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Strategic Independence and Perfect Bayesian Equilibria

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This paper evaluates different refinements of subgame perfection, which rely on different restrictions on players' assessments, using a simple and intuitive independence property for conditional probability systems on the space of strategy profiles. This independence property is necessary for full consistency of assessments, and it is equivalent to full consistency in games with observable deviators. Furthermore, while every conditional system on the strategies satisfying the independence property corresponds to a generally reasonable extended assessment as defined by Fudenberg and Tirole [*J. Econ. Theory* **53** (1991), 236–260], such extended assessments may violate independence, full consistency, and invariance with respect to interchanging of essentially simultaneous moves. *Journal of Economic Literature* Classification Number: C72.

1. INTRODUCTION

It is well known that the notion of subgame perfection (see Selten [20]) does not rule out unreasonable equilibria in extensive games with imperfect information. This can be illustrated by the parameterized example represented in Fig. 1.[1] For every value of u, (R', R'', R''') is an equilibrium profile. Since there are no proper subgames this is also a subgame perfect equilibrium, but assume that $u > 1$. Then action R''' at information set h is clearly irrational because for every conditional probability distribution on h the expected utility of R''' is strictly less than the expected utility of L'''. Knowing this, player II should choose L''.

This unreasonable equilibrium can be ruled out by the requirement of sequential rationality (Kreps and Wilson [13]). Consider a (possibly

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[1] The graphical conventions about extensive games are borrowed from Kreps and Wilson [13]. In particular, numbers in parentheses near branches represent the transition probabilities implied by the proposed equilibrium and numbers in brackets near decision nodes represent conditional probabilities at an information set.

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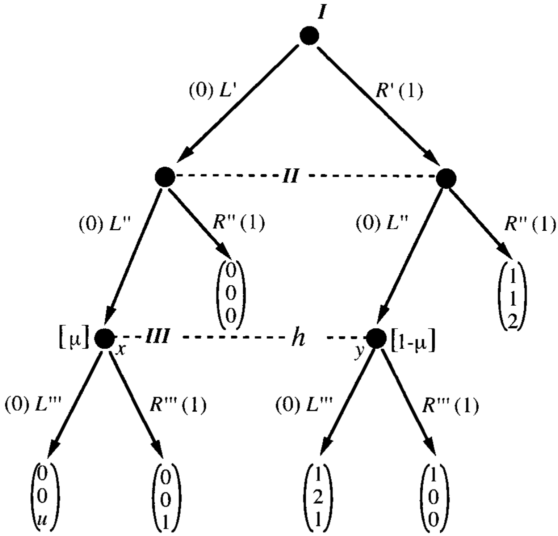

FIG. 1. Nash (and subgame perfect) equilibrium (R', R'', R''') does not satisfy sequential rationality at h, if $u > 1$.

randomized) equilibrium profile π; for each information set h assign a probability distribution $\mu(\cdot\,|\,h)$ on the nodes of h. The array μ of such distributions, called *system of beliefs*, must be derived from π *via* Bayes rule whenever possible. The *assessment* (μ, π) is *sequentially rational* if at each information set the choice prescribed by π maximizes the conditional expected payoff given (μ, π).

Note that in assessment (μ, R', R'', R''') the value of μ at h is unrestricted by Bayes rule. If $u > 1$, this is immaterial, but if $u \leqslant 1$ we can always find a μ such that (μ, R', R'', R''') is sequentially rational. Are all these specifications of μ equally reasonable?

In this paper we analyze some restrictions on assessments, whereby Bayes rule is applied whenever possible. When coupled with sequential rationality, they provide corresponding refinements of subgame perfection, which may be generically called *perfect Bayesian equilibria*. All of these restrictions provide the same answer to the previous question: the conditional probability of node x given h should be zero; thus, the unique "reasonable" equilibrium of the game in Fig. 1 is (R', L'', L'''). It will be shown that these restrictions are related to the game-theoretic principle of strategic independence: the strategic choices of different players should be regarded as stochastically independent events. In the example, the prior probability of player I's strategy L' is zero. The conditional probability of x given h is just the conditional probability of L' after observing that player

II chose L''. If the strategic choices of I and II are independent, the conditional probability of L' given L'' must be equal to the prior probability of L', that is, zero.

In their seminal paper Kreps and Wilson [13] proposed a topological condition which they called "consistency." Following Myerson [17, p. 173], we use the phrase "full consistency," because the mere word consistency might suggest a weaker property than that proposed in [13]. An assessment (μ, π) is *fully consistent* if it is the limit of some sequence (μ^k, π^k), whereby π^k is strictly randomized and μ^k is the system of beliefs derived from π^k via Bayes rule. A fully consistent and sequentially rational assessment is called *sequential equilibrium.* Full consistency is implicit in Selten's [20] notion of "trembling hand perfection." Selten considers perturbed games where every action can be chosen by mistake because of "trembles." Trembles at different information sets are mutually independent. A "trembling hand perfect equilibrium" is a limit of equilibria of perturbed games as the "trembling probabilities" go to zero. It is easily shown that for every trembling hand perfect equilibrium profile π there is a system of beliefs μ such that (μ, π) is a sequential equilibrium. Kreps and Wilson [13] show that for "generic" terminal nodes payoffs a sequential equilibrium corresponds to a trembling hand perfect equilibrium.

There are two problems with the notion of full consistency and the corresponding concept of sequential equilibrium. One is practical: it may be difficult to prove that a given assessment is (or is not) the limit of some appropriate sequence. A second, more important problem is theoretical: trembles do not play any direct role in the sequential rationality condition. For this reason some sequential equilibria are not trembling hand perfect (for example, in simultaneous games there is no difference between Nash and sequential equilibria, but trembling hand perfection rules out equilibria with weakly dominated strategies). Therefore, the sequential equilibrium concept needs a rationale for full consistency which does not rely on trembles.

Kreps and Wilson [13] motivated their notion of full consistency by showing that it yields intuitive restrictions in many instances, but they did not provide a characterization. Furthermore, it turned out that in some games sequential equilibria do not satisfy an apparently very intuitive property: structural consistency. This fact has been interpreted as a violation of the principle that each player believes that her opponents' strategic choices are mutually independent (see Kreps and Ramey [12]). On the other hand, many economic applications use different and more intuitive notions of perfect Bayesian equilibrium, which do not rely on topological restrictions on beliefs (see, e.g., Fudenberg and Tirole [8, 9]).

This situation prompted an effort to provide a non-topological, easily interpretable characterization of full consistency. Bonanno [5], Kohlberg and Reny [11], Fudenberg and Tirole [8], and Swinkels [21] are prominent examples of this line of research. Fudenberg and Tirole [8] provides an intuitive characterization of full consistency for multi-stage games of incomplete information with observed actions. The key condition is that a player's action cannot signal private information that the player does not possess when choosing that action. In a final section devoted to general extensive games the authors extend this condition and propose the notion of a "generally reasonable extended assessment." It is easily shown that every fully consistent assessment corresponds to a generally reasonable extended assessment. Fudenberg and Tirole [8, Proposition 6.1] claim that also the converse proposition holds, but we will show that this is not true.

The present paper shows that these restrictions on players' beliefs are better understood by connecting them to the principle that different players choose their strategies independently. We propose an intuitive and simple independence property for conditional probability systems (Myerson [17; 18, Chapter 1]) over product spaces.[2] When the product space is interpreted as the space of strategic profiles of a game, the property is interpreted as a *strategic independence* principle: information about player k's strategic behavior is irrelevant for probability assessments exclusively concerning player j's strategic behavior ($j \neq k$). This property is used in the analysis of finite extensive games with perfect recall.

Following McLennan [14], we also define a full consistency property for conditional systems which is equivalent to full consistency of assessments as defined by Kreps and Wilson [13].[3] It is easy to show that every fully consistent conditional system—hence, every fully consistent assessment—has the strategic independence property. Kohlberg and Reny [11] show with a counterexample that the converse proposition is not true. However we prove that if the extensive form has *observable deviators*, i.e. if at each information set h it is possible to identify the players who deviated from any given strategic profile not reaching h (as is the case in the game of Fig. 1), then strategic independence is equivalent to full consistency of assessments.

These results contrast with Kreps and Ramey's opinion that fully consistent assessments may violate independence because they may be structurally inconsistent, but we argue that structural consistency is neither necessary nor sufficient for independence and that it yields unreasonable restrictions on beliefs.

[2] It has come to our attention that Hammond [10] had originally put forward a similar independence property.

[3] McLennan [15] considers the agent strategic form instead of the strategic form. By perfect recall, this difference is immaterial.

We show that a conditional system satisfying strategic independence induces a generally reasonable extended assessment on the game tree, but "general reasonableness" is not sufficient for strategic independence even if attention is restricted to games with observable deviators. This is proved with an example, which also shows that, unlike the set of fully consistent assessments, the set of generally reasonable extended assessments is not invariant to the transformation of interchanging essentially simultaneous moves.[4] Both this example and the previously mentioned example by Kohlberg and Reny [11] imply that generally reasonable extended assessments may violate full consistency, but we show that "universal reasonableness," a property in the same spirit of Fudenberg and Tirole [8], is indeed equivalent to strategic independence and full consistency in multi-stage games with observed deviators in the agent form.

Our approach is related to Kohlberg and Reny [11] and Swinkels [21]. Kohlberg and Reny provide a complete characterization of full consistency, taking the point of view of an external observer assessing the play of many separate and identical games. They consider relative probability systems satisfying coordinate-wise symmetry (or exchangeability) and independence[5] across games. It is shown that a single-game relative probability system is fully consistent if and only if, for all k, it is the marginal of a k-games symmetric and independent system. The key difference between their approach and ours is that they take as primitive independence between the outcomes of separate games, while we take as primitive independence between the strategies of different players. Swinkels [21] provides an additional viewpoint on [11]. He expands the original strategies space in a different way adding "calibration" devices, which "elicit" the magnitude of infinitesimals not revealed by the system of relative probabilities on the original space. Building on [11], Swinkels shows that a system of relative probabilities on the original space is fully consistent if and only if it can be extended to every larger, device-augmented product space so that independence is satisfied.

The rest of the paper is organized as follows. Section 2 introduces independence and full consistency for conditional systems over product spaces and shows that full consistency implies independence. Section 3 introduces

[4] Recall that when a game is represented in extensive form, hence with a tree, originally simultaneous moves must be arranged in an arbitrary order, which should not influence the game theoretic analysis.

[5] Actually, Kohlberg and Reny use a slightly different condition called "individual quasi-independence" (IQI) by Swinkels [21], who calls "quasi-independence" (QI) the independence property put forward in this paper. QI implies IQI and the two properties are equivalent for product spaces with two coordinates. Swinkels [21] shows (with an example credited to Myerson) that if there are three or more coordinates QI is strictly stronger than IQI. Both [11] and [21] call "independent product" a fully consistent conditional system.

the necessary game-theoretic notation, relates conditional systems on the strategy space to assessments, and shows that full consistency is strictly stronger than strategic independence in general games, while these two properties are equivalent in games with observable deviators. Section 4 criticizes the notion of structural consistency. Section 5 shows that strategic independence is strictly stronger than general reasonableness and defines the notion of "universal reasonableness," which is shown to be equivalent to strategic independence and full consistency in multi-stage games with observable deviators in the agent form. (The latter proposition is proved in the Appendix.) Section 6 offers some remarks on the assumption of common vs heterogeneous conditional expectations.

