

THE LOGIC OF BELIEF PERSISTENCE

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INTRODUCTION

The principle of belief persistence, or conservativity principle, states that 'When changing beliefs in response to new evidence, you should continue to believe as many of the old beliefs as possible' (Harman, 1986, p. 46). In particular, this means that if an individual gets new information, she has to accommodate it in her new belief set (the set of propositions she believes), and, if the new information is not inconsistent with the old belief set, then (1) the individual has to maintain all the beliefs she previously had and (2) the change should be minimal in the sense that every proposition in the new belief set must be deducible from the union of the old belief set and the new information (see, e.g., Gärdenfors, 1988; Stalnaker, 1984). We focus on this minimal notion of belief persistence and characterize it both semantically and syntactically.

A 'possible world' semantic formalization of the principle easily comes to mind. The set of all the propositions that the individual believes corresponds to the set of states of the world that she considers possible and is a subset of the set of states that are not ruled out by the individual's information (or knowledge). It is required that, if the individual considers a state possible and her new information does not exclude this state, then she continue to consider it possible. Furthermore, if the individual regards a particular state as impossible, then she should continue to regard it as impossible unless her new information excludes *all* the states that she previously regarded as possible. This is closely related to the

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well-known conditionalization rule to update probability measures. If an individual has probabilistic beliefs, the set of states that she considers possible is simply the support of her subjective probability measure. Let P_o be the probability measure representing the agent's beliefs before she receives information E and P_n her subjective probability measure after she learns E . The 'qualitative part' of the conditionalization rule states the following:

$$(C) \text{ if } \text{supp}(P_o) \cap E \neq \emptyset, \text{ then } \text{supp}(P_n) = \text{supp}(P_o) \cap E,$$

where $\text{supp}(P)$ denotes the support of the probability measure P .¹

Despite the apparent simplicity of the semantic formalization, a syntactic characterization of this minimal notion of belief persistence is not obvious. One would like to state something like the following: if the new evidence does not contradict the conjunction of all the propositions believed by the individual, then she incorporates this new evidence in her belief set while maintaining all her previous beliefs. But the belief set is typically infinite and in standard propositional (modal) logic the infinite conjunction of formulae is not a well-formed formula.

We get around this problem by considering an axiom system with knowledge and belief operators for each date satisfying standard properties, including the entailment axiom 'knowledge implies belief'. Given this, we are able to characterize persistence of beliefs with the following axiom which involves only the belief operators:

(PB) The individual believes ϕ at date t if and only if she believes at date t that she will believe ϕ at date $t + 1$.

In Section 2 we begin by studying belief change from a *semantic* point of view. We consider a class of structures or frames that represent the paradigm in the economics of information literature. There is a countable set of dates. At each date knowledge is represented by a partition of the set of states. Beliefs are based on knowledge, in the sense that the set of states that the individual considers possible at a state ω is a subset of the cell of the information partition that contains ω . Furthermore, within each cell of the information partition beliefs are the same at every state (that is, the individual knows her own beliefs). We show that the belief persistence condition can be split into two independent properties. The first property is equivalent to the requirement that if the individual at date $t + 1$ believes ϕ , then there be a ψ which she knows at $t + 1$ and which at date t she believed to imply ϕ (Proposition 2). The second property is

¹ The 'quantitative part' states that, for any two subsets $A, B \subseteq \text{supp}(P_o) \cap E$, the relative likelihood of A versus B does not change, i.e., $P_n(A)P_o(B) = P_o(A)P_n(B)$. There is an ongoing debate in the philosophical literature as to whether or not conditionalization is a requirement of rationality: see, for example, Brown (1976), Jeffrey (1983), Howson and Urbach (1989), Maher (1993) and Teller (1973).

equivalent to the condition that, if what is known at date $t+1$ does not contradict what is believed at date t , then everything which is believed at date t should still be believed at date $t+1$ (Proposition 3). From these two propositions the main result of Section 2 follows, namely that the qualitative part of the conditionalization rule is equivalent to the requirement that, if the new information is consistent with the old beliefs, then the set of propositions that are believed at date $t+1$ be the deductive closure of the union of the set of propositions believed at time t and the set of propositions known at time $t+1$ (Corollary 1).

In Section 3 we turn to a *syntactic* analysis, employing the language of modal logic, which was first used in an epistemic context by the philosopher Hintikka (1962) and whose usefulness has recently been recognized by economists and game theorists.² The formal language that we consider is the one that comes closest to the dynamic models developed in the information economics literature. In particular, time enters the analysis explicitly only through the epistemic operators. The main result of this section (Proposition 5) states that the system obtained by adding axiom (PB) to standard axioms for knowledge and belief and their interaction is sound and complete with respect to the class of models based on frames that satisfy property (C).

Section 4 discusses the notion of perfect recall and the possibility of relaxing the truth axiom for knowledge. Section 5 considers related literature.

2. THE SEMANTICS OF BELIEF CHANGE

We shall consider a class of structures that represent the paradigm in the economics of information literature.³ Let Ω be a (nonempty and possibly infinite) set of states. At every date $t \in \mathbb{N}$ (where \mathbb{N} denotes the set of natural numbers),⁴ the individual has a partition of the set Ω , representing her information or knowledge. We represent this partition by means of an *information correspondence* $\mathcal{K}_t : \Omega \rightarrow 2^\Omega$ [where 2^Ω is the set of *events*, that is, subsets of Ω] that satisfies the following properties: $\forall t \in \mathbb{N}, \forall \omega, \omega' \in \Omega$

reflexivity: $\omega \in \mathcal{K}_t(\omega)$

transitivity: if $\omega' \in \mathcal{K}_t(\omega)$ then $\mathcal{K}_t(\omega') \subseteq \mathcal{K}_t(\omega)$.

² For recent contributions and a list of references see the special issue of *Theory and Decision* on 'Logic and the epistemic foundations of game theory' (1994, Vol. 37). See also the recent survey by Dekel and Gul (1996).

³ See, for example, Geanakoplos (1994, pp. 1456–8). For the importance of the interaction between knowledge and belief in game theory see Dekel and Gul (1996).

⁴ We model time as discrete and take the set of dates to be \mathbb{N} . The latter assumption is without loss of generality since the case of a finite number of dates can be modeled by assuming that there are no changes in information and beliefs after a certain date.

euclideaness: if $\omega' \in \mathcal{K}_t(\omega)$ then $\mathcal{K}_t(\omega) \subseteq \mathcal{K}_t(\omega')$.⁵

For every state $\omega \in \Omega$, $\mathcal{K}_t(\omega)$ is the set of states that, according to her information, the individual cannot rule out at ω .

To represent the individual's beliefs, we define, for every date t , a *belief correspondence* $\mathcal{B}_t : \Omega \rightarrow 2^\Omega$ that satisfies the following properties: $\forall t \in \mathbb{N}, \forall \omega, \omega' \in \Omega$

seriality: $\mathcal{B}_t(\omega) \neq \emptyset$

transitivity: if $\omega' \in \mathcal{B}_t(\omega)$ then $\mathcal{B}_t(\omega') \subseteq \mathcal{B}_t(\omega)$

euclideaness: if $\omega' \in \mathcal{B}_t(\omega)$ then $\mathcal{B}_t(\omega) \subseteq \mathcal{B}_t(\omega')$.

Thus $\mathcal{B}_t(\omega)$ is the set of states that the individual considers possible at ω . Furthermore, beliefs are based on information and depend only on it, in the following sense: $\forall t \in \mathbb{N}, \forall \omega, \omega' \in \Omega$

(R1) $\mathcal{B}_t(\omega) \subseteq \mathcal{K}_t(\omega)$

(R2) if $\omega' \in \mathcal{K}_t(\omega)$ then $\mathcal{B}_t(\omega') = \mathcal{B}_t(\omega)$.

It will be shown below (Remark 1) that (R1) corresponds to the requirement that the individual always believe what she knows and (R2) to the requirement that the individual know her own beliefs.⁶

Whenever Ω is a non-empty set, $\mathcal{K}_t : \Omega \rightarrow 2^\Omega$ is reflexive, transitive and euclidean, $\mathcal{B}_t : \Omega \rightarrow 2^\Omega$ is serial, transitive and euclidean and together they satisfy (R1) and (R2), we call the structure $\langle \Omega, \{\mathcal{K}_t\}_{t \in \mathbb{N}}, \{\mathcal{B}_t\}_{t \in \mathbb{N}} \rangle$, a *KB-frame* ('KB' stands for 'Knowledge and Belief'). We denote by \mathcal{F}_{KB} the set of KB-frames.

Belief and knowledge pertain to propositions. Events, that is, subsets of Ω should be thought of as representing propositions. In order to establish the interpretation of events as propositions we need to introduce the notion of a model based on a frame.

We consider a language with two operators for every $t \in \mathbb{N}$: B_t and K_t . The intended interpretation of $B_t\phi$ is 'at time t the individual *believes* ϕ ' and the interpretation of $K_t\phi$ is 'at time t the individual *knows* ϕ '. The alphabet of the language consists of: (1) a finite or countable set Π of

⁵ Transitivity is made redundant by reflexivity and euclideaness (see Chellas, 1984, p. 85). However, throughout the paper we shall allow for some redundancies when they add clarity to the exposition or make things look more familiar in the light of the existing literature.

