambda}H(\lambda,\alpha) > 0$. This implies that $\arg\max_{\lambda\in[0,1]} H(\lambda,\alpha) = \{1\}$.

Case 2 ($\bar{s} = 0$). Then $\sigma(\bar{s},\alpha) = 0$, $b(\lambda,\alpha) = \lambda b^*$, and $s(\lambda,\alpha) = \tau(\lambda,\alpha)$. We have two sub-cases.

First, for $0 < \lambda \leqslant 1-\alpha$, $\tau(\lambda,\alpha) = \frac{\alpha}{1-\alpha}\lambda$. Define $f_{j|i}^{\max}(s_i) = \max_{x\in[0,1]} f_{j|i}(x \mid s_i) < \infty$; then, since $\frac{\lambda}{1-\alpha} \leqslant 1$,

$$H(\lambda,\alpha) = (\bar{v} - \lambda b^*)F_{j|i}\left(\frac{\alpha}{1-\alpha}\lambda \,\middle|\, s_i\right) \leqslant \bar{v}f_{j|i}^{\max}(s_i)\alpha.$$

For α sufficiently small, the r.h.s. is smaller than $H(1,\alpha) = \bar{v} - b^* > 0$. Hence, for all $\lambda \in (0, 1-\alpha]$, $H(1,\alpha) > H(\lambda,\alpha)$.

For $\lambda \in (1-\alpha, 1)$, $\tau(\lambda,\alpha) = (1 - \frac{1-\alpha}{\alpha}) + \frac{1-\alpha}{\alpha}\lambda$, so

$$\frac{\partial}{\partial\lambda}H(\lambda,\alpha) = -b^*F_{j|i}\big(\tau(\lambda,\alpha) \,\big|\, s_i\big) + (\bar{v} - \lambda b^*)f_{j|i}\big(\tau(\lambda,\alpha) \,\big|\, s_i\big)\frac{1-\alpha}{\alpha}.$$

Thus, with $f_{j|i}^{\min}(s_i)$ as above, $\frac{\partial}{\partial\lambda}H(\lambda,\alpha) \geqslant -b^* + (\bar{v} - b^*) f_{j|i}^{\min}(s_i)\frac{1-\alpha}{\alpha}$, which is positive for α sufficiently small. Thus, $H(1,\alpha) > H(\lambda,\alpha)$ for α small and $\lambda \in (1-\alpha, 1)$.

Since $H(0,\alpha) = 0$, we can again conclude that $\arg\max_{\lambda\in[0,1]} H(\lambda,\alpha) = \{1\}$, and the proof of the claim is complete.

The claim implies that b^* is the unique maximizer of $\pi(b, s_i; \mathbf{g}_{-i}^{(\alpha)})$ in the region $[\bar{b}^{(\alpha)}, b^*]$. Clearly, every $b > b^*$ is dominated by b^* given $\mathbf{g}_{-i}^{(\alpha)}$.

If $\bar{s} = 0$, then $\bar{b}^{(\alpha)} = 0$, so we are done. Otherwise, notice that $\bar{b}^{(\alpha)} = \sigma(\bar{s},\alpha)b^*$ implies that $[\mathbf{g}_{-i}^{(\alpha)} \leqslant b] = [\mathbf{B}_{-i}^{(\alpha)} \leqslant b]$ for all $b < \bar{b}^{(\alpha)}$. To see this, suppose $\mathbf{g}_{-i}^{(\alpha)}(s_{-i}) \leqslant b$, and fix $j \neq i$. If $\mathbf{g}(s_j) = \mathbf{B}(s_j)$, then $\mathbf{B}^{(\alpha)}(s_j) \leqslant b$ follows immediately. Suppose instead that $\mathbf{g}(s_j) = b^*$. Note that, since we are considering a bid $b < \bar{b}^{(\alpha)}$, $\sigma(s_j,\alpha)b^* = \sigma(s_j,\alpha)\mathbf{g}(s_j) = \mathbf{g}^{(\alpha)}(s_j) \leqslant b < \bar{b}^{(\alpha)} = \sigma(\bar{s},\alpha)b^*$. Then we get $\sigma(s_j,\alpha) < \sigma(\bar{s},\alpha)$, and since $\sigma(\cdot,\alpha)$ is increasing, $s_j < \bar{s}$; but then $\mathbf{B}(s_j) \leqslant b^*$, so $\mathbf{g}(s_j) = \mathbf{B}(s_j)$, and both must be equal to b^*; moreover, $\mathbf{B}^{(\alpha)}(s_j) \leqslant b$. Conversely, if $\mathbf{B}_{-i}^{(\alpha)}(s_{-i}) \leqslant b$, then *a fortiori* $\mathbf{g}_{-i}^{(\alpha)}(s_{-i}) \leqslant b$, because, for all $j \neq i$, $\mathbf{g}^{(\alpha)}(s_j) = \sigma(s_j,\alpha)\mathbf{g}(s_j) \leqslant \sigma(s_j,\alpha)\mathbf{B}(s_j) = \mathbf{B}^{(\alpha)}(s_j)$.