2. Independence and Full Consistency for Conditional Systems on Product Spaces

Consider a finite set of "states" S. We are interested in coherently assessing probability distributions conditional on every possible event $E \subseteq S$. For every $E \subseteq S$ let $\Delta(E)$ be the set of probability distributions on S with supports included in E. A *conditional* (*probability*) *system* on S is a map $\sigma: 2^S \times (2^S \backslash \{\varnothing\}) \to [0, 1]$ such that for all $E \in 2^S \backslash \{\varnothing\}$, $\sigma(\cdot \mid E) \in \Delta(E)$, and for all $A, B, C \in (2^S \backslash \{\varnothing\})$

$$A \subseteq B \subseteq C \qquad \text{implies} \qquad \sigma(A \mid C) = \sigma(A \mid B)\, \sigma(B \mid C). \tag{2.1}$$

The set of conditional systems on S is denoted $\Delta^*(S)$.

We regard $\Delta^*(S) \subseteq \times_{\varnothing \neq E \subseteq S} \Delta(E)$ as a subset of an Euclidean space of dimension $|S|(2^{|S|} - 1)$ endowed with the relative topology. Clearly $\Delta^*(S)$ is a compact set. Let $\Delta^{*o}(S)$ denote the set of strictly positive vectors of $\Delta^*(S)$. Equation (2.1) implies that $\Delta^{*o}(S)$ is isomorphic to the set $\Delta^o(S)$ of strictly positive probability distributions on S. Myerson [17, Theorem 1] shows that $\Delta^*(S)$ is the closure of $\Delta^{*o}(S)$.

Now assume that S is a product space: $S = S_1 \times S_2 \times \cdots \times S_n$. We are going to formulate an independence property for conditional systems $\sigma \in \Delta^*(S)$. Let us begin with a simple probability measure $\sigma \in \Delta(S)$. We know from elementary probability theory that σ satisfies stochastic independence if it is a product measure, i.e., if there are n probability measures $\sigma_i \in \Delta(S_i)$, $i = 1, ..., n$, such that for each $s = (s_1, ..., s_n)$, $\sigma(s) = \prod_{i=1}^{i=n} \sigma_i(s_i)$. The same property can be characterized in terms of conditional probabilities. In the following, $\{J, K\}$ denotes a non-trivial bipartition of the index set $\{1, 2, ..., n\}$, S_J denotes the Cartesian product $\times_{j \in J} S_j$ and s_J denotes an element of S_J. It may be checked that σ is a product probability measure if and only if for all $\{J, K\}$, $A_J, B_J \subseteq S_J$, $C_K, D_K \subseteq S_K$ such that $\sigma(B_J \times C_K) > 0$ and $\sigma(B_J \times D_K) > 0$, the following holds:

$$\sigma(A_J \times C_K \mid B_J \times C_K) = \sigma(A_J \times D_K \mid B_J \times D_K). \tag{2.2}$$

The meaning of (2.2) is that the marginal conditional probabilities about group J are independent of information which exclusively concerns the complementary group K.

If σ is a conditional probability system, the conditional probabilities in (2.2) are well defined even if $\sigma(B_J \times C_K) = 0$ and/or $\sigma(B_J \times D_K) = 0$. Therefore, a natural extension of the notion of stochastic independence seems to require that (2.2) also hold in this case (cf. Hammond [10]).

DEFINITION 2.1. A conditional system $\sigma \in \Delta^*(S_1 \times \cdots \times S_n)$ has the *independence property* if for all $\{J, K\}$ and all non-empty sets $A_J, B_J \subseteq S_J$, $C_K, D_K \subseteq S_K$, Eq. (2.2) holds. The set of conditional systems on $(S_1 \times \cdots \times S_n)$ with the independence property is denoted $I\Delta^*(S_1 \times \cdots \times S_n)$.

Myerson [18, Chap. 1] and Blume *et al.* [4] show that conditional systems can be derived axiomatically in a decision theoretic framework with subjective and objective uncertainty. Battigalli and Veronesi [3] show that the independence condition given in Definition 2.1 can be obtained by adding a natural stochastic independence axiom.[6] According to this axiom a decision maker characterized by a conditional system σ thinks that information about the K-component of the state is irrelevant for decisions whose consequences only depend on the J-component even if he observes unexpected events.

Assume that $S = S_1 \times \cdots \times S_n$ is the space of strategic profiles and let $\sigma \in \Delta^*(S)$ represent the players' conditional expectations in an equilibrium. Independence of the prior $\sigma(\cdot \mid S)$ is a necessary property of any refinement of the Nash equilibrium concept, but the notion of *interim* independence given in Definition 2.1 may be more controversial. For example, Kreps and Ramey [12] make a quite convincing case for expecting correlation between the moves of two opponents in a subgame off the equilibrium path (see also the notion of "*c*-perfection" in Fudenberg *et al.* [6]). we are not claiming here that players' expectations should always satisfy (2.2), but we think that this condition is a useful benchmark corresponding to a meaningful behavioral assumption.

[6] Specifically, it is shown that a (non-Archimedean) preference relation satisfying axioms 1–3, 4″, 5′, and 6 (where axiom 6 refers to stochastic independence) of Blume *et al.* [4] is lexicographically represented by a utility function and a conditional system satisfying the independence property given in Definition 2.1. An analogous result holds for Myerson's [17] (Archimedean) decision theory, but the stochastic independence axiom must be strengthened.

Note that $I\Delta^*(S_1 \times \cdots \times S_n)$ is defined by closed conditions and it is a subset of a compact set. Therefore $I\Delta^*(S_1 \times \cdots \times S_n)$ is compact.

For every conditional system $\sigma \in \Delta^*(S_1 \times \cdots \times S_n)$ we write $\sigma_J(A_J \mid B_J) := \sigma(A_J \times S_K \mid B_J \times S_K)$. σ_J is the *marginal* of σ on S_J. It is straightforward to show that $\sigma_J \in \Delta^*(S_J)$. Note that, unlike simple probability measures, a conditional system σ satisfying the independence property is not determined by its marginal conditional systems σ_i, unless they are strictly positive. However, the joint measure conditional on a Cartesian (or rectangular) set can be derived by multiplication in the usual way.

PROPOSITION 2.1. *A conditional system* $\sigma \in \Delta^*(S_1 \times \cdots \times S_n)$ *has the independence property if and only if for every partition* $\{J, K, L, ...\}$ *of* $\{1, 2, ..., n\}$, *every possible event* $E \subseteq (S_1 \times \cdots \times S_n)$ *of the form* $E = (E_J \times E_K \times E_L \times \cdots)$, *and every state* $s = (s_J, s_K, s_L, ...) \in (E_J \times E_K \times E_L \cdots)$, *the following holds*:

$$\sigma(s \mid E) = \sigma_J(s_J \mid E_J)\, \sigma_K(s_K \mid E_K)\, \sigma_L(s_L \mid E_L) \cdots. \tag{2.3}$$

Proof. (If) Set $E = B_J \times C_K$ and $E = B_J \times D_K$ in (2.3) and derive (2.2).

(Only if) Equation (2.3) follows from repeated applications of (2.1) and (2.2)

$$\begin{aligned}
\sigma(s \mid E) &= \sigma(s \mid E_J \times \{s_K\} \times \{s_L\} \times \cdots)\, \sigma(E_J \times \{s_K\} \times \{s_L\} \cdots \mid E) \\
&= \sigma_J(s_J \mid E_J)\, \sigma(E_J \times \{s_K\} \times \{s_L\} \cdots \mid E) \\
&= \sigma_J(s_J \mid E_J)\, \sigma(E_J \times \{s_K\} \times \{s_L\} \cdots \mid E_J \times E_K \times \{s_L\} \times \cdots) \\
&\quad \times \sigma(E_J \times E_K \times \{s_L\} \times \cdots \mid E) \\
&= \sigma_J(s_J \mid E_J)\, \sigma_K(S_K \mid E_K)\, \sigma(E_J \times E_K \times \{s_L\} \times \cdots \mid E) = \cdots,
\end{aligned}$$

where the first equality follows from (2.1), the second follows from (2.2), the third follows from (2.1), the fourth follows from (2.2), and so on. ∎

It is easily checked that every strictly positive product distribution on $(S_1 \times \cdots \times S_n)$ generates *via* Bayes rule a conditional system $\sigma \in \Delta^{*o}(S_1 \times \cdots \times S_n)$ satisfying the independence property. By Proposition 2.1, every strictly positive conditional system $\sigma \in I\Delta^*(S_1 \times \cdots \times S_n)$ is generated by a product prior distribution $\sigma(\cdot \mid S)$. Then $I\Delta^*(S_1 \times \cdots \times S_n) \cap \Delta^{*o}(S_1 \times \cdots \times S_n)$ is exactly the set of conditional systems induced by strictly positive product distributions. Let $\Psi(S_1 \times \cdots \times S_n)$ denote the closure of this set. Note that also $\Psi(S_1 \times \cdots \times S_n)$ is a compact set. The conditional systems in $\Psi(S_1 \times \cdots \times S_n)$ are called *fully consistent* (cf. McLennan [14, 15]), because it is shown in the next section that they are equivalent to fully consistent assessments if $(S_1 \times \cdots \times S_n)$ is the set of strategic profiles of a game with perfect recall. Since $I\Delta^*(S_1 \times \cdots \times S_n)$ is closed, we have:

LEMMA 2.1. *Every fully consistent conditional system has the independence property, i.e.,* $\Psi(S_1 \times \cdots \times S_n) \subseteq I\Delta^*(S_1 \times \cdots \times S_n)$.

It is shown in Section 3 that the converse does not hold.

3. Strategic Independence and Consistent Assessments

We consider a finite extensive game with perfect recall. We focus for simplicity on games without chance moves. Games with chance moves and games with incomplete information are analyzed in Battigalli [2, Section 6].

We use the following quite standard notation:

Notation	Terminology
$N = \{1, ..., n\}$	Set of players
$-i = N \backslash \{i\}$	Set of player i's opponents
X	Set of nodes
Z	Set of terminal nodes
$u_i: Z \to \mathbb{R}$	i's payoff function
H_i	i's collection of information sets
$\mu(\cdot \mid h) \in \Delta(h)$	Belief at h
$M = \times_{i \in N}(\times_{h \in H_i} \Delta(h))$	Set of systems of beliefs
$A(h)$ $(h \in H_i)$	Set of i's actions at h
$\mathscr{A}_i = \bigcup_{h \in H_i} A(h)$	Set of i's actions
$S_i = \{s_i: H_i \to \mathscr{A}_i, s_i(h) \in A(h)\}$	Set of i's pure strategies
$\pi(\cdot \mid h) \in \Delta(A(h))$	Randomized choice at h
$\Pi = \times_{i \in N}(\times_{h \in H_i} \Delta(A(h)))$	Set of behavioral profiles
$\mathscr{s}(x, a)$, $x \in h$, $a \in A(h)$	The immediate follower of x after a
$Z(Y)$, $Y \subseteq X$	Set of terminal successors of nodes in Y
$S(Y)$, $Y \subseteq X$	Set of strategic profiles inducing a play that reaches a node in Y
$S_i(Y)$, $S_{-i}(Y)$	Projections of $S(Y)$ on S_i and S_{-i}

The sets $Z(Y)$, $S(Y)$, $S_i(Y)$, and $S_{-i}(Y)$ have well-known properties. It is trivially true that $S(Y) = S(Z(Y))$. For every node $x \in X$, $S(x) = S_1(x) \times \cdots \times S_n(x)$. By perfect recall, $S_i(x) = S_i(h)$ for all $h \in H_i$ and all $x \in h$. This implies that $S(h) = S_i(h) \times S_{-i}(h)$, for all $h \in H_i$. If $x \in h \in H_i$ and $a \in A(h)$, $S(\mathscr{s}(x, a)) = \{s_i \in S_i(h) : s_i(h) = a\} \times S_{-i}(x)$.

An assessment is a pair $(\mu, \pi) \in M \times \Pi$. Since assessments are lists of conditional probabilities of actions and nodes, they can be derived from richer conditional systems on S or Z. An assessment is a parsimonious description of players' expectations, since it lists just the necessary elements appearing in the sequential rationality condition, but the analysis of richer systems of conditional probabilities allows the formulation of reasonable restrictions on assessments.