⁶ In the economics of information literature beliefs are usually represented by a collection of probability measures, one for each cell of the information partition. In this case one can interpret $\mathcal{B}_t(\omega)$ as the *support* of the probability measure over the cell of the partition that contains state ω . It is easy to verify that, with this interpretation, the belief correspondence \mathcal{B}_t indeed satisfies seriality, transitivity and euclideaness, as well as (R1) and (R2). See also Halpern (1991).

sentence letters (representing atomic propositions),⁷ (2) the connectives \neg (for 'not'), \vee (for 'or'), and, for every $t \in \mathbb{N}$, B_t and K_t , (3) the bracket symbols (and). The set Φ of formulae is obtained from the sentence letters by closing with respect to negation, disjunction and the operators B_t and K_t .⁸ As is customary, we shall often omit the outermost brackets (e.g., we shall write $\phi \vee \psi$ instead of $(\phi \vee \psi)$) and use the following (metalinguistic) abbreviations: $\phi \wedge \psi$ for $\neg(\neg\phi \vee \neg\psi)$ (the symbol \wedge stands for 'and'), $\phi \rightarrow \psi$ for $(\neg\phi) \vee \psi$ (the symbol \rightarrow stands for 'if . . . then . . .') and $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ (the symbol \leftrightarrow stands for 'if and only if').

Given a frame $F \in \mathcal{F}_{KB}$ one obtains a model based on it by adding a function $f : \Pi \rightarrow 2^\Omega$ that associates with every sentence letter π the set of states at which π is true. For every formula $\phi \in \Phi$, the truth set of ϕ in \mathcal{M} , denoted by $\|\phi\|^{\mathcal{M}}$, is defined recursively as follows:

- (1) If $\phi = (\pi)$ where π is a sentence letter, then $\|\phi\|^{\mathcal{M}} = f(\pi)$,
- (2) $\|\neg\phi\|^{\mathcal{M}} = -\|\phi\|^{\mathcal{M}}$ (where '-' denotes complement)
- (3) $\|\phi \vee \psi\|^{\mathcal{M}} = \|\phi\|^{\mathcal{M}} \cup \|\psi\|^{\mathcal{M}}$,
- (4) For all $t \in \mathbb{N}$

$$\|K_t\phi\|^{\mathcal{M}} = \{\omega \in \Omega : \mathcal{K}_t(\omega) \subseteq \|\phi\|^{\mathcal{M}}\}, \text{ and}$$

$$\|B_t\phi\|^{\mathcal{M}} = \{\omega \in \Omega : \mathcal{B}_t(\omega) \subseteq \|\phi\|^{\mathcal{M}}\}.$$
⁹

If $\omega \in \|\phi\|^{\mathcal{M}}$ we say that ϕ is true at state ω in model \mathcal{M} . An alternative notation for $\omega \in \|\phi\|^{\mathcal{M}}$ is $\models_\omega^{\mathcal{M}} \phi$ and an alternative notation for $\omega \notin \|\phi\|^{\mathcal{M}}$ is $\not\models_\omega^{\mathcal{M}} \phi$. A formula ϕ is valid in model \mathcal{M} if and only if $\models_\omega^{\mathcal{M}} \phi$ for all $\omega \in \Omega$.

For every $F \in \mathcal{F}_{KB}$ we denote by $\mathbb{M}(F)$ the set of models based on F .

REMARK 1. Let $F \in \mathcal{F}_{KB}$ and $\mathcal{M} \in \mathbb{M}(F)$. Then, for every $t \in \mathbb{N}$ and $\phi \in \Phi$, by (R1), the formula $K_t\phi \rightarrow B_t\phi$ is valid in \mathcal{M} , and, by (R2), the formula $B_t\phi \rightarrow K_tB_t\phi$ is valid in \mathcal{M} .¹⁰

Within our framework, the qualitative part of the conditionalization rule can be restated as follows:

⁷ Each sentence letter $\pi \in \Pi$ should be interpreted as a partial description of the world at a specific point in time (and location). Different sentence letters may refer to different times and location.

⁸ Thus Φ is obtained recursively as follows: (i) for every sentence letter π , $(\pi) \in \Phi$, (ii) if $\phi, \psi \in \Phi$ then $(\neg\phi) \in \Phi$, $(\phi \vee \psi) \in \Phi$ and, for every $t \in \mathbb{N}$, $(B_t\phi) \in \Phi$ and $(K_t\phi) \in \Phi$.

⁹ Thus at state ω (and time t) the individual believes ϕ if and only if ϕ is true at all the states that she considers possible at ω . Mongin (1994) points out the awkwardness of this (standard) informal explication of Kripke structures and argues in favor of a semantics based on neighborhood structures.

¹⁰ We comment on the axiom schemata $K_t\phi \rightarrow B_t\phi$ and $B_t\phi \rightarrow K_tB_t\phi$ in the next section. Remark 1 can be verified directly. It is also a consequence of Theorem 4.3 in van der Hoek (1993, p. 183). We are grateful to Joe Halpern for bringing this paper to our attention.

(C) $\forall t \in \mathbb{N}, \forall \omega \in \Omega$, if $B_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \neq \emptyset$ then $B_{t+1}(\omega) = B_t(\omega) \cap \mathcal{K}_{t+1}(\omega)$.

The main purpose of this paper is to provide a characterization and axiomatization of (C). Given (R1), (C) can be split into two parts, (P1) and (P2), where

(P1) $\forall \omega \in \Omega, \forall t \in \mathbb{N}$, $B_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \subseteq B_{t+1}(\omega)$.

(P2) $\forall \omega \in \Omega, \forall t \in \mathbb{N}$, if $B_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \neq \emptyset$ then $B_{t+1}(\omega) \subseteq B_t(\omega)$.

Propositions 1–3 below provide two different characterizations of (P1) and (P2). All the proofs are given in the appendix.

PROPOSITION 1. Fix an arbitrary frame $F \in \mathcal{F}_{KB}$. Then:

(1) F satisfies (P1) if and only if

$$\forall \mathcal{M} \in \mathbb{M}(F), \forall t \in \mathbb{N}, \forall \omega \in \Omega, \forall \phi \in \Phi, \models_{\omega}^{\mathcal{M}} (B_t B_{t+1} \phi \rightarrow B_t \phi);$$

(2) F satisfies (P2) if and only if

$$\forall \mathcal{M} \in \mathbb{M}(F), \forall t \in \mathbb{N}, \forall \omega \in \Omega, \forall \phi \in \Phi, \models_{\omega}^{\mathcal{M}} (B_t \phi \rightarrow B_t B_{t+1} \phi).$$

Thus, condition (C) holds in a frame $F \in \mathcal{F}_{KB}$ if and only if the formula $B_t \phi \leftrightarrow B_t B_{t+1} \phi$ (which says that at date t the individual believes ϕ if and only if she believes that at date $t+1$ she will believe ϕ) is valid in every model based on F . Note that this formula involves only the belief operators, while (C) is a condition that involves both the belief and the knowledge correspondences. This reduction of (C) to a condition only on beliefs is possible because of the interaction between knowledge and belief imposed by (R1) and (R2) (cf. Remark 1).

In order to obtain a further characterization of (C) we introduce the notions of consistency, belief set and knowledge set.

DEFINITION 1. A set of formulae $S \subseteq \Phi$ is *consistent* if, for every formula ϕ ,

$$\phi \in S \Rightarrow \neg \phi \notin S.$$

DEFINITION 2. Given a model \mathcal{M} , for every $t \in \mathbb{N}$ and $\omega \in \Omega$ let

$$\beta^{\mathcal{M}}(t, \omega) = \{\phi \in \Phi : \models_{\omega}^{\mathcal{M}} B_t \phi\} \text{ and}$$

$$\kappa^{\mathcal{M}}(t, \omega) = \{\phi \in \Phi : \models_{\omega}^{\mathcal{M}} K_t \phi\}.$$

Thus $\phi \in \beta^{\mathcal{M}}(t, \omega)$ if and only if $B_t(\omega) \subseteq \|\phi\|^{\mathcal{M}}$. Similarly, $\phi \in \kappa^{\mathcal{M}}(t, \omega)$ if and only if $\mathcal{K}_t(\omega) \subseteq \|\phi\|^{\mathcal{M}}$. We call $\beta^{\mathcal{M}}(t, \omega)$ the individual's *belief set* at date t and state ω in model \mathcal{M} and $\kappa^{\mathcal{M}}(t, \omega)$ her *knowledge set* (at t and ω). Note that, for every model \mathcal{M} based on a frame $F \in \mathcal{F}_{KB}$, and for all $t \in \mathbb{N}$

and $\omega \in \Omega$, $\beta^{\mathcal{M}}(t, \omega)$ and $\kappa^{\mathcal{M}}(t, \omega)$ are consistent, since \mathcal{B}_t and \mathcal{K}_t are serial (the latter is reflexive, hence serial).¹¹

Proposition 2 below says that (P1) is equivalent to the requirement that, moving from date t to date $t + 1$, every new belief be deducible from the knowledge acquired at date $t + 1$ and the beliefs at date t .

PROPOSITION 2. Fix an arbitrary frame $F \in \mathcal{F}_{KB}$. Then the following are equivalent:

- (1) F satisfies (P1),
- (2) $\forall \mathcal{M} \in \mathbb{M}(F), \forall t \in \mathbb{N}, \forall \omega \in \Omega, \forall \phi \in \Phi,$

if $\phi \in \beta^{\mathcal{M}}(t + 1, \omega)$ then there exists a $\psi \in \kappa^{\mathcal{M}}(t + 1, \omega)$ such that $(\psi \rightarrow \phi) \in \beta^{\mathcal{M}}(t, \omega)$,

(that is, if the individual at date $t + 1$ believes ϕ , then there must be something which she knows at $t + 1$ and which at date t she believed to imply ϕ).