We conclude that, for $b \in [0, \bar{b}^{(\alpha)})$, $\pi(b, s_i; \mathbf{g}_{-i}^{(\alpha)}) = \pi(b, s_i; \mathbf{B}_{-i}^{(\alpha)})$. For any such b, Lemma 17, claim (3) implies that $\lim_{\alpha\downarrow 0}\pi(b, s_i; \mathbf{B}_{-i}^{(\alpha)}) \leqslant \bar{\pi}(b, s_i; \mathbf{B}_{-i}) \leqslant \sup_{b'\geqslant 0}\bar{\pi}(b', s_i; \mathbf{B}_{-i})$. By assumption, the latter quantity is smaller than $\mathrm{E}[\mathbf{v}_i \mid s_i] - b^* = \pi(b^*, s_i; \mathbf{g}_{-i}^{(\alpha)})$; therefore, for sufficiently small α, and for all $b \in [0, \bar{b}^{(\alpha)})$, $\pi(b, s_i; \mathbf{g}_{-i}^{(\alpha)}) < \pi(b^*, s_i; \mathbf{g}_{-i}^{(\alpha)})$. □

A.1.4. Proof of Proposition 8

(3) Note that, since f is continuous, by Dominated Convergence, $\mathrm{E}[\mathbf{v}_i \mid s_i]$ and $\pi(b, s_i; \mathbf{B}_{-i})$ are continuous in s_i; also, the latter function is continuous in b by Lemma 14. By the Maximum Theorem, the function $s_i \mapsto \max_{b\geqslant 0}\pi(b, s_i; \mathbf{B}_{-i})$ is thus continuous, and this implies that $\phi^{\mathbf{B}}$ is continuous.

(4) *Observation*. By Assumption 4 and Remark 2, for all $s_i, s_i' \in [0,1]$ such that $s_i > s_i'$, $\mathrm{E}[\mathbf{v}_i \mid s_i] > \mathrm{E}[\mathbf{v} \mid s_i']$. Similarly, for every $b \geqslant 0$, if $\Pr[\mathbf{B}_{-i} \leqslant b \mid s_i] > 0$ (respectively $\Pr[\mathbf{B}_{-i}^{\max} > b \mid s_i] > 0$), then $\mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{B}_{-i} \leqslant b]$ (respectively $\mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{B}_{-i}^{\max} > b]$) is increasing in s_i.

Choose $s_i > 0$ and $b \in \arg\max_{y\geqslant 0}\pi(y, s_i; \mathbf{B}_{-i})$.

Claim. *Assume that* $\Pr[\mathbf{B}_{-i} \leqslant b \mid s_i] < 1$, *so* $b < \mathbf{B}(1)$. *Then, for all* $s_i' < s_i$, $\phi^{\mathbf{B}}(s_i') < \phi^{\mathbf{B}}(s_i)$.

To prove the claim, note first that, if $\pi(b, s_i; \mathbf{B}_{-i}) = 0$, then $\phi^{\mathbf{B}}(s_i) = \mathrm{E}[\mathbf{v}_i \mid s_i] > \mathrm{E}[\mathbf{v}_i \mid s_i'] \geqslant \phi^{\mathbf{B}}(s_i')$, where the strict inequality follows from the initial Observation, and the weak inequality from part (1).