An extended assessment is a triple (λ, μ, π), whereby λ is a conditional system and (μ, π) is derived from λ. In the spirit of traditional equilibrium analysis we assume here that an extended assessment describes the players' *common* conditional expectations or reflects the point of view of an outside observer, but most of our analysis can also be applied to solution concepts with heterogeneous conditional expectations (see Section 6). We analyze extended assessments $(\sigma, \mu, \pi) \in \Delta^*(S) \times M \times \Pi$ in the present and next sections, and extended assessments $(\nu, \mu, \pi) \in \Delta^*(Z) \times M \times \Pi$ in Section 5.

DEFINITION 3.1. A *strategic extended assessment* is a triple $(\sigma, \mu, \pi) \in \Delta^*(S) \times M \times \Pi$ such that

$$\forall i \in N, \forall h \in H_i, \forall x \in h, \qquad \mu(x \mid h) = \sigma(S(x) \mid S(h)) \tag{3.1}$$

$$\forall i \in N, \forall h \in H_i, \forall a \in A(h), \qquad \pi(a \mid h) = \sigma(\{s \in S(h)\colon s_i(h) = a\} \mid S(h)). \tag{3.2}$$

The following lemma states that full consistency of conditional systems on $S_1 \times \cdots \times S_n$ is equivalent to full consistency of assessments.

LEMMA 3.1. *An assessment (μ, π) is fully consistent if and only if there exists a fully consistent conditional system $\sigma \in \Psi(S_1 \times \cdots \times S_n)$ such that (σ, μ, π) is a strategic extended assessment.*

Proof. Under perfect recall Eq. (3.2) defines a one-to-one correspondence between strictly positive profiles of mixed strategies—i.e., strictly positive product distributions on $S_1 \times \cdots \times S_n$—and strictly positive profiles of behavioral strategies, such that two corresponding profiles are realization equivalent; that is, they induce the same probability distribution on terminal nodes.

(*Only if*) Let (μ, π) be fully consistent. By definition (μ, π) is the limit of some sequence of strictly positive assessments $\{(\mu^k, \pi^k)\}_0^\infty$ where μ^k is derived from π^k. Let $\{\sigma^k(\cdot \mid S)\}_0^\infty$ be the corresponding sequence of strictly positive product priors and let σ^k be the conditional system derived from prior $\sigma^k(\cdot \mid S)$. Note that $\sigma^k \in \Psi(S_1 \times \cdots \times S_n)$ and (σ^k, μ^k, π^k) satisfies (3.2) by construction. Since $\sigma^k(\cdot \mid S)$ and π^k are realization equivalent, (σ^k, μ^k, π^k) satisfies (3.1) too. By compactness $\{\sigma^k\}_0^\infty$ has a cluster point $\sigma \in \Psi(S_1 \times \cdots \times S_n)$ and (σ, μ, π) satisfies (3.1) and (3.2).

(*If*) Assume that (σ, μ, π) is a strategic extended assessment and $\sigma \in \Psi(S_1 \times \cdots \times S_n)$. By definition, there is a sequence $\{\sigma^k\}_0^\infty$ of strictly positive conditional systems derived *via* Bayes rule from strictly positive product priors $\sigma^k(\cdot \mid S)$, which correspond to strictly positive behavioral profiles π^k. Since $\sigma^k(\cdot \mid S)$ and π^k are realization equivalent, they induce the same system of beliefs μ^k and $(\mu, \pi) = \lim_{k \to \infty}(\mu^k, \pi^k)$. Therefore (μ, π) is fully consistent. ∎

The following corollary, saying that full consistency implies strategic independence is an immediate consequence of Lemma 2.1 and Lemma 3.1.

COROLLARY 3.1. *An assessment (μ, π) is fully consistent only if it is part of a strategic extended assessment (σ, μ, π) where σ has the independence property.*

Kohlberg and Reny [11] show that the converse of Corollary 3.1 does not hold for general extensive games. An analysis of their example is included here for completeness.

Consider the extensive form of Fig. 2. Players I and II choose simultaneously among three actions. Player III observes a joint signal about their actions. In this example it does not matter how many actions

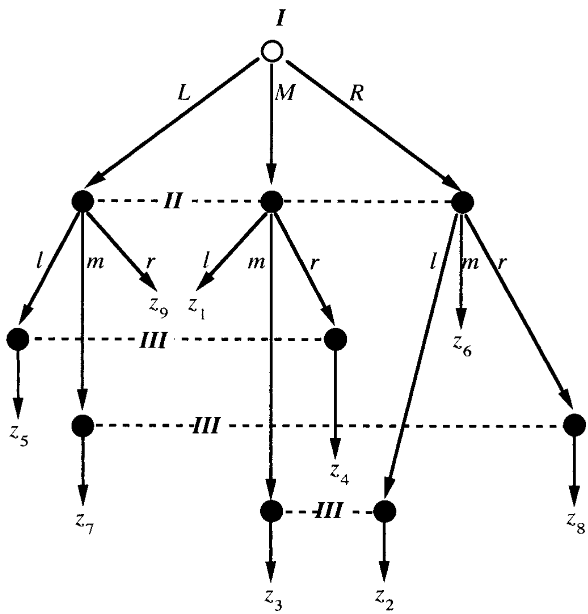

FIG. 2. The strategic extended assessment (σ, μ, π), where $\sigma(z_m \mid \{z_l, z_m\}) = 0$ for $m > l$, does not satisfy full consistency.

are available for III. We assume for simplicity that III has only one action at each information set. All this implies that there is a one-to-one correspondence between the set $S_{\mathrm{I}} \times S_{\mathrm{II}}$ and the set Z of terminal nodes.

We consider a conditional system where all conditional probabilities are either 1 or 0. By Eq. (2.1), the list of all conditional probabilities of the form $\sigma(s \mid \{s, t\})$ determines the whole conditional system. Furthermore, the binary relation $\geqslant \subseteq S \times S$ defined by

$$\forall s, t \in S, \quad s \geqslant t \Leftrightarrow \sigma(s \mid \{s, t\}) > 0 \text{ (i.e., } t > s \Leftrightarrow (\sigma(t \mid \{s, t\}) = 1 \wedge s \neq t)) \quad (3.3)$$

is a complete preorder on the set of states S (see McLennan [14, Lemma 2.3]). The conditional system of this example is given by

$$\forall l, m \in \{1, 2, ..., 9\}, l < m \Rightarrow z_l > z_m \text{ (or } l < m \Rightarrow \sigma(z_m \mid \{z_l, z_m\}) = 0). \quad (3.4)$$

This conditional system can be obtained, for example, by taking the limit as $k \to \infty$ of the strictly positive probabilities

$$\sigma^k(z_l) = C(k)(1/k)^{l-1}, \; C(k) = (1 - 1/k)[1 - (1/k)^9]^{-1}, \; l = 1, 2, ..., 9.$$

We show that σ satisfies the independence property, but the assessment (μ, π) derived from σ is not fully consistent.

Figure 3 represents the extensive form given by Fig. 2 with a 3×3 matrix. Boxes with the same figure (circle, square, or diamond) correspond to nodes of the same information set. In Fig. 3 we can also read the conditional system given by (3.4). In this case the independence property is just the following: the boxes of each row (column) are ordered by the conditional system in the same way: $l > m > r$ $(M > R > L)$. Therefore, looking at Fig. 3, it is straightforward to check that condition (2.2) holds.

We now show that this conditional system is not equivalent to a fully consistent assessment. Consider a sequence $\{\sigma_{\mathrm{I}}^k \times \sigma_{\mathrm{II}}^k\}_0^\infty$ of strictly positive

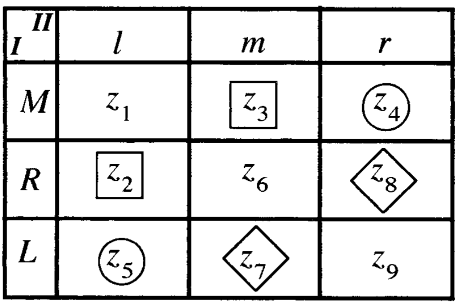

FIG. 3. A strategic representation of the extensive form and assessment of Fig. 2, showing that σ satisfies the independence property.

product distributions on $S_{\mathrm{I}} \times S_{\mathrm{II}}$ and let $\{\sigma^k\}_0^\infty$ be the associated sequence of conditional systems. Note that

$$\sigma^k(z_2 \mid \{z_2, z_3\})\, \sigma^k(z_7 \mid \{z_7, z_8\})\, \sigma^k(z_4 \mid \{z_4, z_5\}) \\ = \sigma^k(z_3 \mid \{z_2, z_3\})\, \sigma^k(z_8 \mid \{z_7, z_8\})\, \sigma^k(z_5 \mid \{z_4, z_5\}). \tag{3.5}$$

(For example, z_2 is (R, l), thus $\sigma^k(z_2 \mid \{z_2, z_3\}) = \sigma_{\mathrm{I}}^k(R)\, \sigma_{\mathrm{II}}^k(l)/\sigma^k(\{z_2, z_3\})$. Expressing each factor in a similar way one obtains (3.5).)

Of course, (3.5) also holds for every limit point σ^∞, but it does not hold if σ^k is replaced by σ, because in this case the left-hand side is 1 and the right-hand side is 0.[7]

Note that in this example the information sets of player III are such that if III observes a deviation from a given strategic profile, he is not able to identify the deviating opponent. In the strategic representation of Fig. 3 this corresponds to the fact that the strategic profiles reaching an information set do not form a Cartesian (or rectangular) set. This turns out to be a crucial feature of the example.

DEFINITION 3.2. A game has *observable deviators*[8] if

$$\forall i \in N, \forall h \in H_i, \qquad S(h) = S_1(h) \times \cdots \times S_n(h). \tag{3.6}$$

The game in Fig. 1 has observable deviators, while the game of Fig. 2 does not. Games of perfect information and, more generally, multi-stage games with observed actions (Fudenberg and Tirole [9, pp. 71–72]) have observable deviators. Since, by perfect recall, $S(h) = S_i(h) \times S_{-i}(h)$ for all $h \in H_i$, two-person games have observable deviators.

PROPOSITION 3.1. *In every game with observable deviators an assessment* (μ, π) *is fully consistent if and only if it is part of a strategic extended assessment* (σ, μ, π) *where* σ *has the independence property.*

Proof. The "only if" part of the proposition immediately follows from Corollary 3.1.

(*If*) By Lemma 3.1, at is sufficient to show that (μ, π) is part of an extended assessment $(\hat{\sigma}, \mu, \pi)$ such that $\hat{\sigma} \in \Psi(S_1 \times \cdots \times S_n)$. Recall that

[7] With a similar example it can be shown that an assessment satisfying independence and sequential rationality need not be a sequential equilibrium (see, e.g., Kohlberg and Reny [11]).

[8] This terminology is borrowed from Fudenberg and Levine [7], who give an equivalent definition. Battigalli [2, Section 6] shows how Definition 3.2 and Proposition 3.1 can be extended to games of incomplete information, using a concept which is akin to Milgrom and Weber's [16] distributional strategies.