Thus, by Proposition 2, condition (P1) is equivalent to ruling out 'arbitrary' expansions of the belief set. Proposition 3 below, on the other hand, shows that (P2) is equivalent to ruling out 'arbitrary' contractions of the belief set: imposing (P2) amounts to requiring that, if what is known at date $t + 1$ is consistent with what was believed at date t , then at date $t + 1$ the individual continue to believe everything that she believed at date t .

PROPOSITION 3. Fix an arbitrary frame $F \in \mathcal{F}_{KB}$. Then the following are equivalent:

- (1) F satisfies (P2),
- (2) $\forall \mathcal{M} \in \mathbb{M}(F), \forall t \in \mathbb{N}, \forall \omega \in \Omega,$

if $\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t + 1, \omega)$ is consistent then $\beta^{\mathcal{M}}(t, \omega) \subseteq \beta^{\mathcal{M}}(t + 1, \omega)$,

(that is, if what is known at date $t + 1$ does not contradict what is believed at date t , then everything which is believed at date t is still believed at date $t + 1$).

DEFINITION 3. Given a set of formulae $S \subseteq \Phi$, its *PL-deductive closure* $[S]^{\text{PL}}$ (where 'PL' stands for 'Propositional Logic') is defined as follows: $\psi \in [S]^{\text{PL}}$ if and only if there exist $\phi_1, \dots, \phi_n \in S$ such that $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$ is a propositional tautology (that is, a theorem of Propositional Logic).

It is well-known (see Chellas, 1984) that for every model \mathcal{M} based on

¹¹ It can be shown (see, in particular, the remark following Definition 3) that $\beta^{\mathcal{M}}(t, \omega)$ is a belief set in the sense of Gärdenfors (1988, p. 24); it also coincides with an 'acceptance state' as defined by Stalnaker (1984, pp. 81–2). The same holds for $\kappa^{\mathcal{M}}(t, \omega)$.

a frame $F \in \mathcal{F}_{KB}$, for every $t \in \mathbb{N}$ and $\omega \in \Omega$, $\beta^{\mathcal{M}}(t, \omega) = [\beta^{\mathcal{M}}(t, \omega)]^{PL}$ and $\kappa^{\mathcal{M}}(t, \omega) = [\kappa^{\mathcal{M}}(t, \omega)]^{PL}$.

The following corollary is the main result of this section.¹² It states that a KB-frame satisfies condition (C) if and only if – for every model based on it – if the union of the old belief set and the new knowledge set is consistent then its PL-deductive closure coincides with the new belief set.

COROLLARY 1. Fix an arbitrary frame $F \in \mathcal{F}_{KB}$. Then the following are equivalent:

- (1) F satisfies (C),
- (2) $\forall \mathcal{M} \in \mathbb{M}(F), \forall t \in \mathbb{N}, \forall \omega \in \Omega$,
if $\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)$ is consistent then $\beta^{\mathcal{M}}(t+1, \omega) = [\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)]^{PL}$.

We now turn to the syntactic analysis, to provide an axiomatization of (C) within the class of KB-frames.

3. THE SYNTAX OF BELIEF CHANGE

Let $\mathbf{K}_{KB}^{\text{time}}$ be the *system* or *calculus* specified by the following axiom schemata and rules of inference:

- (1) A suitable axiomatization of propositional calculus,
- (2) the following schemata¹³ (the names are as in Chellas, 1984), for every $t \in \mathbb{N}$,

| | | |
|---------|---|---------------------------------------|
| (K_K) | $K_t(\phi \rightarrow \psi) \rightarrow (K_t\phi \rightarrow K_t\psi),$ | |
| (K_B) | $B_t(\phi \rightarrow \psi) \rightarrow (B_t\phi \rightarrow B_t\psi),$ | |
| (T_K) | $K_t\phi \rightarrow \phi$ | (truth or veridicality of knowledge), |
| (4_K) | $K_t\phi \rightarrow K_tK_t\phi$ | (positive introspection of knowledge) |
| (5_K) | $\neg K_t\phi \rightarrow K_t\neg K_t\phi$ | (negative introspection of knowledge) |
| (D_B) | $B_t\phi \rightarrow \neg B_t\neg\phi$ | (consistency of beliefs) |
| (4_B) | $B_t\phi \rightarrow B_tB_t\phi$ | (positive introspection of beliefs) |

¹² We are grateful to Tim Van Zandt for prompting us to prove this result.

¹³ The axiomatization given below is not minimal. In particular, the positive and negative introspection axioms for the belief operator are redundant, since they can be derived from the remaining axioms (semantically, if \mathcal{K}_t is reflexive and euclidean, \mathcal{B}_t is serial and (R1) and (R2) are satisfied, then \mathcal{B}_t is transitive and euclidean: cf. Kraus and Lehmann, 1988). Furthermore, it is well known (cf. Chellas, 1984) that the positive introspection axiom for knowledge can be derived from the other two axioms for knowledge. We chose this non-minimal formulation in order to make the discussion of Section 4 easier to follow.

(5_B) $\neg B_t \phi \rightarrow B_t \neg B_t \phi$ (negative introspection of beliefs)

(A1) $K_t \phi \rightarrow B_t \phi$ (entailment axiom)

(A2) $B_t \phi \rightarrow K_t B_t \phi$ (knowledge of own beliefs)

(3) the rule of inference *Modus Ponens*:

$$\text{MP} \quad \frac{\phi, \phi \rightarrow \psi}{\psi}$$

(4) the rule of inference *Necessitation*: for every $t \in \mathbb{N}$,

$$\frac{\phi}{K_t \phi} \quad \text{and} \quad \frac{\phi}{B_t \phi}.$$

Unlike the other axioms, perhaps (A1) and (A2) are less well-known. (A1) says that the individual believes everything that she knows. It is referred to in the philosophical literature as ‘the entailment thesis’. (A2) says that if the individual believes something then she knows that she believes it. Both axioms were first discussed by Hintikka (1962), who defended (A1) but rejected (A2). Lenzen (1978) provides a detailed discussion of these two axioms and refutes attempts to reject them. Concerning the entailment thesis, Lenzen concludes (p. 27) that it ‘is sounder than its critics claim. It may be regarded as a cornerstone of epistemic logic’. Concerning (A2), Lenzen finds Hintikka’s rejection surprising; he endorses (A2) after a detailed discussion of the arguments that have been put forward against it. Both axioms are implicit in the economics of information literature, where beliefs are represented by subjective probability measures conditional on the information received. Indeed (A1) corresponds to the requirement that the subjective probability measure assign probability 1 to the information received and (A2) corresponds to the assumption that, within each element of the information partition, the subjective probability measure be independent of the state (cf. Remark 1).

Let Σ be an extension of K_{KB}^{time} , that is, Σ is either K_{KB}^{time} itself or a system obtained by adding one or more axiom schemata to K_{KB}^{time} . Let \mathbb{M} be a class of models. Then we say that Σ is *sound* with respect to \mathbb{M} if every theorem of Σ (that is, every axiom and every formula that can be obtained, in a finite number of steps, from the axioms using the rules of inference¹⁴) is valid in every model in \mathbb{M} and we say that Σ is *complete* with respect to \mathbb{M} if every formula that is valid in every model in \mathbb{M} is a theorem of Σ .

¹⁴ Formally: ϕ is a *theorem* of Σ if and only if there exist $m \geq 1$ formulae ϕ_1, \dots, ϕ_m such that $\phi_m = \phi$ and, for every $k \leq m$, ϕ_k is either an axiom or is obtained from some ϕ_i and ϕ_j (with $i, j < k$) by applying one of the rules of inference.

REMARK 2. Let $\mathbb{M}(\mathcal{F}_{KB})$ be the class of models based on the frames in \mathcal{F}_{KB} . Then $\mathbf{K}_{KB}^{\text{time}}$ is sound and complete with respect to $\mathbb{M}(\mathcal{F}_{KB})$.¹⁵

We can now turn to the main purpose of this section, which is the axiomatization of (C), the qualitative version of the conditionalization rule. Consider the following axiom schemata, which were first introduced in Proposition 1: for all $t \in \mathbb{N}$,

$$(A3) \ B_t B_{t+1} \phi \rightarrow B_t \phi$$

$$(A4) \ B_t \phi \rightarrow B_t B_{t+1} \phi$$

(A3) says that if at date t the individual believes that she will believe ϕ at date $t + 1$, then she must believe ϕ at date t . (A4) says the converse: if the individual believes ϕ at date t , then she must also believe, at date t , that she will believe ϕ at date $t + 1$.

First we show that (A3) and (A4) are equivalent to more general axioms which look more than one date into the future. Consider the following axiom schemata:

$$\forall t \in \mathbb{N}, \forall k \in \mathbb{N}, k \geq 1$$

$$(A3') \ B_t B_{t+k} \phi \rightarrow B_t \phi$$

$$(A4') \ B_t \phi \rightarrow B_t B_{t+k} \phi.$$

(A3') says that if the individual believes at date t that she will believe ϕ at any future date, then she believes ϕ at date t , while (A4') says the converse. Clearly, (A3) and (A4) are instances (with $k = 1$) of (A3') and (A4'), respectively. Let $\mathbf{K}_{KB}^{\text{time}} + (A3) + (A4)$ be the system obtained by adding (A3) and (A4) to $\mathbf{K}_{KB}^{\text{time}}$. By Proposition 4 below, $\mathbf{K}_{KB}^{\text{time}} + (A3) + (A4)$ is equivalent to the system obtained by adding to $\mathbf{K}_{KB}^{\text{time}}$ all the instances of (A3') and (A4') (for the definition of a theorem see Footnote 14).