Thus, assume $\pi(b, s_i; \mathbf{B}_{-i}) > 0$, so in particular $\Pr[\mathbf{B}_{-i} \leqslant b \mid s_i] \in (0,1)$ because $\mathbf{B}_{-i} > 0$. Then $\mathrm{E}[\mathbf{v}_i \mid s_i] = \mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{B}_{-i} \leqslant b]\Pr[\mathbf{B}_{-i} \leqslant b \mid s_i] + \mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{B}_{-i}^{\max} > b]\Pr[\mathbf{B}_{-i}^{\max} > b \mid s_i]$ and

$$\phi^{\mathbf{B}}(s_i) = \mathrm{E}\big[\mathbf{v}_i \bigm| s_i, \mathbf{B}_{-i}^{\max} > b\big]\Pr\big[\mathbf{B}_{-i}^{\max} > b \bigm| s_i\big] + b\Pr[\mathbf{B}_{-i} \leqslant b \mid s_i].$$

Since $\pi(b, s_i; \mathbf{B}_{-i}) > 0$, $b < \mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{B}_{-i} \leqslant b] \leqslant \mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{B}_{-i}^{\max} > b]$, where the second inequality follows from Remark 2 by an argument analogous to the one used in the proof of Lemma 16(2). Also note that the indicator function of the event $[\mathbf{B}_{-i}^{\max} > b]$ is nondecreasing, because $\mathbf{B}_{-i}^{\max}$ is nondecreasing; therefore, for $s_i' < s_i$, $\Pr[\mathbf{B}_{-i}^{\max} > b \mid s_i] \geqslant \Pr[\mathbf{B}_{-i}^{\max} > b \mid s_i']$. Note that, since f is bounded away from zero, $\Pr[\mathbf{B}_{-i}^{\max} > b \mid s_i'] > 0$; therefore,

$$\begin{aligned}
\phi^{\mathbf{B}}(s_i) &\geqslant \mathrm{E}\big[\mathbf{v}_i \bigm| s_i, \mathbf{B}_{-i}^{\max} > b\big]\Pr\big[\mathbf{B}_{-i}^{\max} > b \bigm| s_i'\big] + b\Pr\big[\mathbf{B}_{-i} \leqslant b \bigm| s_i'\big] \\
&> \mathrm{E}\big[\mathbf{v}_i \bigm| s_i', \mathbf{B}_{-i}^{\max} > b\big]\Pr\big[\mathbf{B}_{-i}^{\max} > b \bigm| s_i'\big] + b\Pr\big[\mathbf{B}_{-i} \leqslant b \bigm| s_i'\big] \\
&= \mathrm{E}\big[\mathbf{v}_i \bigm| s_i'\big] - \mathrm{E}\big[\mathbf{v}_i - b \bigm| s_i', \mathbf{B}_{-i} \leqslant b\big]\Pr\big[\mathbf{B}_{-i} \leqslant b \bigm| s_i'\big] \\
&\geqslant \mathrm{E}\big[\mathbf{v}_i \bigm| s_i'\big] - \max_{x\geqslant 0}\pi\big(x, s_i'; \mathbf{B}_{-i}\big) = \phi^{\mathbf{B}}(s_i'),
\end{aligned}$$

where the strict inequality follows from the initial Observation, and the proof of the claim is complete.

Consider now signals s_i, s_i' such that $s_i' < s_i$ and $\phi^{\mathbf{B}}(s_i') = \mathbf{B}(1)$. Then it must be the case that $\arg\max_{x \geqslant 0} \pi(x, s_i; \mathbf{B}_{-i}) = \{\mathbf{B}(1)\}$: otherwise, by the preceding Claim, $s_i' < s_i$, $b \in \arg\max_{y \geqslant 0} \pi(y, s_i; \mathbf{B}_{-i})$ and $b < \mathbf{B}(1)$ would imply

$$\begin{aligned}\phi^{\mathbf{B}}(s_i') < \phi^{\mathbf{B}}(s_i) &= \mathrm{E}[\mathbf{v}_i \mid s_i] - \pi^*(s_i, \mathbf{B}_{-i}) \leqslant \mathrm{E}[\mathbf{v}_i \mid s_i] - \pi\big(\mathbf{B}(1), s_i; \mathbf{B}_{-i}\big) \\ &= \mathrm{E}[\mathbf{v}_i \mid s_i] - \big(\mathrm{E}\big[\mathbf{v}_i \bigm| s_i, \mathbf{B}_{-i} \leqslant \mathbf{B}(1)\big] - \mathbf{B}(1)\big) = \mathbf{B}(1),\end{aligned}$$

a contradiction. Thus, we must have $\phi^{\mathbf{B}}(s_i) = \phi^{\mathbf{B}}(s_i') = \mathbf{B}(1)$. Let

$$s^{\mathbf{B}} = \min\Big(1, \inf\Big\{s_i \in [0,1]\colon\ \mathbf{B}(1) \in \arg\max_{x \geqslant 0} \pi(x, s_i; \mathbf{B}_{-i})\Big\}\Big).$$