$\sigma_i \in \Delta^*(S_i)$ denotes the marginal of σ on S_i. By [17, Theorem 1], each σ_i is the limit of some sequence of conditional systems $\{\sigma_i^k\}_0^\infty$ generated by strictly positive priors $\sigma_i^k(\cdot \mid S_i)$. Let $\hat{\sigma}^k \in \Psi(S_1 \times \cdots \times S_n)$ be the strictly positive conditional system derived by the product prior $\sigma_1^k(\cdot \mid S_1) \times \cdots \times \sigma_n^k(\cdot \mid S_n)$. By compactness, $\{\hat{\sigma}^k\}_0^\infty$ has a cluster point $\hat{\sigma} \in \Psi(S_1 \times \cdots \times S_n) \subseteq I\Delta^*(S_1 \times \cdots \times S_n)$ and $\hat{\sigma}$ has the same marginals of σ. We now show that since (σ, μ, π) satisfies (3.1) and (3.2) also $(\hat{\sigma}, \mu, \pi)$ does, thug completing the proof. By assumption, $S(h) = S_1(h) \times \cdots \times S_n(h)$ for all h. Hence we can apply Proposition 2.1 to the conditional systems σ and $\hat{\sigma}$ and use the equality $\sigma_j = \hat{\sigma}_j$ (for all j). For all i, $h \in H_i$, $x \in h$,

$$\mu(x \mid h) = \sigma(S(x) \mid S(h)) = \prod_1^n \sigma_j(S_j(x) \mid S_j(h)) = \hat{\sigma}(S(x) \mid S(h)).$$

Furthermore, by perfect recall, independence, and equality $\sigma_i = \hat{\sigma}_i$, for all $h \in H_i$, $a \in A(h)$,

$$\begin{aligned}
\pi(a \mid h) &= \sigma(\{s \in S(h) \colon s_i(h) = a\} \mid S(h)) \\
&= \sigma(\{s_i \in S_i(h) \colon s_i(h) = a\} \times S_{-i}(h) \mid S_i(h) \times S_{-i}(h)) \\
&= \sigma_i(\{s_i \in S_i(h) \colon s_i(h) = a\} \mid S_i(h)) \\
&= \hat{\sigma}_i(\{s_i \in S_i(h) \colon s_i(h) = a\} \mid S_i(h)) \\
&= \hat{\sigma}(\{s_i \in S_i(h) \colon s_i(h) = a\} \times S_{-i}(h) \mid S_i(h) \times S_{-i}(h)) \\
&= \hat{\sigma}(\{s \in S(h) \colon s_i(h) = a\} \mid S(h)). \quad \blacksquare
\end{aligned}$$

4. A Comment on Structural Consistency

It is sometimes argued that fully consistent assessments may violate independence, because there may be "unreached" information sets h such that the conditional distribution $\mu(\cdot \mid h)$ cannot be derived from a product prior distribution (or a profile of behavioral strategies) *via* Bayes rule.[9] Such assessments are called *structurally inconsistent*. There are games where all sequential equilibrium assessments are structurally inconsistent (see Kreps

[9] We quote Kreps and Ramey [12, p. 1341]: "In particular, the Nash character of conjectures is lost, a player's conjectures concerning different opponents' strategies need no longer be statistically independent. (...) Of course, if one accepts this sort of correlation in conjectured strategies, then other sorts of correlation should be considered. (...) We wish to present a philosophy of eclecticism: all form of correlation should be considered, depending on the context."

and Ramey [12]). But according to our approach this is not a violation of independence, because our independence condition is necessary for full consistency.

We argue that structural consistency on the one hand does not yield all the restrictions implied by an intuitive game-theoretic notion of independence; on the other hand, it yields non-necessary and unreasonable restrictions. This supports the opinion of some leading game theorists (see, e.g., Van Damme [22, pp. 110–111]).

The first part of this claim is very easy to support. Consider the extensive form represented in Fig. 1. It is straightforward to check that any value of μ at the unreached information set h can be derived from some uncorrelated prior or, equivalently, from some profile of behavioral strategies. Hence this assessment is structurally consistent, but we have shown that independence (and consistency) is violated unless $\mu(x|h)=0$.

Now consider the extensive form in Fig. 4 (cf. Kreps and Ramey [12, Fig. 1]). We argue that in this case structural consistency may yield non-necessary and unintuitive restrictions on players' beliefs.

Figure 4 shows (part of) an assessment (μ, π), where $\pi(R')=\pi(R'')=1$ and $\mu(y|h)=\mu(z|h)=1/2$. In this extensive form we have the following facts:

(a) If a conditional distribution $\mu^*(\cdot|h)$ can be derived from an uncorrelated prior, then $\mu^*(y|h)>0$ and $\mu^*(z|h)>0$ imply that $\mu^*(x|h)>0$. Hence the conditional distribution $\mu(\cdot|h)$ cannot be derived from an uncorrelated prior and the assessment (μ, π) is not structurally consistent.

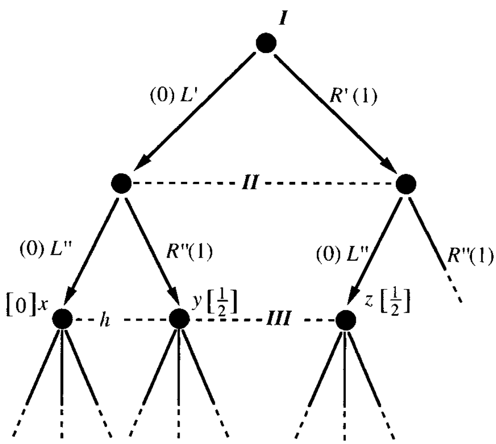

FIG. 4. A fully consistent assessment which violates structural consistency.

(b) But it is easy to check that the assessment (μ, π) is fully consistent and it can be derived form an independent conditional system using (3.1) and (3.2).

(c) For each independent conditional system σ on $S_{\mathrm{I}} \times S_{\mathrm{II}}$ (we may disregard player III) we have

$$\sigma((R', R'') \mid S_{\mathrm{I}} \times S_{\mathrm{II}}) = 1 \Rightarrow \sigma((L', L'') \mid \{(L', L''), (L', R''), (R', L'')\} = 0,$$

because by (2.1) and (2.2)

$$\begin{aligned} &\sigma((R', R'') \mid S_{\mathrm{I}} \times S_{\mathrm{II}}) \\ &\quad = 1 \Rightarrow \sigma((L', L'') \mid S_{\mathrm{I}} \times \{L''\}) = \sigma(\{L'\} \times S_{\mathrm{II}} \mid S_{\mathrm{I}} \times S_{\mathrm{II}}) = 0 \\ &\quad\quad \Rightarrow \sigma((L', L'') \mid \{(L', L''), (L', R''), (R', L'')\} \\ &\quad = \sigma((L', L'') \mid S_{\mathrm{I}} \times \{L''\})\, \sigma(S_{\mathrm{I}} \times \{L''\} \mid \{(L', L''), \\ &\quad\quad (L', R''), (R', L'')\}) = 0. \end{aligned}$$

Therefore, if an assessment (μ^*, π^*) of this extensive form is derived from an independent conditional system and $\pi^*(R') = \pi^*(R'') = 1$, we must have $\mu^*(x \mid h) = \sigma((L', L'') \mid \{(L', L''), (L', R''), (R', L'')\}) = 0$.

(d) By (a) and (c) we deduce that if (μ^*, π^*) satisfies independence, structural consistency, and $\pi^*(R') = \pi^*(R'') = 1$, there are only two possibilities: either $\mu^*(y \mid h) = 1$ or $\mu^*(z \mid h) = 1$.

(e) But in this extensive form players I and II are symmetric and we should at least allow for the possibility that the assessment reflects this symmetry. If (μ^*, π^*) is symmetric, $\mu^*(y \mid h) = \mu^*(z \mid h)$.

(f) By (c) and (e) it follows that the proposed assessment (μ, π) is the only one which satisfies independence, symmetry, and $\pi(R') = \pi(R'') = 1$.

It may be the case that in every solution of a game with this extensive form R' and R'' are chosen with probability one (this happens in the first example of Kreps and Ramey [12]). Then we deduce from (a)–(f) that in such a case every structurally consistent assessment which supports the solution violates either independence or symmetry.[10]

We conclude that structural consistency is neither necessary nor sufficient for independence and that it may yield unreasonable restrictions on beliefs.

[10] Note that this extensive form has *not* observable deviators. By Proposition 2.1, in every game with observable deviators every assessment derived from a conditional system with the independence property must satisfy structural consistency.

5. Strategic Independence and Tree-Extended Assessments

We now consider conditional systems on the set Z of terminal nodes instead of the set S of strategy profiles. The advantage of this approach is that the involved events are observable (at least in principle), while subsets of S do not always correspond to observable events.

Following [8], we use the letter ν for such systems, while we continue to use σ to denote a conditional system on $S_1 \times \cdots \times S_n$. Recall that $\mathscr{s}(x, a)$ is the immediate successor of node x following action a, and $Z(Y)$ is the set of terminal successors of nodes in Y.

Definition 5.1. A *tree-extended assessment* is a triple $(\nu, \mu, \pi) \in \Delta^*(Z) \times M \times \Pi$ such that

$$\forall i \in N,\ \forall h \in H_i,\ \forall x \in h, \qquad \mu(x \mid h) = \nu(Z(x) \mid Z(h)) \tag{5.1}$$

$$\forall i \in N,\ \forall h \in H_i,\ \forall x \in h,\ \forall a \in A(h), \qquad \pi(a \mid h) = \nu(Z(\mathscr{s}(s, a)) \mid Z(x)). \tag{5.2}$$

Equation (5.2) says that it must be possible to derive a unique profile of behavioral strategies from the given conditional system ν. Note that in games of imperfect information this condition is not satisfied by every $\nu \in \Delta^*(Z)$.

The following lemma says that a conditional system on $S_1 \times \cdots \times S_n$ satisfying the independence property always induces a tree-extended assessment.

Lemma 5.1. *Consider a strategic extended assessment* (σ, μ, π) *such that* $\sigma \in I\Delta^*(S_1 \times \cdots \times S_n)$ *and define* $\nu(\cdot \mid \cdot)\colon 2^Z \times (2^Z \setminus \varnothing) \to [0, 1]$ *as*

$$\forall W \in 2^Z, \quad \forall Y \in 2^Z \setminus \{\varnothing\}\ \nu(W \mid Y) = \sigma(S(W) \mid S(Y)). \tag{5.3}$$

Then $\nu \in \Delta^*(Z)$ *and* (ν, μ, π) *is a tree-extended assessment.*

Proof. Since σ is a conditional system and $W \subseteq Y$ implies that $S(W) \subseteq S(Y)$, it is straightforward to check that ν is also a conditional system. Equation (5.1) follows by (3.1) and (5.3). We now show that (5.2) holds.

Let $x \in h \in H_i$, $a \in A(h)$. We know that, by perfect recall, $S_i(x) = S_i(h)$ and $S(h) = S_i(h) \times S_{-i}(h)$. Therefore,

$$\begin{aligned}
\pi(a \mid h) &= \sigma(\{s \in S(h)\colon s_i(h) = a\} \mid S(h)\}) \\
&= \sigma(\{s_i \in S_i(x)\colon s_i(h) = a\} \times S_{-i}(h) \mid S_i(x) \times S_{-i}(h)) \\
&= \sigma(\{s_i \in S_i(x)\colon s_i(h) = a\} \times S_{-i}(x) \mid S_i(x) \times S_{-i}(x)) \\
&= \sigma(S(Z(\mathscr{s}(x, a))) \mid S(Z(x))) = \nu(Z(x, a) \mid Z(x)).
\end{aligned}$$

The first equality is (3.2), the second holds by perfect recall. The third equality is the crucial one, and it follows from the independence property (2.2). The fourth equality is just a consequence of the definitions and the fifth follows from (5.3). ∎

The following proposition shows that all the restrictions on assessments we are considering, when coupled with sequential rationality, correspond to refinements of subgame perfection. As the proof shows, this is due to the fact that a tree-extended assessment satisfies Bayes rule in all subgames.[11]

PROPOSITION 5.1. *Let (ν, π, μ) be a tree-extended assessment and assume that (μ, π) is sequentially rational, then π is a subgame perfect equilibrium profile.*

Proof. Let (ν, μ, π) be a tree-extended assessment, and let $P^\pi(y \mid x)$ denote the probability of reaching node y from a predecessor x of y. Then, by (2.1), (5.1), and (5.2) for all subgames Γ_x with root x, all information sets h in Γ_x, and all nodes $y \in h$, the following holds:

$$\mu(y \mid h)\left(\sum_{t \in h} P^\pi(t \mid x)\right) = \nu(Z(y) \mid Z(h))\, \nu(Z(h) \mid Z(x))$$

$$= \nu(Z(y) \mid Z(x)) = P^\pi(y \mid x). \tag{5.4}$$

This simply means that (μ, π) satisfies Bayes rule in each subgame. We now show that, given this fact, if π is not subgame perfect, (μ, π) is not sequentially rational. This establishes the proposition.