PROPOSITION 4. Every instance of (A3') and (A4') is a theorem of $\mathbf{K}_{KB}^{\text{time}} + (A3) + (A4)$.

The following proposition is the main result of this section. It states that the conjunction of (A3) and (A4) provides an axiomatization of (C) within the class of KB-frames. Let \mathcal{F}_{KB}^C be the subset of \mathcal{F}_{KB} consisting of those frames that satisfy (C) and let $\mathbb{M}(\mathcal{F}_{KB}^C)$ be the class of models based on frames in \mathcal{F}_{KB}^C .

PROPOSITION 5. $\mathbf{K}_{KB}^{\text{time}} + (A3) + (A4)$ is sound and complete with respect to $\mathbb{M}(\mathcal{F}_{KB}^C)$.

¹⁵ This is a straightforward extension (to our temporal framework) of a result by Kraus and Lehmann (1988). See Section 5 for further discussion.

4. DISCUSSION

In the information economics literature it is standard to assume that the individual, as time progresses, never forgets what she knew before and possibly acquires more information. Such an assumption, called 'perfect recall',¹⁶ is captured semantically by the requirement that the information partition at time $t+1$ be a refinement of the information partition at time t :

$$(R5) \quad \forall t \in \mathbb{N}, \forall \omega \in \Omega, \quad \mathcal{K}_{t+1}(\omega) \subseteq \mathcal{K}_t(\omega)$$

REMARK 3. Let $F \in \mathcal{F}_{KB}$ be a frame that satisfies (R5). Then the following schema is valid in every model based on F :

$$(A5) \quad K_t \phi \rightarrow K_{t+1} \phi.$$

Furthermore, if (A5) is added to $\mathbf{K}_{KB}^{\text{time}}$ (respectively $\mathbf{K}_{KB}^{\text{time}} + (A3) + (A4)$), then the resulting system is sound and complete with respect to the class of models based on KB-frames that satisfy (R5) (respectively (C) and (R5)).¹⁷

(A5) says that if the individual knows ϕ at date t then she still knows ϕ at date $t+1$. There are other axioms, besides (A5), that could be proposed as an expression of the notion of perfect recall. For example, one could find (A6) or (A7) below more appealing:

$$(A6) \quad \neg K_t \phi \rightarrow K_{t+1} \neg K_t \phi$$

$$(A7) \quad K_t \phi \rightarrow K_{t+1} K_t \phi$$

(A6) says that if the individual doesn't know something at date t then at date $t+1$ she knows that she did not know, while (A7) says that if the individual knows something at date t then at date $t+1$ she will know that she knew. Perhaps not surprisingly, the following proposition shows that, granting our basic system, these three axioms are equivalent.

PROPOSITION 6. The following three systems are equivalent:

- (1) $\mathbf{K}_{KB}^{\text{time}} + (A5)$, (2) $\mathbf{K}_{KB}^{\text{time}} + (A6)$, and (3) $\mathbf{K}_{KB}^{\text{time}} + (A7)$.

Note that the perfect recall axiom (A5) is *not* a theorem of $\mathbf{K}_{KB}^{\text{time}} + (A3) + (A4)$ (and neither is *belief* in perfect recall: $B_t(K_t \phi \rightarrow K_{t+1} \phi)$).¹⁸ For

¹⁶ In the game theoretic literature this assumption is sometimes called 'perfect recall of information sets', whereas 'perfect recall' refers to a stronger assumption (see, for example, Piccione and Rubinstein, 1995).

¹⁷ This result can be obtained as an application of Theorem 4.3 in van der Hoek (1993, p. 183).

¹⁸ Consider the following KB-frame: $\Omega = \{\alpha, \beta, \gamma\}$, $\mathcal{K}_0(\alpha) = \mathcal{K}_0(\beta) = \{\alpha, \beta\}$, $\mathcal{B}_0(\alpha) = \mathcal{B}_0(\beta) = \{\alpha\}$, $\mathcal{K}_0(\gamma) = \mathcal{B}_0(\gamma) = \{\gamma\}$, and, for all $t \geq 1$, $\mathcal{K}_t(\alpha) = \mathcal{B}_t(\alpha) = \{\alpha\}$, $\mathcal{K}_t(\beta) = \mathcal{K}_t(\gamma) = \{\beta, \gamma\}$, $\mathcal{B}_t(\beta) = \mathcal{B}_t(\gamma) = \{\gamma\}$. This KB-frame satisfies (C). Let \mathcal{M} be a model based on this frame where $\|\pi\| = \{\gamma\}$. Then both $K_0 \pi \rightarrow K_1 \pi$ and $B_0(K_0 \pi \rightarrow K_1 \pi)$ are false at γ . Thus, by

the sake of parsimony, the results of Sections 2 and 3 were obtained *without* assuming perfect recall. But in a context in which perfect recall is not assumed, condition (C) has no motivation. Consider, for example, a temporal sequence of information partitions *with imperfect recall* and suppose that the individual derives her conditional probabilistic beliefs from a *full support prior*. Then, considering the supports of the conditional probability distributions, one obtains a KB-frame (in particular, (R1) and (R2) will be automatically satisfied) in which condition (C) is violated. Another clear example is the simplest instance of imperfect recall, namely the case of pure loss of information, where a cell of the information partition at time $t+1$ contains the union of two or more cells of the information partition at time t . In this case (R2) is incompatible with (C).¹⁹

One might want to consider a weaker system than $\mathbf{K}_{\text{KB}}^{\text{time}} + (\text{A3}) + (\text{A4})$ where the truth axiom $K_t\phi \rightarrow \phi$ is *not* postulated for the knowledge operator and is replaced by the consistency axiom $K_t\phi \rightarrow \neg K_t\neg\phi$ (semantically the information function \mathcal{K}_t is serial but not necessarily reflexive). Such a system would represent situations where beliefs are still based on information, but information is not necessarily veridical, that is, the individual might receive inaccurate information and form beliefs based on it.²⁰ In such a system, both ‘knowledge’ and belief would satisfy the logic of KD45. Axiom (A1) would still establish the epistemic priority of the beliefs represented by K_t (information) over the beliefs represented by B_t , so that the receipt of new information might prompt a belief revision. Let $\mathbf{K}_{\text{KD45}}^{\text{time}}$ denote the system obtained from $\mathbf{K}_{\text{KB}}^{\text{time}}$ by replacing the truth axiom for K_t with the consistency axiom. Define a *KD45-frame* as a structure $\langle \Omega, \{\mathcal{K}_t\}_{t \in \mathbb{N}}, \{\mathcal{B}_t\}_{t \in \mathbb{N}} \rangle$, where Ω is a non-empty set, $\mathcal{K}_t : \Omega \rightarrow 2^\Omega$

Proposition 5, neither (A5) nor the schema $B_t(K_t\phi \rightarrow K_{t+1}\phi)$ are theorems of $\mathbf{K}_{\text{KB}}^{\text{time}} + (\text{A3}) + (\text{A4})$.

¹⁹ Furthermore, restricting attention (for simplicity) to the case of two dates, the following is true. Let $F = \langle \Omega, \mathcal{K}_0, \mathcal{K}_1 \rangle$ be such that \mathcal{K}_t ($t = 0, 1$) is reflexive and euclidean. If F satisfies (R5), then, for every serial, transitive and euclidean (STE) \mathcal{B}_0 that satisfies (R1) and (R2) there is an STE \mathcal{B}_1 that satisfies (R1), (R2) such that $\langle \Omega, \mathcal{K}_0, \mathcal{K}_1, \mathcal{B}_0, \mathcal{B}_1 \rangle$ satisfies (C). On the other hand, if F violates (R5) then there is an STE \mathcal{B}_0 that satisfies (R1) and (R2) such that, for every STE \mathcal{B}_1 that satisfies (R1) and (R2), $\langle \Omega, \mathcal{K}_0, \mathcal{K}_1, \mathcal{B}_0, \mathcal{B}_1 \rangle$ violates (C).

²⁰ For example, the analysis of extensive games usually involves two different epistemic levels: a player receives information which is encoded in the description of the extensive game (information sets encode information about past moves and the payoff functions encode information about preferences: this corresponds to \mathcal{K} in our framework), and forms beliefs at every information set (in the form of a conditional probability distribution) concerning unobserved past moves and future moves (this corresponds to our \mathcal{B}). Although it is usually assumed that a player’s information is veridical, in general it need not be. For example, a player might be mistaken about the preferences of her opponents. (In their textbook on game theory Osborne and Rubinstein, 1994, p. 135, write: ‘In our view a model should attempt to capture the features of reality that the players *perceive*; it should not necessarily aim to describe the reality that an outside observer perceives, though obviously there are links between the two perceptions’.)

and $B_t : \Omega \rightarrow 2^\Omega$ are serial, transitive and euclidean and together they satisfy (R1) and (R2). Let $\mathcal{F}_{\text{KD45}}$ be the class of KD45-frames, $\mathcal{F}_{\text{KD45}}^C$ the subset of those frames that satisfy (C) and $\mathbb{M}(\mathcal{F}_{\text{KD45}}^C)$ the class of models based on frames in $\mathcal{F}_{\text{KD45}}^C$. The following proposition states that the conjunction of (A3) and (A4) no longer characterizes (C).