Then $\phi^{\mathbf{B}}$ is increasing on $[0, s^{\mathbf{B}})$ and constant on $[s^{\mathbf{B}}, 1]$. □

A.2. Rationalizability

Lemma 18. *Let* $\mathbf{b}, \mathbf{B}$ *be bid functions such that* $\mathbf{b}$ *is increasing,* $\mathbf{B}$ *is nondecreasing, and* $\mathbf{b} \geqslant \mathbf{B} > 0$. *Then, for every bid* $b \in [0, \mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{b}_{-i} < b]]$, $\pi(b, s_i; \mathbf{b}) \leqslant \bar{\pi}(b, s_i; \mathbf{B})$. *In particular,* $\pi^*(s_i; \mathbf{b}) \leqslant \max_{b \geqslant 0} \bar{\pi}(b, s_i; \mathbf{B})$.

Proof. Note that

$$\begin{aligned}\bar{\pi}(b, s_i; \mathbf{B}) = {} & \mathrm{E}[\mathbf{v}_i - b \mid s_i, \mathbf{b}_{-i} < b] \Pr[\mathbf{b}_{-i} < b \mid s_i] \\ & + \sum_{\emptyset \neq J \subset N \setminus \{i\}} \mathrm{E}[\mathbf{v}_i - b \mid s_i,\ \mathbf{b}_{N \setminus J \cup \{i\}} < b,\ \mathbf{B}_J < b \leqslant \mathbf{b}_J] \\ & \cdot \Pr[\mathbf{b}_{N \setminus J \cup \{i\}} < b,\ \mathbf{B}_J < b \leqslant \mathbf{b}_J \mid s_i] \\ & + \max\big\{0, \big(\mathrm{E}\big[\mathbf{v}_i \bigm| s_i, \mathbf{B}^{\max}_{-i} = b\big] - b\big) \Pr\big[\mathbf{B}^{\max}_{-i} = b \bigm| s_i\big]\big\}.\end{aligned}$$

Since $\mathbf{b}$ is increasing, the first term on the right-hand side is $\pi(b, s_i; \mathbf{b})$. Since $\mathbf{b}$ and $\mathbf{B}$ are nondecreasing, by Result 2, $\mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{b}_{N \setminus J \cup \{i\}} < b, \mathbf{B}_J < b \leqslant \mathbf{b}_J] \geqslant \mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{b}_{-i} < b]$ for all J, as in the proof of Lemma 16, part (1). Since, moreover, by assumption $b \leqslant \mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{b}_{-i} < b]$, the second term in the above expression is nonnegative. Since the third is clearly also nonnegative, the first claim follows.

The second claim holds trivially if $\pi^*(s_i; \mathbf{b}_{-i}) = 0$; otherwise, it follows by observing that, if $b^* \in \arg\max_{b \geqslant 0} \pi(b, s_i; \mathbf{b})$ and $\pi^*(s_i; \mathbf{b}_{-i}) > 0$, then surely $b^* \in (0, \mathrm{E}[\mathbf{v}_i \mid s_i, \mathbf{b}_{-i} < b^*])$. □

Lemma 19. *Let* $\{\mathbf{B}^k\}_{k=1}^{\infty}$ *be a weakly decreasing sequence of continuous, increasing and positive functions, and let* $\mathbf{B} = \lim_{k \to \infty} \mathbf{B}^k > 0$. *Define* $\mathbf{B}^k_{-i} = (\mathbf{B}^k, \mathbf{B}^k \ldots)$ *and* $\mathbf{B}_{-i} = (\mathbf{B}, \mathbf{B} \ldots)$. *Then, for every signal* $s_i \in S_i$, $\lim_{k \to \infty} \pi^*(s_i; \mathbf{B}^k_{-i}) = \inf_{\mu \in \Delta^+(\mathcal{M}_{-i}, \mathbf{B}_{-i})} \pi^*(s_i; \mu)$.