Assume that π is not subgame perfect. Then there are a game Γ_x with root x, an information set $h \in H_i$ in Γ_x, and actions $a, b \in A(h)$ such that $\pi(a|h) > 0$ and

$$\sum_{y \in h} P^\pi(y \mid x)\left[\sum_{z \in Z(\mathscr{s}(y, a))} P^\pi(z \mid \mathscr{s}(y, a))\, u_i(z)\right]$$

$$< \sum_{y \in h} P^\pi(y \mid x)\left[\sum_{z \in Z(\mathscr{s}(y, b))} P^\pi(z \mid \mathscr{s}(y, b))\, u_i(z)\right]. \tag{5.5}$$

[11] Let a sequentially rational assessment (μ, π) satisfy Bayes rule on the *equilibrium path*. Then we say that (μ, π) is a *weak sequential equilibrium*. This equilibrium concept is not a refinement of subgame perfection, because Bayes rule may be violated in subgames off the equilibrium path (cf. Myerson [18, pp. 170–171]).

Equation (5.5) implies that $\sum_{t \in h} P^{\pi}(t \mid x) > 0$, i.e., h is on the equilibrium path of Γ_x. Hence, (5.4) and (5.5) yield

$$\sum_{y \in h} \mu(y \mid h) \left[\sum_{z \in Z(\jmath(y, a))} P^{\pi}(z \mid \jmath(y, a))\, u_i(z) \right] < \sum_{y \in h} \mu(y \mid h) \left[\sum_{z \in Z(\jmath(y, b))} P^{\pi}(z \mid \jmath(y, b))\, u_i(z) \right], \tag{5.6}$$

which implies that (μ, π) is not sequentially rational. ∎

Definition 5.2. (cf. Fudenberg and Tirole [8]). A tree-extended assessment (ν, π, μ) is *generally reasonable* if

$$\forall i \in N, \quad \forall h \in H, \quad \forall a \in A(h), \quad \forall x, y \in h$$
$$\nu(Z(\jmath(x, a)) \mid Z(\{\jmath(x, a), \jmath(y, a)\})) = \nu(Z(x) \mid Z(\{x, y\})). \tag{5.1$'$}$$

Equation (5.1′) says that, according to ν, the choice of player i at $h \in H_i$ cannot affect the relative probabilities of the events "x is reached" and "y is reached," because i cannot distinguish x from y. We now show that this pattern of conditional probabilities on nodes and actions is implied by the independence property of conditional systems on strategies.

Proposition 5.2. *Consider a strategic extended assessment* (σ, μ, π) *such that* $\sigma \in I\Delta^*(S_1 \times \cdots \times S_n)$ *and derive* $\nu \in \Delta^*(Z)$ *as in Lemma* 5.1. *Then* (ν, μ, π) *is a generally reasonable extended assessment.*

Proof. Let $y, x \in h \in H_i$, $a \in A(h)$. Then

$$\begin{aligned}
&\nu(Z(\jmath(x, a)) \mid Z(\{\jmath(x, a), \jmath(y, a)\})) \\
&\quad = \sigma(S(Z(\jmath(x, a))) \mid S(Z(\{\jmath(x, a), \jmath(y, a)\}))) \\
&\quad = \sigma(\{s_i \in S_i(x) : s_i(h) = a\} \times S_{-i}(x) \mid \{s \in S(\{x, y\}) : s_i(h) = a\}) \\
&\quad = \sigma(\{s_i \in S_i(h) : s_i(h) = a\} \times S_{-i}(x) \mid \{s_i \in S_i(h) : s_i(h) = a\} \\
&\qquad \times S_{-i}(\{x, y\})) \\
&\quad = \sigma(S_i(x) \times S_{-i}(x) \mid S_i(x) \times S_{-i}(\{x, y\})) \\
&\quad = \sigma(S_i(x) \times S_{-i}(x) \mid S_i(\{x, y\}) \times S_{-i}(\{x, y\})) = \sigma(S(x) \mid S(\{x, y\})) \\
&\quad = \nu(Z(x) \mid Z(\{x, y\})).
\end{aligned}$$

The first equality follows by (5.3), the second is definitional and the third follows by perfect recall. The fourth equality is the crucial one and it follows by the independence property (2.2). The fifth equality follows again by perfect recall, the sixth is definitional, and the last one follows by (5.3). ▮

Fudenberg and Tirole [8, Proposition 6.1] show that every fully consistent assessment is part of a generally reasonable extended assessment. This result is also a consequence of Corollary 3.1 and Proposition 5.2. In the same proposition, Fudenberg and Tirole also claim that a generally reasonable extended assessment contains a fully consistent assessment, but the example of Fig. 2 shows that strategic independence—which is sufficient for general reasonableness—is not sufficient for full consistency. Hence Proposition 6.1 of Fudenberg and Tirole [8] must be false.[12]

It remains to be ascertained whether general reasonableness at least is equivalent to strategic independence. The answer is again negative. Furthermore, general reasonableness is strictly weaker than strategic independence even in the restricted class of games with observable deviators, where strategic independence is equivalent to full consistency.

Figures 5 and 6 represent a game which begins with simultaneous moves by players I and II. If II plays Across (A'') the game ends; otherwise player III observes I's choice (and observes II not choosing A'') and chooses a, b, or c. Figure 5 arbitrarily assigns the first move to player I and shows a pure strategies equilibrium assessment with $(s_{\mathrm{I}}, s_{\mathrm{II}}, s_{\mathrm{III}}) = [L', A'', (a, \text{ if } L'; b, \text{ if } R')]$. This assessment of Fig. 5 is sequentially rational if and only if $\hat{\mu} \geqslant 2/3$ and $\hat{v} \leqslant 1/3$.

Such an assessment (μ, π) is part of a generally reasonable extended assessment (v, μ, π). In order to construct an appropriate $v \in \Delta^*(Z)$ we can proceed as follows. Consider for simplicity the case in which $0 < \hat{\mu}, \hat{v} < 1$. Let $\varepsilon = 1/k$, where $k \to \infty$. Assign vanishing transition probabilities to arcs of the game tree as indicated in Fig. 5. (Note the "correlated trembles.") All the other probabilities of actions are $1 - o(1)$ as $k \to \infty$. Derive the corresponding v^k and let v be the limit of v^k. It may be checked that (v, μ, π) is a generally reasonable extended assessment. In particular,

$$v(Z(x) \mid Z(\{x, y\})) = 0 = v(Z(\mathcal{s}(x, L'')) \mid Z(\{\mathcal{s}(x, L''), \mathcal{s}(y, L'')\}))$$

$$= v(Z(\mathcal{s}(x, R'')) \mid Z(\{\mathcal{s}(x, R''), \mathcal{s}(y, R'')\})).$$

But strategic independence is violated because in Fig. 5 player III's opinion about player II's choice depends on the observed choice of player I. In this game strategic independence implies that III cannot rationally implement the strategy (a, if L';b, if R'). (Note, however, that the proposed

[12] All the equations in [8, Proof of Proposition 6.1] are correct, but the claim saying that the last equation implies the thesis is incorrect.

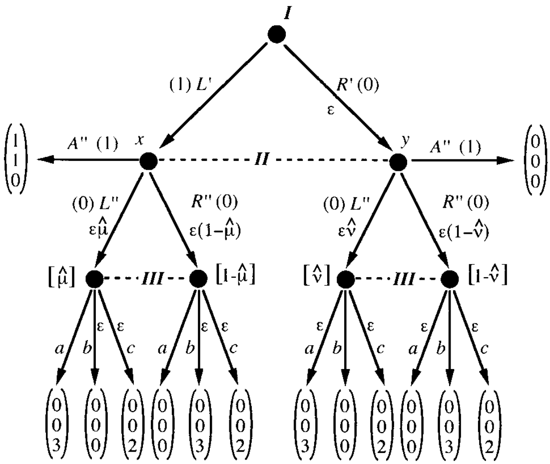

FIG. 5. A game with observable deviators where a generally reasonable extended assessment violates full consistency and independence.

path (L', A'') is a sequential equilibrium path; this point is commented on in Section 6.)

Now consider Fig. 6. Here the first move is arbitrarily given to player II and general reasonableness makes player III have the same belief about player II after different choices of player I. Therefore the set of generally reasonable extended assessments is not invariant—in an obvious sense—to

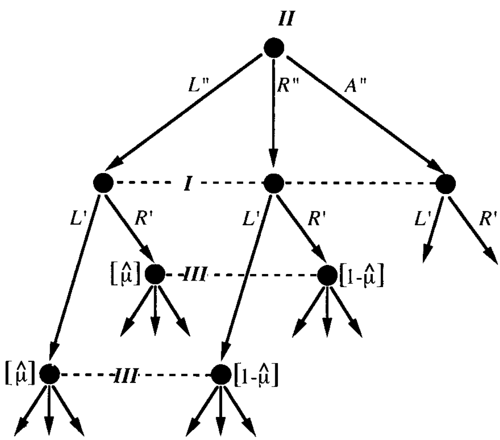

FIG. 6. The extensive form of Fig. 5 after interchanging the order of player I and II's "simultaneous" moves. Here general reasonableness is equivalent to independence and full consistency.

the transformation of interchanging essentially simultaneous moves.[13] This happens because in a generally reasonable extended assessment player i's action a cannot signal something about "past" moves of j which i does not observe, but a can signal something about the relative likelyhood of "future" unexpected moves of j even if j does not observe i's action. This may introduce an asymmetry in the restrictions on beliefs given by general reasonableness when we impose an arbitrary order to simultaneous moves.

Note that strategic extended assessments satisfying independence must be "invariant" with respect to interchanging of simultaneous moves because—up to obvious isomorphisms—such transformations do not change the strategy space and the sets of strategies reaching any given information set.

The introduction of the following property eliminates the problem just illustrated for the example above:

$$\forall i \in N, \quad \forall h \in H_i, \quad \forall x, y \in h, \quad \forall a, b \in A(h),$$
$$\nu(Z(\mathscr{s}(x, a)) \mid Z(\{\mathscr{s}(x, a), \mathscr{s}(x, b)\})) = \nu(Z(\mathscr{s}(y, a)) \mid Z(\{\mathscr{s}(y, a), \mathscr{s}(y, b)\})). \tag{5.2'}$$

Equation (5.2′) is a completion of (5.2) in the following sense: if (ν, μ, π) is a tree-extended assessment satisfying (5.2′), for each h, ν uniquely induces a whole conditional system $\pi(\cdot \mid \cdot; h) \in \Delta^*(A(h))$ with prior $\pi(\cdot \mid h)$ instead of the simple probability measure $\pi(\cdot \mid h)$. The conditional system $\pi(\cdot \mid \cdot; h)$ is derived from the "binary" conditional probabilities

$$\pi(a \mid \{a, b\}; h) = \nu(Z(\mathscr{s}(x, a) \mid Z(\{\mathscr{s}(x, a), \mathscr{s}(x, b)\}),$$

where $x \in h$ can be arbitrarily chosen.

Definition 5.2′. A tree-extended assessment (ν, μ, π) is *universally reasonable* if it satisfies (5.1′) and (5.2′).

Note that for every universally reasonable assessment of the extensive form in Fig. 5 $\hat{\mu} = \hat{\nu}$.

The next proposition shows that every conditional system satisfying strategic independence necessarily induces a universally reasonable extended assessment.