PROPOSITION 7. $\mathbf{K}_{\text{KB}}^{\text{time}} + (\text{A3}) + (\text{A4})$ is sound and complete with respect to a proper subset of $\mathbb{M}(\mathcal{F}_{\text{KD45}}^C)$.²¹

$\mathbf{K}_{\text{KB}}^{\text{time}} + (\text{A3}) + (\text{A4})$ is complete with respect to $\mathbb{M}(\mathcal{F}_{\text{KD45}}^C)$ but not sound. The following example shows that soundness fails. Let F be the following KD45-frame: $\Omega = \{\alpha, \beta\}$, $\mathcal{K}_0(\alpha) = \mathcal{K}_0(\beta) = \mathcal{B}_0(\alpha) = \mathcal{B}_0(\beta) = \{\beta\}$ and for all $t \geq 1$, $\mathcal{K}_t(\alpha) = \mathcal{K}_t(\beta) = \mathcal{B}_t(\alpha) = \mathcal{B}_t(\beta) = \{\alpha\}$. Then F satisfies (C) (vacuously). Let \mathcal{M} be a model based on F where $\|\pi\| = \{\beta\}$. Then at α both $B_0\pi \rightarrow B_0B_1\pi$ and $B_0B_1\neg\pi \rightarrow B_0\neg\pi$ are false. Hence this model does not validate (A3) and (A4).

5. RELATED LITERATURE

The notion of belief change has been an active area of research in philosophy and artificial intelligence. Of particular relevance to this paper is the literature that attempts to describe how an agent should accommodate a new belief (possibly inconsistent with her other beliefs) about a static world (Alchourrón, Gärdenfors and Makinson, 1985, Gärdenfors, 1988; Gärdenfors and Makinson, 1988; see also Stalnaker, 1984). In this literature an agent's beliefs are represented as a logically closed set of formulae in some language, and then constraints are put on how these beliefs can change in order to accommodate a new formula (representing new information). The main goal of this approach, as Gärdenfors (1988, p. 2) states, is to present *rationality criteria* for belief changes. The guiding principle is that (1988, p. 8) 'when evaluating changes of belief, we require that the change be the *minimal* one needed to accommodate the epistemic input that generates the change'. One of the results in this literature is the representation theorem for belief expansion. First we need to introduce some notation and definitions. Gärdenfors (p. 24) defines a belief set as a *proper* subset β of the set of formulae such that (using our notation: cf. Definition 3) $\beta = [\beta]^{\text{PL}}$. Let β be a belief set and ψ a formula such that $\neg\psi \notin \beta$. The formula ψ represents an 'epistemic input', for example, a piece of information received by the individual (thus the condition $\neg\psi \notin \beta$ says that the epistemic input does not contradict her previous beliefs). The new formula ψ must be accommodated in the belief set, leading to a new belief set β_ψ^+ , called the *expansion* of β by ψ . Gärdenfors (1988, pp. 48–51) gives a list of 'rationality conditions' relating to β and β_ψ^+ and proves (Theorem 3.1, p. 51) that those conditions are

²¹ This proper subset is explicitly characterized in the proof contained in the appendix.

satisfied if and only if (in our notation) $\beta_{\psi}^{+} = [\{\psi\} \cup \beta]^{PL}$. The close relationship between this result and Corollary 1 is apparent. However, unlike ours, Gärdenfors' approach is primarily concerned with the case in which the epistemic input contradicts the individual's prior beliefs. In this case 'revision' necessarily differs from 'expansion'. Also, Gärdenfors' work is not based on an explicit representation of time; instead potential later belief states are defined by the new information that might be received. In contrast, we index the belief states by time, leaving the content of the information that induces the change in the belief state implicit. Furthermore, Gärdenfors' analysis is not carried out within a two-level structure that explicitly distinguishes between information (evidence, knowledge) and beliefs.

Our analysis is based on a logic with two modal operators, the knowledge operator, representing information, and the belief operator. We already mentioned that the *atemporal* relationship between knowledge and belief was first discussed by Hintikka (1962) and Lenzen (1978).²² Kraus and Lehmann (1988) presented a formal analysis focused on soundness and completeness results. They also analyzed an atemporal multi-agent logic with operators for common knowledge and common belief. In the last part of the paper the authors considered the possibility of extending the logic to include time. In particular, they addressed the question of how to characterize the notion of persistence of beliefs: 'if person *i* believes *something*, he will keep on believing it until he knows it is false' (1988, p. 107, our emphasis). If 'something' were interpreted as the conjunction of all the propositions the individual believes, this would correspond to our property (P2) (see Proposition 3). But Kraus and Lehmann mean 'an arbitrary proposition'. In fact, they put forward an axiom schema that in our framework would be written as follows:²³

$$(A^*) \quad B_t\phi \rightarrow B_{t+1}\phi \vee K_{t+1}\neg\phi.$$

(A*) says that if at date *t* the individual believes ϕ , then at date *t* + 1 either

²² One property of systems that model the interaction of knowledge and belief is that if one adds to (A1) and (A2) the axiom schema $B\phi \rightarrow BK\phi$ (if you believe something then you believe that you know it) then knowledge and belief become identical, that is, one obtains the theorem $B\phi \leftrightarrow K\phi$. This point was first discussed by Lenzen (1978) and later taken up by van der Hoek (1993) and Halpern (forthcoming). Van der Hoek provides an extensive analysis of the causes of this 'problem' and of a similar system that allows one to introduce the axiom $B\phi \rightarrow BK\phi$ without obtaining a collapse of belief into knowledge. Halpern shows that the problem can be avoided if the axiom schema $B\phi \rightarrow BK\phi$ is weakened so that it applies only to non-modal formulae.

²³ While time enters our analysis explicitly only through the knowledge and belief operators, Kraus and Lehmann introduced a time operator \bigcirc in the syntax, where $\bigcirc\phi$ would be interpreted as 'at the next date ϕ '. Thus (A*) below does not coincide with any of the axioms they considered. However, it can be viewed as the translation, within our framework, of their axiom (A21).

she knows that ϕ is false or she still believes ϕ . It should be clear that in non trivial models based on KB-frames (A^*) cannot be valid. To see this, suppose that at date t the individual believes that both ϕ and ψ are true, and at date $t + 1$ she learns that either ϕ or ψ is false, but according to her new knowledge neither ϕ nor ψ can be ruled out as false. By (A^*) the individual should believe both ϕ and ψ at date $t + 1$, but this would contradict either (A1) (entailment axiom) or (D_B) (consistency of beliefs).²⁴ Besides (A^*), Kraus and Lehmann briefly discussed a number of possible axioms (all of them different from our (A3) and (A4)) and concluded by saying that 'An open problem is: find a natural family of models for which the systems considered above are complete'.²⁵

Some of the papers mentioned above deal with the more general case where there are $n \geq 1$ individuals, whereas we have restricted attention to the case of one individual. It should be clear, however, that our approach can be extended to the multi-agent case in order to analyze dynamic, interactive epistemic systems.

APPENDIX

Proof of Proposition 1. Fix an arbitrary frame $F \in \mathcal{F}_{KB}$. First we show that (P1) is equivalent to the following:

(R3) $\forall \omega, \omega' \in \Omega, \forall t \in \mathbb{N}$, if $\omega' \in \mathcal{B}_t(\omega)$ then there exists an $\omega'' \in \mathcal{B}_t(\omega)$ such that $\omega' \in \mathcal{B}_{t+1}(\omega'')$.

(R3) \Rightarrow (P1). Fix arbitrary t , ω and ω' such that $\omega' \in \mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega)$. We want to show that $\omega' \in \mathcal{B}_{t+1}(\omega)$. By (R3) there exists an $\omega'' \in \mathcal{B}_t(\omega)$ such that $\omega' \in \mathcal{B}_{t+1}(\omega'')$. Since \mathcal{B}_{t+1} is euclidean, $\omega' \in \mathcal{B}_{t+1}(\omega')$. By (R2), since $\omega' \in \mathcal{K}_{t+1}(\omega)$, $\mathcal{B}_{t+1}(\omega') \subseteq \mathcal{B}_{t+1}(\omega)$. Hence $\omega' \in \mathcal{B}_{t+1}(\omega)$.

(P1) \Rightarrow (R3). Fix arbitrary t , ω and ω' such that $\omega' \in \mathcal{B}_t(\omega)$. We want to show that there exists an $\omega'' \in \mathcal{B}_t(\omega)$ such that $\omega' \in \mathcal{B}_{t+1}(\omega'')$. Take $\omega'' = \omega'$. Then we only have to show that $\omega' \in \mathcal{B}_{t+1}(\omega')$. By reflexivity of

²⁴ For example, consider a model where: $\Omega = \{\alpha, \beta, \gamma\}$, $\forall \omega \in \Omega, \mathcal{K}_0(\omega) = \Omega, \mathcal{B}_0(\omega) = \{\alpha\}$, for all $t \geq 1, \mathcal{K}_t(\alpha) = \{\alpha\}, \mathcal{K}_t(\beta) = \mathcal{K}_t(\gamma) = \mathcal{B}_t(\beta) = \mathcal{B}_t(\gamma) = \{\beta, \gamma\}, \|\phi\| = \{\alpha, \beta\}, \|\psi\| = \{\alpha, \gamma\}$. Then $B_0\phi, B_0\psi, \neg K_1\neg\phi, \neg K_1\neg\psi$ and $K_1\neg(\phi \wedge \psi)$ hold at β and γ . Thus (A1) implies that $B_1\neg(\phi \wedge \psi)$ holds at β and γ , while (A^*) would imply that $B_1(\phi \wedge \psi)$ holds at β and γ .