Proof. Define $\bar{\pi}^*(s_i, \mathbf{B}_{-i}) = \max_{b \geqslant 0} \bar{\pi}(b, s_i; \mathbf{B}_{-i})$ for notational convenience. Then, Lemma 18 applied to the pairs $\mathbf{B}^k, \mathbf{B}^{k+1}$ and $\mathbf{B}^{k+1}, \mathbf{B}$ implies that $0 \leqslant \pi^*(s_i, \mathbf{B}^k_{-i}) \leqslant \pi^*(s_i, \mathbf{B}^{k+1}_{-i}) \leqslant \bar{\pi}^*(s_i, \mathbf{B}_{-i})$ for all k. Therefore, the sequence $\{\pi^*(s_i, \mathbf{B}^k_{-i})\}_{k=1}^{\infty}$ has a limit, and if $\bar{\pi}^*(s_i, \mathbf{B}_{-i}) = 0$, this limit is zero. Now assume that $\bar{\pi}^*(s_i, \mathbf{B}_{-i}) > 0$ and let $b^* \in \arg\max_{b \geqslant 0} \bar{\pi}(b, s_i; \mathbf{B}_{-i})$. Since $\mathbf{B}_{-i} > 0$, $\pi(0, s_i, \mathbf{B}_{-i}) = 0 < \bar{\pi}^*(s_i, \mathbf{B}_{-i})$. Therefore $b^* > 0$, and—by Lemma 16(2)—if $\Pr[\mathrm{B}^{\max}_{-i} = b^* \mid s_i] > 0$, then $\mathrm{E}[\mathbf{v}_i - b^* \mid s_i, \mathbf{B}^{\max}_{-i} = b] \geqslant 0$. Hence, regardless of whether or not b^* ties with positive probability, $\bar{\pi}(b^*, s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i - b^* \mid s_i, \mathbf{B}_{-i} \leqslant b^*] \Pr[\mathbf{B}_{-i} \leqslant b^* \mid s_i]$.

Now let $\{b_\ell\}_{\ell=1}^{\infty}$ be a sequence of bids such that $\Pr[\mathrm{B}^{\max}_{-i} = b_\ell \mid s_i] = 0$ for all integers ℓ, and $b_\ell \downarrow b^*$. Thus, for each ℓ, $\bar{\pi}(b_\ell, s_i; \mathbf{B}_{-i}) = \mathrm{E}[\mathbf{v}_i - b^* \mid s_i, \mathbf{B}_{-i} < b_\ell] \Pr[\mathbf{B}_{-i} < b_\ell \mid s_i]$. Also, for all ℓ and k, the indicator functions of the sets $[\mathbf{B}^k_{-i} < b_\ell]$ converge pointwise on S_{-i} to the indicator function of $[\mathbf{B}_{-i} < b_\ell]$.[28] Thus,

[28] Consider s_j such that $\mathbf{B}(s_j) < b_\ell$; then, for k large enough, $\mathbf{B}^k < b_\ell$. If instead $\mathbf{B}(s_j) \geqslant b_\ell$, then also $\mathbf{B}^k(s_j) > b_\ell$. Thus, pointwise convergence obtains for all $s_j \in S_j$.

$[\mathbf{v}_i(s_i, s_{-i}) - b_\ell]1_{[\mathbf{B}^k_{-i} < b_\ell]}(s_{-i}) \to [\mathbf{v}_i(s_i, s_{-i}) - b_\ell]1_{[\mathbf{B}_{-i} < b_\ell]}(s_{-i})$ pointwise on S_{-i}; hence, by Dominated Convergence, $\bar{\pi}(b_\ell, s_i; \mathbf{B}^k_{-i}) \to \bar{\pi}(b_\ell, s_i; \mathbf{B}_{-i})$.

Similarly, note that the indicator functions of the sets $[\mathbf{B}_{-i} < b_\ell]$ converge pointwise on S_{-i} to the indicator function of $[\mathbf{B}_{-i} \leqslant b^*]$.[29] Thus, $[\mathbf{v}_i(s_i, s_{-i}) - b_\ell]1_{[\mathbf{B}_{-i} < b_\ell]}(s_{-i}) \to [\mathbf{v}_i(s_i, s_{-i}) - b^*]1_{[\mathbf{B}_{-i} \leqslant b^*]}(s_{-i})$ pointwise on S_{-i}; hence, by Dominated Convergence, $\bar{\pi}(b_\ell, s_i; \mathbf{B}_{-i}) \to \bar{\pi}(b^*, s_i; \mathbf{B}_{-i})$.