Proposition 5.2′. *Consider a strategic extended assessment (σ, μ, π) such that $\sigma \in I\Delta^*(S_1 \times \cdots \times S_n)$ and derive $\nu \in \Delta^*(Z)$ as in Lemma* 5.1. *Then (ν, μ, π) is a universally reasonable extended assessment.*

[13] A similar example shows that also Bonanno's [5] perfect Bayesian equilibrium concept has the same drawback.

Proof. We know by Proposition 2.1 that under the stated assumptions (ν, μ, π) is generally reasonable; i.e., it satisfies (5.1′). We now show that (ν, μ, π) satisfies also (5.2′).

Let $x, y \in h \in H_i$, $a, b \in A(h)$. We know that, by perfect recall, $S_i(x) = S_i(y) = S_i(h)$. Furthermore, it is true by definition that $S(t) = S_i(t) \times S_{-i}(t)$ for each $t \in h$. Therefore,

$$\begin{aligned}
&\nu(Z(\mathscr{s}(x, a) \mid Z(\{\mathscr{s}(x, a), \mathscr{s}(x, b)\})\\
&\quad = \sigma(\{s \in S(x) \colon s_i(h) = a\} \mid \{s \in S(x) \colon s_i(h) = a \vee s_i(h) = b\})\\
&\quad = \sigma(\{s_i \in S_i(h) \colon s_i(h) = a\} \times S_{-i}(x) \mid \{s_i \in S_i(h) \colon s_i(h)\\
&\quad = a \vee s_i(h) = b\} \times S_{-i}(x))\\
&\quad = \sigma(\{s_i \in S_i(h) \colon s_i(h) = a\} \times S_{-i}(y) \mid \{s_i \in S_i(h) \colon s_i(h)\\
&\quad = a \vee s_i(h) = b\} \times S_{-i}(y))\\
&\quad = \sigma(\{s \in S(y) \colon s_i(h) = a\} \mid \{s \in S(y) \colon s_i(h)\\
&\quad = a \vee s_i(h) = b\})\\
&\quad = \nu(Z(\mathscr{s}(y, a) \mid Z(\{\mathscr{s}(y, a), \mathscr{s}(y, b)\}),
\end{aligned}$$

where we have applied respectively (5.3), perfect recall, independence (i.e. (2.2)), again perfect recall, and again (5.3). ▮

Now the obvious question is whether universal reasonableness is equivalent to strategic independence. The following example shows that, somewhat surprisingly, universal reasonableness can be strictly weaker than strategic independence even in games with observable deviators.

Consider the extensive form with observable deviators depicted in Fig. 7: player I chooses the order of two essentially simultaneous moves by II and III, who do not observe I's choice. Therefore there are "crossing information sets." Player IV gets the move if and only if II and III choose left. Figure 7 also shows "correlated trembles" inducing limit beliefs where IV's conditional probability about I's choice is (1/2, 1/2), although IV is sure at the beginning of the game that I chooses A. Therefore strategic independence and full consistency are violated, but it may be checked that the tree-extended assessment induced by these "trembles" is universally reasonable.

However, we can state an equivalence result for a restricted class of games, i.e., multi-stage games with observable deviators in the agent form.

A game has a *multi-stage* structure if all the nodes of any given information set have the same number of predecessors. This means that at each information set h the player moving at h knows how many actions have been chosen in order to reach h. (This is the class of extensive games

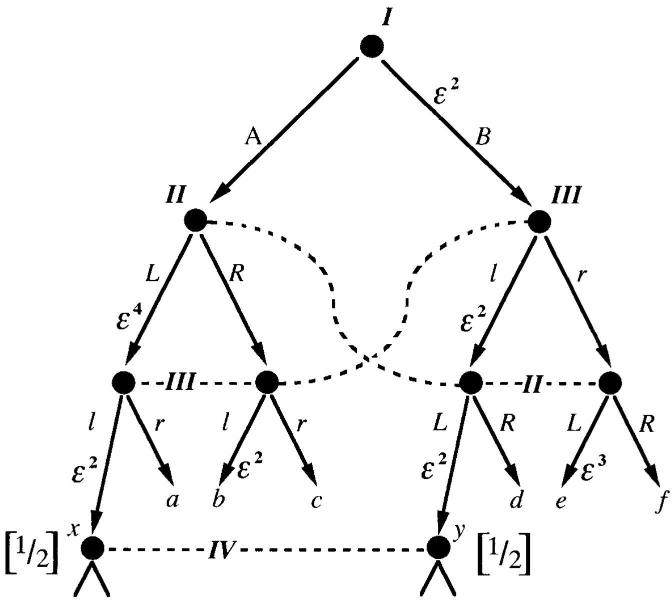

FIG. 7. An extensive form with observable deviators and "crossing information sets" where a universally reasonable extended assessment violates full consistency and independence.

defined in Von Neumann and Morgenstern's [23] seminal contribution.) Note that the game of Fig. 7 does not have a multi-stage structure.

A game Γ has *observable deviators in the agent form* if the corresponding game, say $\mathscr{A}(\Gamma)$, whereby distinct information sets belong to distinct players (called *agents* by Selten [20]) has observable, deviators. Clearly this condition is stronger than requiring that the original game have observable deviators. It is required that when an information set h is unexpectedly reached the player moving at h is able to observe who deviated from the expected path and also at which information sets the deviations have occurred. Games with observable deviators in the agent form are characterized in the Appendix.

All the examples preceding Fig. 7 are multi-stage games. The games depicted in Fig 1, 5, 6, and 7 have observable deviators in the agent form. Games of perfect information and, more generally, multi-stage games with observed actions have observable deviators in the agent form. Although two-person games of perfect recall have observable deviators, they need not have observable deviators in the agent form.

PROPOSITION 5.3. *In every multi-stage game with observable deviators in the agent form the following statements are equivalent*:

(i) (μ, π) *is part of a universally reasonable tree-extended assessment* (ν, μ, π),

(ii) (μ, π) *is part of a strategic extended assessment* (σ, μ, π) *such that* $\sigma \in I\Delta^*(S_1 \times \cdots \times S_n)$,

(iii) (μ, π) *is fully consistent.*

Proof. It is shown in the Appendix that (i) implies (iii). By Corollary 3.1 and Proposition 5.2′ this establishes the equivalence of (i), (ii), and (iii). ∎

We have not been able to ascertain whether the equivalence between universal reasonableness and strategic independence holds for general multi-stage games either with or without observable deviators.

6. Common and Heterogeneous Conditional Expectations

The example of Fig. 5 is somewhat weakened by the fact that, although the perfect Bayesian equilibrium $[L', A'', (a \text{ if } L'; b \text{ if } R')]$ cannot be supported by a fully consistent assessment, there is a sequential equilibrium with the same outcome. It can be shown that this is a general property; that is:

> For every multi-stage game with observable deviators in the agent form, every generally reasonable and sequentially rational extended assessment $(\hat{\nu}, \hat{\mu}, \hat{\pi})$ induces the same probability measure over terminal nodes as some sequential equilibrium (μ^*, π^*).

Here we prove this claim for games with the extensive form depicted in Fig. 5 (or Fig. 8 below) independently of payoffs. This gives a hint about the general proof.[14] Fix a sequentially rational and generally reasonable extended assessment $(\hat{\nu}, \hat{\mu}, \hat{\pi})$. First note that we may assume without loss of generality that condition (5.2′) holds at every information set except $\{x, y\}$, because $(\hat{\nu}, \hat{\mu}, \hat{\pi})$ satisfies (5.2). We consider two cases:

(i) either $\hat{\pi}(A'') < 1$ or $0 < \hat{\nu}(Z(x) \mid Z(\{x, y\})) < 1$,

(ii) $\hat{\pi}(A'') = 1$ and $\hat{\nu}(Z(x) \mid Z(\{x, y\})) \in \{0, 1\}$.

In case (i), $(\hat{\nu}, \hat{\mu}, \hat{\pi})$ is also universally reasonable and, by Proposition 5.3, $(\hat{\mu}, \hat{\pi})$ is a sequential equilibrium. This is easily checked if $\hat{\pi}(A'') < 1$, because in this case (5.2) implies (5.2′). If $0 < \hat{\nu}(Z(x) \mid Z(\{x, y\})) < 1$, apply the following:

[14] A general proof is available on request.

Remark 6.1. For every $\nu \in \Delta^*(Z)$, if (5.1′) holds, then for all h, x, $y \in h$, a, $b \in A(h)$, if $0 < \nu(Z(x) \mid Z(\{x, y\})) < 1$ then

$$\nu(Z(\mathscr{s}(x, a)) \mid Z(\{\mathscr{s}(x, a), \mathscr{s}(x, b)\})) = \nu(Z(\mathscr{s}(y, a)) \mid Z(\{\mathscr{s}(y, a), \mathscr{s}(y, b)\})).[15]$$

In case (ii), without loss of generality assume that $\hat{\nu}(Z(x)|Z(\{x, y\})) = 1$. Then for every action a by player I or II the probability of reaching player III's right information set given $(\hat{\mu}, \hat{\pi}/a)$ is zero. Thus we can modify the belief and the local strategy of III after R' without affecting the incentives of I and II. The outcome-equivalent sequential equilibrium (μ^*, π^*) is obtained assigning to the right information set of player III the same belief (and corresponding best reply) of the left information set.

This outcome-equivalence result may suggest that the weakness of "general reasonableness" is in some sense irrelevant, but we argue that the result crucially depends on the full strength of the common expectations assumption.

According to some authors it is plausible to assume that different players have different beliefs off the equilibrium path (see, e.g., Fudenberg and Tirole [9, pp. 332–333], Bonanno [5]). In the present context we can consider *heterogeneous extended assessments*, i.e., $(n+2)$-*tuples* $(\nu_1, ..., \nu_n, \mu, \pi)$ whereby each ν_i agrees with π as in (5.2) and agrees with μ only at information sets $h \in H_i$ and not at information sets $h' \in H_j$, $j \neq i$ (cf. (5.1)). In other words, μ gives the beliefs of each player i at his own information sets; the other beliefs for i are implicit in ν_i. Property (5.2) implies that the players hold the same beliefs at every information set h which can be reached with positive π-probability from the root of some subgame. Furthermore, it is easy to check that if $(\nu_1, ..., \nu_n, \mu, \pi)$ is a sequentially rational heterogeneous assessment, then π is a subgame perfect equilibrium (consider the proof of Proposition 5.1: for all $h \in H_i$, replace ν with ν_i in (5.4)). A heterogeneous extended assessment $(\nu_1, ..., \nu_n, \mu, \pi)$ is *generally reasonable* if each ν_i satisfies (5.1′) and is *universally reasonable* if each ν_i

[15] This is quite easily proved. In order to simplify the notation consider the conditional system $\rho \in \Delta^*(\{x, y\} \times \{a, b\})$ induced by ν (e.g., $\rho((x, a) \mid \{x\} \times \{a, b\}) = \nu(Z(\mathscr{s}(x, a)) \mid Z(\{\mathscr{s}(x, a), \mathscr{s}(x, b)\}))$) and write marginal and prior probabilities as follows: $\rho((x, a) \mid \{x\} \times \{a, b\}) = \rho(a \mid x)$, $\rho(\{x\} \times \{a, b\} \mid \{x, y\} \times \{a, b\}) = \rho(x)$, etc. We must show that $0 < \nu(Z(x) \mid Z(\{x, y\})) < 1$ implies that $\rho(a \mid x) = \rho(a \mid y)$. By (5.1′) ρ is such that

$$\rho(x \mid a) = \rho(x \mid b) = \rho(x) = \nu(Z(x) \mid Z(\{x, y\})).$$

If $0 < \nu(Z(x) \mid Z(\{x, y\})) < 1$, then

$$\rho(a \mid x) = \rho(x \mid a)\, \rho(a)/\rho(x) = \rho(a),$$

$$\rho(a \mid y) = \rho(y \mid a)\, \rho(a)/\rho(y) = \rho(a),$$

satisfies (5.1′) and (5.2′). The following example shows that, even in multi-stage games with observable deviators in the agent form, there are perfect Bayesian equilibrium outcomes supported by generally reasonable heterogeneous assessments, which *cannot* be supported by *universally* reasonable heterogeneous assessments.[16]

Consider the game depicted in Fig. 8 (a modification of Fig. 5). The pure strategy profile $[A', A'', (a \text{ if } L';\ b \text{ if } R')]$ is part of a sequentially rational and generally reasonable heterogeneous assessment $(\nu_{\mathrm{I}}, \nu_{\mathrm{II}}, \nu_{\mathrm{III}}, \mu, \pi)$, whereby L' and R' are equally likely according to ν_{II} while L' as infinitely more likely than R' according to ν_{III} (Figure 8 shows η-trembles inducing ν_{II} and correlated ε-trembles inducing ν_{III}). But outcome A' is not induced by any *universally reasonable* and sequentially rational heterogeneous assessment. In every such assessment player III has the same belief after L' and after R' and strategy (a if L'; b if R') cannot be a sequential best response. If player II assigns zero probability to this strategy, then A'' cannot be a best response. If player I assigns zero probability to A'', then A' cannot be a best response.