²⁵ Van der Hoek and Meyer (1995, p. 96) review the axioms proposed by Kraus and Lehmann (1988) and state 'As far as is known to us, it is still an open problem to find a natural family of models with respect to which these systems are complete'. In a very recent paper Friedman and Halpern (1995) provide a general semantic analysis of belief change in terms of knowledge and plausibility, the latter being a generalization of probabilistic beliefs (an agent believes ϕ if he knows that $\neg\phi$ is implausible). In the last section they introduce time into their analysis and give an axiom (which they call COH, for coherence) which is somewhat related to our axioms (A3) and (A4). However, in what would be their analogue of our framework, the system containing axiom COH is sound but not complete (Halpern, private communication).

\mathcal{K}_{t+1} , $\omega' \in \mathcal{K}_{t+1}(\omega')$. By euclideaness of \mathcal{B}_t , since $\omega' \in \mathcal{B}_t(\omega)$, $\omega' \in \mathcal{B}_t(\omega')$. By (P1), since $\omega' \in \mathcal{B}_t(\omega') \cap \mathcal{K}_{t+1}(\omega')$, $\omega' \in \mathcal{B}_{t+1}(\omega')$.

Next we show that (P2) is equivalent to the following:

(R4) $\forall \omega, \omega', \omega'' \in \Omega$, $\forall t \in \mathbb{N}$, if $\omega' \in \mathcal{B}_t(\omega)$ and $\omega'' \in \mathcal{B}_{t+1}(\omega')$ then $\omega'' \in \mathcal{B}_t(\omega)$.

(R4) \Rightarrow (P2). Fix arbitrary t , ω and ω' such that $\omega' \in \mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega)$ (so that $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \neq \emptyset$). We want to show that $\mathcal{B}_{t+1}(\omega) \subseteq \mathcal{B}_t(\omega)$. Fix an $\hat{\omega} \in \mathcal{B}_{t+1}(\omega')$ (it exists because \mathcal{B}_{t+1} is serial). By (R4), since $\omega' \in \mathcal{B}_t(\omega)$ and $\hat{\omega} \in \mathcal{B}_{t+1}(\omega')$, $\hat{\omega} \in \mathcal{B}_t(\omega)$. By (R2), since $\omega' \in \mathcal{K}_{t+1}(\omega)$, $\mathcal{B}_{t+1}(\omega') \subseteq \mathcal{B}_{t+1}(\omega)$. Hence $\hat{\omega} \in \mathcal{B}_{t+1}(\omega)$. It follows from euclideaness of \mathcal{B}_{t+1} that $\mathcal{B}_{t+1}(\omega) \subseteq \mathcal{B}_{t+1}(\hat{\omega})$. Now choose an arbitrary $\omega'' \in \mathcal{B}_{t+1}(\omega)$. Then $\omega'' \in \mathcal{B}_{t+1}(\hat{\omega})$. By (R4), since $\hat{\omega} \in \mathcal{B}_t(\omega)$ and $\omega'' \in \mathcal{B}_{t+1}(\hat{\omega})$, $\omega'' \in \mathcal{B}_t(\omega)$. Thus $\mathcal{B}_{t+1}(\omega) \subseteq \mathcal{B}_t(\omega)$.

(P2) \Rightarrow (R4). Fix arbitrary t , ω , ω' and ω'' such that $\omega' \in \mathcal{B}_t(\omega)$ and $\omega'' \in \mathcal{B}_{t+1}(\omega')$. We want to show that $\omega'' \in \mathcal{B}_t(\omega)$. By reflexivity of \mathcal{K}_{t+1} , $\omega' \in \mathcal{K}_{t+1}(\omega')$. Since $\omega' \in \mathcal{B}_t(\omega)$, by euclideaness and transitivity of \mathcal{B}_t , $\mathcal{B}_t(\omega) = \mathcal{B}_t(\omega')$. Thus $\omega' \in \mathcal{B}_t(\omega')$. By (P2), since $\omega' \in \mathcal{B}_t(\omega') \cap \mathcal{K}_{t+1}(\omega')$, $\mathcal{B}_{t+1}(\omega') \subseteq \mathcal{B}_t(\omega')$. Hence $\omega'' \in \mathcal{B}_t(\omega) = \mathcal{B}_t(\omega')$.

To complete the proof we invoke Theorem 4.3 in van der Hoek (1993, p.183), which, adapted to our framework, states that a frame satisfies (R3) if and only if for every model \mathcal{M} based on it and for all t and ϕ , the formula $(\mathcal{B}_t \mathcal{B}_{t+1} \phi \rightarrow \mathcal{B}_t \phi)$ is valid in \mathcal{M} and it satisfies (R4) if and only if for every model \mathcal{M} based on it and for all t and ϕ , the formula $(\mathcal{B}_t \phi \rightarrow \mathcal{B}_t \mathcal{B}_{t+1} \phi)$ is valid in \mathcal{M} . ■

Proof of Proposition 2. (1) \Rightarrow (2). Let F be a frame that satisfies (P1). Fix an arbitrary model $\mathcal{M} \in \mathbb{M}(F)$ and arbitrary $t \in \mathbb{N}$, $\omega \in \Omega$ and $\phi \in \beta(t+1, \omega)$ (to simplify the notation we omit the superscript \mathcal{M}). Then $\models_{\omega} \mathcal{B}_{t+1} \phi$. By (R2) (cf. Remark 1) $\models_{\omega} (\mathcal{B}_{t+1} \phi \rightarrow \mathcal{K}_{t+1} \mathcal{B}_{t+1} \phi)$. Hence $\models_{\omega} \mathcal{K}_{t+1} \mathcal{B}_{t+1} \phi$. Thus

$$\mathcal{B}_{t+1} \phi \in \kappa(t+1, \omega). \quad (\text{a.1})$$

By euclideaness of \mathcal{B}_{t+1} , for every $\alpha \in \Omega$, $\models_{\alpha} \mathcal{B}_{t+1}(\mathcal{B}_{t+1} \phi \rightarrow \phi)$ (it is a theorem of the logic K5 (see Section 3), hence valid in every euclidean model). Thus $\models_{\omega} \mathcal{B}_t \mathcal{B}_{t+1}(\mathcal{B}_{t+1} \phi \rightarrow \phi)$. Since F satisfies (P1), by (1) of Proposition 1, $\models_{\omega} \mathcal{B}_t \mathcal{B}_{t+1}(\mathcal{B}_{t+1} \phi \rightarrow \phi) \rightarrow \mathcal{B}_t(\mathcal{B}_{t+1} \phi \rightarrow \phi)$. Thus $\models_{\omega} \mathcal{B}_t(\mathcal{B}_{t+1} \phi \rightarrow \phi)$, that is,

$$(\mathcal{B}_{t+1} \phi \rightarrow \phi) \in \beta(t, \omega). \quad (\text{a.2})$$

Hence, by (a.1) and (a.2), (2) is satisfied with $\psi = \mathcal{B}_{t+1} \phi$.

Not (1) \Rightarrow not (2). Suppose that F does not satisfy (P1). Then there exist t and ω such that $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \not\subseteq \mathcal{B}_{t+1}(\omega)$, that is, there exists an $\alpha \in \mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega)$ such that $\alpha \notin \mathcal{B}_{t+1}(\omega)$. Let π be a sentence letter and $\mathcal{M} \in \mathbb{M}(F)$ be a model where $\|\pi\| = \mathcal{B}_{t+1}(\omega)$. Then $\pi \in \beta(t+1, \omega)$. Choose

an arbitrary $\psi \in \kappa(t+1, \omega)$. Then $\mathcal{K}_{t+1}(\omega) \subseteq \|\psi\|$. Since $\alpha \in \mathcal{B}_t(\omega)$ and $\alpha \in \mathcal{K}_{t+1}(\omega) \cap -\mathcal{B}_{t+1}(\omega) \subseteq \|\psi\| \cap \|\neg\pi\| = \|\psi \wedge \neg\pi\| = \|\neg(\psi \rightarrow \pi)\|$, $(\psi \rightarrow \pi) \notin \beta(t, \omega)$, that is, (2) is violated. ■

Remark A.1. The proof of Proposition 2 does not make use of reflexivity of \mathcal{K}_t nor of transitivity of \mathcal{B}_t and \mathcal{K}_t . Thus Proposition 2 holds in a larger class than \mathcal{F}_{KB} .

To prove Proposition 3 we need the following Lemma.

Lemma A.1. Fix an arbitrary frame $F \in \mathcal{F}_{\text{KB}}$ and arbitrary $t \in \mathbb{N}$ and $\omega \in \Omega$. Then the following are equivalent:

- (1) $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \neq \emptyset$,
- (2) for every $\mathcal{M} \in \mathbb{M}(F)$, $\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t, \omega)$ is consistent.

Proof. Not (2) \Rightarrow not (1). Let $\mathcal{M} \in \mathbb{M}(F)$ be a model where $\beta(t, \omega) \cup \kappa(t+1, \omega)$ is inconsistent. Then, since both $\beta(t, \omega)$ and $\kappa(t, \omega)$ are consistent, there must be a formula ϕ such that $\phi \in \beta(t, \omega)$ and $\neg\phi \in \kappa(t, \omega)$, that is, $\mathcal{B}_t(\omega) \subseteq \|\phi\|$ and $\mathcal{K}_{t+1}(\omega) \subseteq \|\neg\phi\| = -\|\phi\|$. Then $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \subseteq \|\phi\| \cap -\|\phi\| = \emptyset$.