Thus, for every $\epsilon > 0$ we can find k and ℓ large enough that $|\bar{\pi}^*(b^*, s_i; \mathbf{B}_{-i}) - \bar{\pi}(b_\ell, s_i; \mathbf{B}^k_{-i})| < \epsilon$. Moreover, $\bar{\pi}(b_\ell, s_i; \mathbf{B}^k_{-i}) \leqslant \pi^*(s_i, \mathbf{B}^k_{-i}) \leqslant \bar{\pi}^*(s_i, \mathbf{B}_{-i})$. Therefore,

$$\bar{\pi}^*(s_i; \mathbf{B}_{-i}) \geqslant \pi^*\left(s_i, \mathbf{B}^k_{-i}\right) \geqslant \pi\left(b_\ell, s_i; \mathbf{B}^k_{-i}\right) \geqslant \bar{\pi}(b^*, s_i; \mathbf{B}_{-i}) - \epsilon,$$

which implies that $\lim_{k\to\infty} \pi^*(s_i, \mathbf{B}^k_{-i}) = \max_{b \geqslant 0} \bar{\pi}(b, s_i; \mathbf{B}_{-i})$. Since $\mathbf{B} > 0$ is nondecreasing, part (3) of Theorem 6 yields the desired result. □

References

Aliprantis, C., Border, K., 1994. Infinite-Dimensional Analysis. Springer-Verlag, Berlin.

Battigalli, P., 1999. Rationalizability in incomplete-information games. Working paper ECO 99/17, European University Institute.

Battigalli, P., Siniscalchi, M., 2000. Rationalizable bidding in general first-price auctions. Working paper IGIER–Università Bocconi. Latest version available at http://faculty.econ.northwestern.edu/faculty/siniscalchi.

Bernheim, D., 1984. Rationalizable strategic behavior. Econometrica 52, 1007–1028.

Chung, K.S., Ely, J., 2000. Efficient and dominance-Solvable auctions with interdependent valuations. CMS–EMS discussion paper No. 1313, Northwestern University.

Cox, J., Smith, V., Walker, J., 1988. Theory and individual behavior of first-price auctions. J. Risk Uncertainty 1, 61–99.

Cox, J., Smith, V., Walker, J., 1992. Theory and misbehavior in first-price auctions: comment. Amer. Econ. Rev. 82, 1392–1412.

Dekel, E., Fudenberg, D., 1990. Rational behavior with payoff uncertainty. J. Econ. Theory 52, 243–267.

Dekel, E., Gul, F., 1997. Rationality and knowledge in game theory. In: Kreps, D., Wallis, K. (Eds.), Advances in Economics and Econometrics, Vol. 1. Cambridge Univ. Press, Cambridge, UK.

Dekel, E., Wolinsky, A., 2000. Rationalizable outcomes of large independent private-value first-price discrete auctions. Mimeo, Northwestern University and Tel Aviv University.

Friedman, D., 1992. Theory and misbehavior in first-price auctions: comment. Amer. Econ. Rev. 82, 1374–1378.

Fudenberg, D., Levine, D., 1998. Theory of Learning in Games. MIT Press, Cambridge, MA.

Goeree, J., Holt, C., Palfrey, T., 2000. Quantal response equilibrium and overbidding in private-value auctions. Mimeo, Caltech.

Harrison, G., 1989. Theory and misbehavior in first-price auctions. Amer. Econ. Rev. 79, 749–762.

Kagel, J., 1995. Auctions: a survey of experimental research. In: Kagel, J., Roth, A. (Eds.), The Handbook of Experimental Economics. Princeton Univ. Press, Princeton.

Kagel, J., Levin, D., 1993. Independent private values auctions: bidder behaviour in first-, second- and third-price auctions with varying numbers of bidders. Econ. J. 103, 868–879.

Kagel, J., Roth, A., 1992. Theory and misbehavior in first-price auctions: comment. Amer. Econ. Rev. 82, 1379–1391.

Klemperer, P., 1998. Auctions with almost common values: the 'Wallet game' and its applications. Europ. Econ. Rev. 42 (3–5), 757–769.

Lipman, B., 1994. A note on the implications of common knowledge of rationality. Games Econ. Behav. 6, 114–129.

Milgrom, P., Weber, R., 1982. A theory of auctions and competitive bidding. Econometrica 50, 1089–1122.

[29] Consider s_j such that $\mathbf{B}(s_j) \leqslant b^*$; then $\mathbf{B}(s_j) < b_\ell$ for all ℓ. If instead $\mathbf{B}(s_j) > b^*$, then, for ℓ large, $\mathbf{B}(s_j) > b_\ell$. In either case, pointwise convergence obtains.

Myerson, R., 1981. Optimal auction design. Math. Oper. Res. 6, 58–73.
Osborne, M., Rubinstein, A., 1994. A Course in Game Theory. MIT Press, Cambridge, MA.
Riley, J., Samuelson, W., 1981. Optimal auctions. Amer. Econ. Rev. 71, 381–392.
Vickrey, W., 1961. Counterspeculation, auctions, and competitive sealed tenders. J. Finance 16, 8–37.