Appendix: Observable Deviators in the Agent Form

The analysis of this Appendix requires some additional notation:

Notation	Terminology
$h \in H$	Information sets, agents
$S_h = \{(a \text{ if } h): a \in A(h)\}$	Strategies of agent h
$H(x)$	Information set containing node x
$A(x) = A[H(x)]$	Actions available at x
$x < y$	Node x is a predecessor of node y
$x \leqslant y$	Either $x < y$ or $x = y$
$A(x \to Y) = \{a \in A(x) \mid \exists y \in Y: \mathcal{s}(x, a) \leqslant y\}$ $(Y \subseteq X)$	Actions which may lead from x to some $y \in Y$
$Z(x, \hat{A}) = \{z \in Z(x) \mid \exists a \in \hat{A}: \mathcal{s}(x, a) \leqslant z\}$ $(\hat{A} \subseteq A(x))$	Terminal successors of actions in $\hat{A} \subseteq A(x)$ from x

Recall that a game has *observable deviators in the agent form* if $S(h) = \mathsf{X}_{g \in H} S_g(h)$ for all $h \in H$, where $S(h)$ now is the set of profiles of

[16] The same game provides an analogous example for corresponding non-equilibrium solution concepts in the spirit of Pearce's [19] rationalizability.

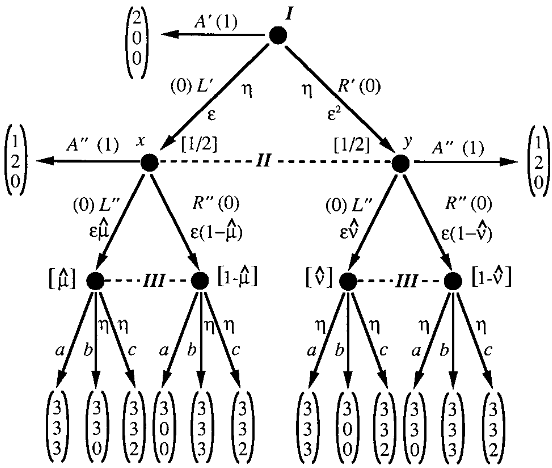

FIG. 8. A' is a perfect Bayesian equilibrium *outcome* supported by a *generally reasonable* heterogeneous assessment, but *not* supportable by any *universally reasonable* heterogeneous assessment.

agents' conditional choices reaching h, and $S_g(h)$ is the set of g's conditional choices not preventing h from being reached.

LEMMA A.1. *If a game has observable deviators in the agent form, then the following holds*:

(A.1) *For all* $h \in H$, $x, y \in h$, $x' < x$, *if* $A(x' \to h) \subset A(x')$ ($\subset$ *means* "*strict inclusion*"), *then there is one* (*and only one*) *node* $y' < y$ *such that* $H(x') = H(y')$ *and* $A(x' \to h) = A(y' \to h)$.[17]

Proof. Assume that (A.1) does not hold. Then there are two possible (non-mutually exclusive) cases.

(i) There are an information set h with two distinct nodes $x, y \in h$ and a node $x' < x$ such that $A(x' \to h) \subset A(x')$ and for every $y' < y$, $H(y') \neq H(x')$. Consider an agent form strategy profile $(s_g)_{g \in H}$ inducing a path through node y. Since the induced path does not intersect $H(x')$, we may assume without loss of generality that $s_{H(x')}$ prescribes an action $a^* \in A(x') \backslash A(x' \to h)$ if $H(x')$ is reached. Now consider a profile $(t_g)_{g \in H}$ inducing a path through x. Clearly $(s_g)_{g \in H} \in S(h)$ and $(t_g)_{g \in H} \in S(h)$; thus $(s_{H(x')}, t_{-H(x')}) \in S_{H(x')}(h) \times S_{-H(x')}(h)$. But $(s_{H(x')}, t_{-H(x')}) \notin S(h)$, because the induced path reaches x' where choice a^* prevents h from being reached.

[17] It may be checked that (A.1) is also a sufficient condition for "observable deviators" (see [2, Appendix]).

(ii) There are an information set h with two distinct nodes $x, y \in h$ and nodes $x' < x$, $y' < y$ such that $H(y') = H(x')$, $A(x' \to h) \subset A(x')$, and $A(x' \to h) \neq A(y' \to h)$. We may assume without loss of generality that there is an action $a^* \in A(y' \to h) \backslash A(x' \to h)$ (if for every choice of x, x', y, $y' A(y' \to h) \backslash A(x' \to h)$ were empty, then h would satisfy condition (A.1)). Let $w \in h$ be a node satisfying $\mathscr{s}(y', a^*) \leqslant w$. Consider an agent form strategy profile $(s_g)_{g \in H}$ inducing a path through $w \in h$ and a profile $(t_g)_{g \in H}$ inducing a path through x. Clearly $(s_g)_{g \in H} \in S(h)$ and $(t_g)_{g \in H} \in S(h)$; thus $(s_{H(x')}, t_{-H(x')}) \in S_{H(x')}(h) \times S_{-H(x')}(h)$. But $(s_{H(x')}, t_{-H(x')}) \notin S(h)$, because the induced path reaches x' where choice a^* prevents h from being reached.

We have shown that in both cases we can find two information sets h and g such that $S(h) \subset S_g(h) \times S_{-g}(h)$. Therefore deviators are not observable in the agent form if (A.1) does not hold.

Recall that a game is multi-stage if all the nodes of any given information set h have the same number τ of predecessors. Nodes, information sets, and actions will be often indexed according to this number τ, which represents the stage reached by the play.

PROPOSITION A.1. *In every multi-stage game with observable deviators in the agent form, a tree-extended assessment* (v, μ, π) *is universally reasonable only if* (μ, π) *is fully consistent.*

Proof. Assume that (v, μ, π) is universally reasonable. Then for all $h' \in H$, $x', y' \in h'$, $\hat{A} \subseteq A(h')$, $a \in \hat{A}$, Eq. (5.1′) implies that

$$v(Z(x', \hat{A}) \mid Z(x', \hat{A}) \cup Z(y', \hat{A})) = v(Z(x') \mid Z(\{x', y'\})) \tag{A.2}$$

and Eq. (5 .2′) implies that

$$v(Z(\mathscr{s}(x, a)) \mid Z(x, \hat{A})) = v(Z(\mathscr{s}(y, a)) \mid Z(y, \hat{A})). \tag{A.3}$$

These equations will be used repeatedly later on.

For every information set h^τ select an arbitrary node $v(h^\tau) \in h^\tau$. As in Fudenberg and Tirole [8, Section 6] we consider a sequence of strictly positive probabilities $v^k \in \Delta^o(Z)$ approaching v (i.e., for all $Z' \subseteq Z'' \subseteq Z$, $v(Z' \mid Z'') = \lim_{k \to \infty} v^k(Z')/v^k(Z''))$ and derive a sequence of behavior strategies as follows: for all h^τ, $c^\tau \in A(h^\tau)$

$$\pi^k(c^\tau \mid h^\tau) = v^k(Z(\mathscr{s}(w(h^\tau), c^\tau)))/v^k(Z(w(h^\tau))). \tag{A.4}$$

Note that the sequence π^k must converge to π, because (ν, μ, π) is a tree-extended assessment: for all h^τ, $c^\tau \in A(h^\tau)$

$$\pi(c^\tau \mid h^\tau) = \nu(Z(\mathscr{s}(w(h^\tau), c^\tau)) \mid (Z(w(h^\tau)))$$
$$= \lim_{k \to \infty} \nu^k(Z(\mathscr{s}(w(h^\tau), c^\tau)))/\nu^k(Z(w(h^\tau))) = \lim_{k \to \infty} \pi^k(c^\tau \mid h^\tau),$$

where the first equality follows from (5.2).

Consider an information set $h^{t+1} = \{x_1^{t+1}, ..., x_m^{t+1}\}$. Let $(x^0 = x_l^0, a_l^0, ..., x_l^t, a_l^t, x_l^{t+1})$ denote the sequence of nodes and actions leading to x_l^{t+1}, where x^0 is the root of the game tree and $x_l^{\tau+1} = \mathscr{s}(x_l^\tau, a_l^\tau)$, $l = 1, ..., m$, $\tau = 0, ..., t$. We have to show that for all $l = 1, ..., m$

$$\mu(x_l^{t+1} \mid h^{t+1}) = \lim_{k \to \infty} \left(\prod_{\tau=0}^{\tau=t} \pi^k(a_l^\tau \mid H(x_l^\tau)) \right) \Bigg/ \left[\sum_{j=1}^{j=m} \left(\prod_{\tau=0}^{\tau=t} \pi^k(a_j^\tau \mid H(x_j^\tau)) \right) \right].$$

Since $\mu(x^{t+1} \mid h^{t+1}) = \nu(Z(x^{t+1} \mid Z(h^{t+1}))$ and ν^k approaches ν, this is equivalent to the following: for all x^{t+1}, $y^{t+1} \in h^{t+1}$ such that $\mu(x^{t+1} \mid h^{t+1}) > 0$

$$\lim_{k \to \infty} \nu^k(Z(y^{t+1}))/\nu^k(Z(x^{t+1}))$$
$$= \lim_{k \to \infty} \left(\prod_{\tau=0}^{\tau=t} \pi^k(b^\tau \mid H(y^\tau)) \right) \Bigg/ \left(\prod_{\tau=0}^{\tau=t} \pi^k(a^\tau \mid H(x^\tau)) \right), \qquad \text{(A.5)}$$

where $(x^0, a^0, ..., x^t, a^t, x^{t+1})$ and $(x^0 = y^0, b^0, ..., y^t, b^t, y^{t+1})$ are the paths leading to x^{t+1} and y^{t+1} respectively.

We prove (A.5) by induction on t + 1. The basis step (with $t + 1 = 0$) is trivial.

Inductive step. Assume that (A.5) holds for $t + 1 = 0, ..., T$. We must show that it holds for $t + 1 = T + 1$.

Fix an information set h^{T+1}. For each pair x^{T+1}, $y^{T+1} \in h^{T+1}$ let $(x^0, a^0, ..., x^T, a^T, x^{T+1})$ and $(x^0 = y^0, b^0, ..., y^T, b^T, y^{T+1})$ be the paths leading to x^{T+1} and y^{T+1} respectively. By (A.1) of Lemma A.1 we can partition the stages $\tau = 0, ..., T$ as follows:

(i) There are stages τ such that for all x^{T+1}, $y^{T+1} \in h^{T+1}$, $H(x^\tau) = H(y^\tau) := h^\tau$ and $A(x^\tau \to h^{T+1}) = A(y^\tau \to h^{T+1})$: $= A(h^\tau \to h^{T+1}) \subset A(h^\tau)$.