Not (1) \Rightarrow not (2). Suppose that $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) = \emptyset$, that is, $\mathcal{K}_{t+1}(\omega) \subseteq -\mathcal{B}_t(\omega)$. Let π be a sentence letter and $\mathcal{M} \in \mathbb{M}(F)$ a model where $\|\pi\| = \mathcal{B}_t(\omega)$. Then $\pi \in \beta(t, \omega)$ and $\mathcal{K}_{t+1}(\omega) \subseteq \|\neg\pi\|$, that is, $\neg\pi \in \kappa(t+1, \omega)$ and, therefore, $\beta(t, \omega) \cup \kappa(t+1, \omega)$ is inconsistent. ■

Proof of Proposition 3. (1) \Rightarrow (2). Let $F \in \mathcal{F}_{\text{KB}}$ be a frame that satisfies (P2). Fix arbitrary $\mathcal{M} \in \mathbb{M}(F)$, $t \in \mathbb{N}$ and $\omega \in \Omega$ such that $\beta(t, \omega) \cup \kappa(t+1, \omega)$ is consistent (once again we drop the superscript \mathcal{M}). Choose an arbitrary $\phi \in \beta(t, \omega)$, i.e. $\models_{\omega} \mathcal{B}_t \phi$. By (2) of Proposition 1, $\models_{\omega} \mathcal{B}_t \phi \rightarrow \mathcal{B}_t \mathcal{B}_{t+1} \phi$. Hence $\models_{\omega} \mathcal{B}_t \mathcal{B}_{t+1} \phi$, that is, $\mathcal{B}_{t+1} \phi \in \beta(t, \omega)$. Since $\beta(t, \omega) \cup \kappa(t+1, \omega)$ is consistent, $\neg \mathcal{B}_{t+1} \phi \notin \kappa(t+1, \omega)$ i.e. $\mathcal{K}_{t+1}(\omega) \not\subseteq -\|\mathcal{B}_{t+1} \phi\|$. Thus there exists an $\alpha \in \mathcal{K}_{t+1}(\omega)$ such that $\models_{\alpha} \mathcal{B}_{t+1} \phi$. By (R2) (cf. Remark 1) $\models_{\alpha} \mathcal{B}_{t+1} \phi \rightarrow \mathcal{K}_{t+1} \mathcal{B}_{t+1} \phi$. Hence $\models_{\alpha} \mathcal{K}_{t+1} \mathcal{B}_{t+1} \phi$ and, therefore, $\models_{\omega} \neg \mathcal{K}_{t+1} \neg \mathcal{K}_{t+1} \mathcal{B}_{t+1} \phi$. Since \mathcal{K}_{t+1} is euclidean, $\models_{\omega} \neg \mathcal{K}_{t+1} \neg \mathcal{K}_{t+1} \mathcal{B}_{t+1} \phi \rightarrow \mathcal{K}_{t+1} \mathcal{B}_{t+1} \phi$. Hence $\models_{\omega} \mathcal{K}_{t+1} \mathcal{B}_{t+1} \phi$. By (R1) (cf. Remark 1) $\models_{\omega} \mathcal{K}_{t+1} \mathcal{B}_{t+1} \phi \rightarrow \mathcal{B}_{t+1} \mathcal{B}_{t+1} \phi$. Thus $\models_{\omega} \mathcal{B}_{t+1} \mathcal{B}_{t+1} \phi$. By euclideaness of \mathcal{B}_{t+1} , $\models_{\omega} \mathcal{B}_{t+1} \mathcal{B}_{t+1} \phi \rightarrow \mathcal{B}_{t+1} \phi$ (it is a theorem of the logic K5 (see Section 3) hence valid in every euclidean frame). Thus $\models_{\omega} \mathcal{B}_{t+1} \phi$, that is, $\phi \in \beta(t+1, \omega)$. Since $\phi \in \beta(t, \omega)$ was chosen arbitrarily, $\beta(t, \omega) \subseteq \beta(t+1, \omega)$.

not (1) \Rightarrow not (2). Let $F \in \mathcal{F}_{\text{KB}}$ be a frame that violates (P2). Then there exist t and ω such that $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \neq \emptyset$ and $\mathcal{B}_{t+1}(\omega) \not\subseteq \mathcal{B}_t(\omega)$, that is, there exists an $\alpha \in \mathcal{B}_{t+1}(\omega)$ such that $\alpha \notin \mathcal{B}_t(\omega)$. Let π be a sentence letter and $\mathcal{M} \in \mathbb{M}(F)$ be a model where $\|\pi\| = \mathcal{B}_t(\omega)$. Thus $\pi \in \beta(t, \omega)$. Since $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \neq \emptyset$, by Lemma A.1 $\beta(t, \omega) \cup \kappa(t+1, \omega)$ is consistent.

Furthermore, $\pi \notin \beta(t+1, \omega)$ since $\alpha \in \mathcal{B}_{t+1}(\omega)$ and $\alpha \in -\mathcal{B}_t(\omega) = \|\neg\pi\|$. Thus $\beta(t, \omega) \not\subseteq \beta(t+1, \omega)$. ■

Remark A.2. The proof of Proposition 3 does not make use of reflexivity of \mathcal{K}_t nor of transitivity of \mathcal{B}_t and \mathcal{K}_t . Thus Proposition 3 holds in a larger class than \mathcal{F}_{KB} .

In order to prove Corollary 1 we need the following lemmas.

Lemma A.2. Fix arbitrary $F \in \mathcal{F}_{KB}$, $t \in \mathbb{N}$, $\omega \in \Omega$ and $\mathcal{M} \in \mathbb{M}(F)$. Then the following are equivalent:

- (1) if $\phi \in \beta^{\mathcal{M}}(t+1, \omega)$ then $\exists \psi \in \kappa^{\mathcal{M}}(t+1, \omega)$ such that $(\psi \rightarrow \phi) \in \beta^{\mathcal{M}}(t, \omega)$,
- (2) $\beta^{\mathcal{M}}(t+1, \omega) \subseteq [\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)]^{PL}$,
- (3) if $\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)$ is consistent then $\beta^{\mathcal{M}}(t+1, \omega) \subseteq [\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)]^{PL}$.

Proof. (1) \Rightarrow (2). Choose an arbitrary $\phi \in \beta^{\mathcal{M}}(t+1, \omega)$. By (1) $\exists \psi \in \kappa^{\mathcal{M}}(t+1, \omega)$ such that $(\psi \rightarrow \phi) \in \beta^{\mathcal{M}}(t, \omega)$. Since $\psi, (\psi \rightarrow \phi) \in \beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)$ and $(\psi \wedge (\psi \rightarrow \phi)) \rightarrow \phi$ is a tautology, $\phi \in [\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)]^{PL}$.

(2) \Rightarrow (1). Fix an arbitrary $\phi \in \beta^{\mathcal{M}}(t+1, \omega)$. By (2) $\phi \in [\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)]^{PL}$, that is, there exist $\phi_1, \dots, \phi_n \in \beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)$ such that $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \phi$ is a tautology. If $\phi_i \in \kappa^{\mathcal{M}}(t+1, \omega)$ for all $i = 1, \dots, n$, then $(\phi_1 \wedge \dots \wedge \phi_n) \in \kappa^{\mathcal{M}}(t+1, \omega)$ (see Chellas, 1984) and, therefore, (1) holds with $\psi = (\phi_1 \wedge \dots \wedge \phi_n)$. If $\phi_i \in \beta^{\mathcal{M}}(t, \omega)$ for all $i = 1, \dots, n$, then $(\phi_1 \wedge \dots \wedge \phi_n) \in \beta^{\mathcal{M}}(t, \omega)$. It follows that $\phi \in \beta^{\mathcal{M}}(t, \omega)$ and therefore (1) holds for arbitrary $\psi \in \kappa^{\mathcal{M}}(t+1, \omega)$. Finally, let $1 \leq k < n$ be such that (after a renumbering of the formulae, if necessary) $\phi_1, \dots, \phi_k \in \beta^{\mathcal{M}}(t, \omega)$ and $\phi_{k'}, \dots, \phi_n \in \kappa^{\mathcal{M}}(t+1, \omega)$. Let $\chi = (\phi_1 \wedge \dots \wedge \phi_k)$ and $\psi = (\phi_{k'} \wedge \dots \wedge \phi_n)$. Then $\chi \in \beta^{\mathcal{M}}(t, \omega)$ and $\psi \in \kappa^{\mathcal{M}}(t+1, \omega)$. Since, by hypothesis, $(\chi \wedge \psi) \rightarrow \phi$ is a tautology and it is equivalent to $\chi \rightarrow (\psi \rightarrow \phi)$, the latter is a tautology and hence belongs to $\beta^{\mathcal{M}}(t, \omega)$. Hence $(\psi \rightarrow \phi) \in \beta^{\mathcal{M}}(t, \omega)$.

(2) \Leftrightarrow (3). That (2) implies (3) is obvious. That (3) implies (2) follows from the fact that if $\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)$ is inconsistent, then there is a formula ϕ such that both ϕ and $\neg\phi$ belong to $\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)$ and, therefore, since, for every formula ψ , $(\phi \wedge \neg\phi) \rightarrow \psi$ is a tautology, $[\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)]^{PL} = \Phi \supseteq \beta^{\mathcal{M}}(t+1, \omega)$. ■

Lemma A.3. Fix arbitrary $F \in \mathcal{F}_{KB}$, $t \in \mathbb{N}$, $\omega \in \Omega$ and $\mathcal{M} \in \mathbb{M}(F)$. Then the following are equivalent:

- (1) $\beta^{\mathcal{M}}(t, \omega) \subseteq \beta^{\mathcal{M}}(t+1, \omega)$,
- (2) $[\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)]^{PL} \subseteq \beta^{\mathcal{M}}(t+1, \omega)$.

Proof. (1) \Rightarrow (2). Assume (1). By (R1) (cf. Remark 1), $\kappa^{\mathcal{M}}(t+1, \omega) \subseteq \beta^{\mathcal{M}}(t+1, \omega)$. Hence $\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega) \subseteq \beta^{\mathcal{M}}(t+1, \omega)$ and, therefore, $[\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)]^{\text{PL}} \subseteq [\beta^{\mathcal{M}}(t+1, \omega)]^{\text{PL}} = \beta^{\mathcal{M}}(t+1, \omega)$.

(2) \Rightarrow (1). Assume (2). Fix an arbitrary $\phi \in \beta^{\mathcal{M}}(t, \omega)$. Since $\phi \rightarrow \phi$ is a tautology, $\phi \in [\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)]^{\text{PL}}$. Hence $\phi \in \beta^{\mathcal{M}}(t+1, \omega)$. ■

Proof of Corollary 1. Let $F \in \mathcal{F}_{\text{KB}}$. F satisfies (C) if and only if it satisfies (P1) and (P2). By Proposition 2 and Lemma A.2, F satisfies (P1) if and only if $\forall \mathcal{M} \in \mathbb{M}(F), \forall t \in \mathbb{N}, \forall \omega \in \Omega$, if $\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)$ is consistent then $[\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)]^{\text{PL}} \supseteq \beta^{\mathcal{M}}(t+1, \omega)$. By Proposition 3 and Lemma A.3, F satisfies (P2) if and only if $\forall \mathcal{M} \in \mathbb{M}(F), \forall t \in \mathbb{N}, \forall \omega \in \Omega$, if $\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)$ is consistent then $[\beta^{\mathcal{M}}(t, \omega) \cup \kappa^{\mathcal{M}}(t+1, \omega)]^{\text{PL}} \subseteq \beta^{\mathcal{M}}(t+1, \omega)$. ■

Proof of Proposition 4. We prove by induction that if

$$B_t \phi \leftrightarrow B_t B_{t+k} \phi$$

is a theorem of $\mathbf{K}_{\text{KB}}^{\text{time}} + (\text{A3}) + (\text{A4})$ (it clearly is for $k = 1$) then so is $B_t \phi \leftrightarrow B_t B_{t+k+1} \phi$.

1. $B_t \phi \leftrightarrow B_t B_{t+k} \phi$ (hypothesis)
2. $B_{t+k} \phi \leftrightarrow B_{t+k} B_{t+k+1} \phi$ ((A3) and (A4))
3. $B_t B_{t+k} \phi \leftrightarrow B_t B_{t+k} B_{t+k+1} \phi$ (2, RE: see Chellas, 1984, p. 17)
4. $B_t \phi \leftrightarrow B_t B_{t+k} B_{t+k+1} \phi$ (1, 3, Propositional Logic)
5. $B_t B_{t+k+1} \phi \leftrightarrow B_t B_{t+k} B_{t+k+1} \phi$ (hypothesis applied to the formula $B_{t+k+1} \phi$)
6. $B_t \phi \leftrightarrow B_t B_{t+k+1} \phi$ (4, 5, Propositional Logic). ■

Proof of Proposition 5. Let $\mathcal{F}_{\text{KB}}^{\text{R3-R4}}$ be the class of KB-frames that satisfy properties (R3) and (R4) given in the proof of Proposition 1 and let $\mathbb{M}(\mathcal{F}_{\text{KB}}^{\text{R3-R4}})$ be the class of models based on frames in $\mathcal{F}_{\text{KB}}^{\text{R3-R4}}$. It follows from Theorem 4.3 in van der Hoek (1993, p. 183) that $\mathbf{K}_{\text{KB}}^{\text{time}} + (\text{A3}) + (\text{A4})$ is sound and complete with respect to $\mathbb{M}(\mathcal{F}_{\text{KB}}^{\text{R3-R4}})$. By Proposition 1, a KB-frame satisfies (R3) and (R4) if and only if it satisfies (P1) and (P2). Since in every KB-frame the conjunction of (P1) and (P2) is equivalent to (C), $\mathcal{F}_{\text{KB}}^{\text{R3-R4}} = \mathcal{F}_{\text{KB}}^{\text{C}}$. ■

Proof of Proposition 6. First we show that (A6) is a theorem of $\mathbf{K}_{\text{KB}}^{\text{time}} + (\text{A5})$:

1. $\neg K_t \phi \rightarrow K_t \neg K_t \phi$ (axiom 5 κ)

2. $K_t \neg K_t \phi \rightarrow K_{t+1} \neg K_t \phi$ (instance of (A5))
3. $\neg K_t \phi \rightarrow K_{t+1} \neg K_t \phi$ (1,2, Propositional Logic).

Next we show that (A7) is a theorem of $\mathbf{K}_{KB}^{\text{time}} + (A6)$:

1. $\neg K_t \phi \rightarrow K_{t+1} \neg K_t \phi$ (axiom (A6))
2. $\neg K_{t+1} \neg K_t \phi \rightarrow K_t \phi$ (1, Propositional Logic)
3. $K_{t+1} \neg K_{t+1} \neg K_t \phi \rightarrow K_{t+1} K_t \phi$ (2, rule RK for K_{t+1} : see Chellas, 1984, p. 19)
4. $\neg K_{t+1} \neg K_t \phi \rightarrow K_{t+1} \neg K_{t+1} \neg K_t \phi$ (axiom 5_K applied to the formula $\neg K_t \phi$)
5. $\neg K_{t+1} \neg K_{t+1} \neg K_t \phi \rightarrow K_{t+1} \neg K_t \phi$ (4, Propositional Logic)
6. $K_{t+1} \neg K_t \phi \rightarrow \neg K_t \phi$ (axiom T_K applied to the formula $\neg K_t \phi$)
7. $\neg K_{t+1} \neg K_{t+1} \neg K_t \phi \rightarrow \neg K_t \phi$ (5, 6, Propositional Logic)
8. $K_t \phi \rightarrow K_{t+1} \neg K_{t+1} \neg K_t \phi$ (7, Propositional Logic)
9. $K_t \phi \rightarrow K_{t+1} K_t \phi$ (8, 3, Propositional Logic).

Next we show that (A5) is a theorem of $\mathbf{K}_{KB}^{\text{time}} + (A7)$:

1. $K_t \phi \rightarrow \phi$ (axiom T for K_t)
2. $K_{t+1} K_t \phi \rightarrow K_{t+1} \phi$ (1, rule RK for K_{t+1} : see Chellas, 1984, p. 19)
3. $K_t \phi \rightarrow K_{t+1} K_t \phi$ (axiom (A7))
4. $K_t \phi \rightarrow K_{t+1} \phi$ (3, 2, Propositional Logic). ■

Proof of Proposition 7. By Theorem 4.3 in van der Hoek (1993, p. 183) $\mathbf{K}_{KB45}^{\text{time}} + (A3) + (A4)$ is sound and complete with respect to the class of frames in \mathcal{F}_{KD45} that satisfy (R3) and (R4) (see the proof of Proposition 1). By Proposition 1, (R3) implies (P1) and (R4) implies (P2) (the proof does not rely on any properties of \mathcal{K}_t : in particular the lack of reflexivity of \mathcal{K}_t is irrelevant). Thus the conjunction of (R3) and (R4) implies the conjunction of (P1) and (P2), which in turn is equivalent to (C) (again, the proof that the conjunction of (P1) and (P2) is equivalent to (C) does not rely on reflexivity of \mathcal{K}_t). To see that, without reflexivity of \mathcal{K}_t , (C) does not imply (R3) and (R4), consider the following KD45-frame: $\Omega = \{\alpha, \beta\}$, $\mathcal{K}_0(\alpha) = \mathcal{K}_0(\beta) = \mathcal{B}_0(\alpha) = \mathcal{B}_0(\beta) = \{\beta\}$ and for all $t \geq 1$, $\mathcal{K}_t(\alpha) = \mathcal{K}_t(\beta) = \mathcal{B}_t(\alpha) = \mathcal{B}_t(\beta) = \{\alpha\}$. It is easily verified that, for every $t \in \mathbb{N}$, \mathcal{K}_t and \mathcal{B}_t are serial, transitive and euclidean and satisfy properties (R1) and (R2). Furthermore, (C) is also (vacuously) satisfied. However, both (R3) and (R4) are violated. (R4) is violated because $\beta \in \mathcal{B}_0(\alpha)$ and $\alpha \in \mathcal{B}_1(\beta)$ and yet

$\alpha \notin \mathcal{B}_0(\alpha)$. (R3) is violated because $\beta \in \mathcal{B}_0(\alpha)$ and yet there does not exist an $\omega \in \mathcal{B}_0(\alpha)$ such that $\beta \in \mathcal{B}_1(\omega)$.

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