(ii) All the other stages τ are such that for all x^{T+1}, $y^{T+1} \in h^{T+1}$, $A(x^\tau \to h^{T+1}) = A(x^\tau)$, $A(y^\tau \to h^{T+1}) = A(y^\tau)$.

Let $\tau = \vartheta$ be the *last* stage of type (i) before $T + 1$ (if such a stage does not exist, let $\vartheta = 0$). Note that $\lim_{k \to \infty} \nu^k(Z(y^{T+1}))/\nu^k(Z(x^{T+1}))$ can be decomposed as

$$\lim_{k\to\infty} \nu^k(Z(y^{T+1}))/\nu^k(Z(x^{T+1}))$$

$$= \lim_{k\to\infty} \frac{\nu^k(Z(y^\vartheta, A(h^\vartheta \to h^{T+1})))}{\nu^k(Z(x^\vartheta, A(h^\vartheta \to h^{T+1})))}$$

$$\times \frac{\nu^k(Z(\jmath(y^\vartheta, b^\vartheta)))/\nu^k(Z(y^\vartheta, A(h^\vartheta \to h^{T+1})))}{\nu^k(Z(\jmath(x^\vartheta, a^\vartheta)))/\nu^k(Z(x^\vartheta, A(h^\vartheta \to h^{T+1})))}$$

$$\times \left(\prod_{\tau=\vartheta+1}^{\tau=T} \frac{\nu^k(Z(\jmath(y^\tau, b^\tau)))/\nu^k(Z(y^\tau))}{\nu^k(Z(\jmath(x^\tau, a^\tau)))/\nu^k(Z(x^\tau))} \right) \tag{A.6}$$

(recall that $x^{\tau+1} = \jmath(x^\tau, a^\tau)$, $y^{\tau+1} = \jmath(y^\tau, b^\tau)$).

Claim. *If* $\mu(x^{T+1} \mid h^{T+1}) > 0$, *then*

$$\forall \tau \in \{\vartheta+1, ..., T\},\ \nu(Z(\jmath(x^\tau, a^\tau)) \mid Z(x^\tau)) > 0, \tag{A.7}$$

$$\nu(Z(\jmath(x^\vartheta, a^\vartheta)) \mid Z(x^\vartheta, A(h^\vartheta \to h^{T+1}))) > 0, \tag{A.8}$$

$$\mu(x^\vartheta \mid h^\vartheta) > 0. \tag{A.9}$$

Proof of the claim. Assume that $\mu(x^{T+1} \mid h^{T+1}) > 0$. Since $\mu(x^{T+1} \mid h^{T+1}) = \nu(Z(x^{T+1} \mid Z(h^{T+1}))$ and ν^k approaches ν, $\lim_{k\to\infty} \nu^k(Z(y^{T+1}))/\nu^k(Z(x^{T+1})) < \infty$ for all $y^{T+1} \in h^{T+1}$.

Choose a path $y^\vartheta, b^\vartheta, ..., y^T, b^T, y^{T+1} \in h^{T+1}$ as

$$\mu(y^\vartheta \mid h^\vartheta) > 0$$

$$\nu(Z(\jmath(y^\vartheta, b^\vartheta)) \mid Z(y^\vartheta, A(h^\vartheta \to h^{T+1}))) > 0,$$

$$\nu(Z(\jmath(y^\tau, b^\tau)) \mid Z(y^\tau)) > 0, \quad \tau = \vartheta+1, ..., T.$$

In particular, it is possible to satisfy the third equality because—by definition of ϑ—$Z(y^\tau, A(y^\tau \to h^{T+1})) = Z(y^\tau)$ for all $\tau = \vartheta+1, ..., T$, and we can always find $b^\tau \in A(y^\tau \to h^{T+1})$ such that $\nu(Z(\jmath(y^\tau, b^\tau)) \mid Z(y^\tau, A(y^\tau \to h^{T+1}))) > 0$.

Equation (A.2) and $\mu(y^\vartheta \mid h^\vartheta) > 0$ imply that

$$\lim_{k\to\infty} \frac{\nu^k(Z(y^\vartheta, A(h^\vartheta \to h^{T+1})))}{\nu^k(Z(x^\vartheta, A(h^\vartheta \to h^{T+1})))} = \lim_{k\to\infty} \frac{\nu^k(Z(y^\vartheta))}{\nu^k(Z(x^\vartheta))} > 0. \tag{A.10}$$

Our choice of y^{T+1} implies that all the factors on the right-hand side of (A.6) have strictly positive limit, and if at least one of the inequalities (A.7)–(A.9) is not satisfied at least one factor has infinite limit. Hence (A.7)–(A.9) must hold, because otherwise $\lim_{k\to\infty} \nu^k(Z(y^{T+1}))/\nu^k(Z(x^{T+1})) = \infty$

for this choice of y^{T+1}, contradicting $\mu(x^{T+1}|h^{T+1})>0$. This concludes the proof of the claim.

Assume that $\mu(x^{T+1}|h^{T+1})>0$ and consider an arbitrary node $y^{T+1}\in h^{T+1}$. Equality (A.10), inequality (A.9), and the induction hypothesis imply that

$$\lim_{k\to\infty}\frac{\nu^k(Z(y^\vartheta, A(h^\vartheta\to h^{T+1})))}{\nu^k(Z(x^\vartheta, A(h^\vartheta\to h^{T+1})))}$$

$$=\lim_{k\to\infty}\left(\prod_{\tau=0}^{\tau=\vartheta-1}\pi^k(b^\tau\,|\,H(y^\tau))\right)\Big/\left(\prod_{\tau=0}^{\tau=\vartheta-1}\pi^k(a^\tau\,|\,H(x^\tau))\right)<\infty. \tag{A.11}$$

We already know that for all τ, w^τ, $c^\tau\in A(w^\tau)$

$$\lim_{k\to\infty}\nu^k(Z(\mathscr{s}(w^\tau, c^\tau)))/\nu^k(Z(w^\tau))$$

$$=\nu(Z(\mathscr{s}(w^\tau, c^\tau))\,|\,Z(w^\tau))$$

$$=\pi(c^\tau\,|\,H(w^\tau))=\lim_{k\to\infty}\pi^k(c^\tau\,|\,H(w^\tau)).$$

This fact and (A.7) imply that for all $\tau=\vartheta+1, ..., T$

$$\lim_{k\to\infty}\frac{\nu^k(Z(\mathscr{s}(y^\tau, b^\tau)))/\nu^k(Z(y^\tau))}{\nu^k(Z(\mathscr{s}(x^\tau, a^\tau)))/\nu^k(Z(x^\tau))}=\lim_{k\to\infty}\frac{\pi^k(b^\tau\,|\,H(y^\tau))}{\pi^k(a^\tau\,|\,H(x^\tau))}<\infty.$$

Therefore we get

$$\lim_{k\to\infty}\left(\prod_{\tau=\vartheta+1}^{\tau=T}\frac{\nu^k(Z(\mathscr{s}(y^\tau, b^\tau)))/\nu^k(Z(y^\tau))}{\nu^k(Z(\mathscr{s}(x^\tau, a^\tau)))/\nu^k(Z(x^\tau))}\right)$$

$$=\lim_{k\to\infty}\left(\prod_{\tau=\vartheta+1}^{\tau=T}\pi^k(b^\tau\,|\,H(y^\tau))\right)\Big/\left(\prod_{\tau=\vartheta+1}^{\tau=T}\pi^k(a^\tau\,|\,H(x^\tau))\right)<\infty. \tag{A.12}$$

By (A.8) and (A.3) respectively we get

$$0<\lim_{k\to\infty}\nu^k(Z(\mathscr{s}(x^\vartheta, a^\vartheta)))/\nu^k(Z(x^\vartheta, A(h^\vartheta\to h^{T+1})))$$

$$=\lim_{k\to\infty}\nu^k(Z(\mathscr{s}(w(h^\vartheta), a^\vartheta)))/\nu^k(Z(w(h^\vartheta), A(h^\vartheta\to h^{T+1})))$$

$$=\lim_{k\to\infty}\frac{\nu^k(Z(\mathscr{s}(w(h^\vartheta), a^\vartheta)))}{\nu^k(Z(w(h^\vartheta)))}\times\frac{\nu^k(Z(w(h^\vartheta)))}{\nu^k(Z(w(h^\tau), A(h^\vartheta\to h^{T+1})))}.$$

The same equalities (but possibly not the inequality) hold with $(x^\vartheta, a^\vartheta)$ replaced by $(y^\vartheta, b^\vartheta)$. Therefore (A.4) yields

$$\lim_{k\to\infty} \frac{v^k(Z(\mathscr{s}(y^\vartheta, b^\vartheta)))/v^k(Z(y^\vartheta, A(h^\vartheta \to h^{T+1})))}{v^k(Z(\mathscr{s}(x^\vartheta, a^\vartheta)))/v^k(Z(x^\vartheta, A(h^\vartheta \to h^{T+1})))}$$

$$= \lim_{k\to\infty} \frac{v^k(Z(\mathscr{s}(w(h^\vartheta), b^\vartheta)))/v^k(Z(w(h^\vartheta)))}{v^k(Z(\mathscr{s}(w(h^\vartheta), a^\vartheta)))/v^k(Z(w(h^\vartheta)))}$$

$$= \lim_{k\to\infty} \frac{\pi^k(b^\vartheta \mid h^\vartheta)}{\pi^k(a^\vartheta \mid h^\vartheta)} < \infty. \tag{A.13}$$

Equations (A.6) and (A.11)–(A.13) imply that

$$\lim_{k\to\infty} v^k(Z(y^{T+1}))/v^k(Z(x^{T+1}))$$

$$= \lim_{k\to\infty} \left(\prod_{\tau=0}^{\tau=T} \pi^k(b^\tau \mid H(y^\tau))\right) \Bigg/ \left(\prod_{\tau=0}^{\tau=T} \pi^k(a^\tau \mid H(x^\tau))\right).$$

References

1. P. Battigalli, Strategic independence, generally reasonable extended assessments, and consistent assessments, mimeo, Istituto di Economia Politica, Bocconi University, (revised) 1992.
2. P. Battigalli, Strategic independence and perfect Bayesian equilibria, Politecnico di Milano, Dip. Econ. Produz., Rapporto Interno 93-011, 1993.
3. P. Battigalli and P. Veronesi, A note on stochastic independence without savage–null events, mimeo, Istituto di Economia Politica, Bocconi University, 1992.
4. L. Blume, A. Brandenburger, and E. Dekel, Lexicographic probabilities and choice under uncertainty, *Econometrica* **59** (1991), 61–79.
5. G. Bonanno, A characterization of sequential equilibrium, mimeo, Department of Economics, UC Davis, 1992.
6. D. Fudenberg, D. Kreps, and D. Levine, On the robustness of equilibrium refinements, *J. Econ. Theory* **44** (1988), 354–380.
7. D. Fudenberg and D. Levine, Self-confirming equilibrium, *Econometrica* **61** (1993), 523–545.
8. D. Fudenberg and J. Tirole, Perfect Bayesian equilibrium and sequential equilibrium, *J. Econ. Theory* **53** (1991), 236–260.
9. D. Fudenberg and J. Tirole, "Game Theory," MIT Press, Cambridge, MA, 1991.
10. P. Hammond, Extended probabilities for decision theory and games, mimeo, Department of Economics, Stanford University, 1987.
11. E. Kohlberg and P. Reny, On the rationale of perfect equilibrium, mimeo, 1992.
12. D. Kreps and G. Ramey, Structural consistency, consistency and sequential rationality, *Econometrica* **55** (1987), 1331–1348.
13. D. Kreps and R. Wilson, Sequential equilibria, *Econometrica* **50** (1982), 863–894.
14. A. McLennan, The space of conditional systems is a ball, *Internat. J. Game Theory* **18** (1989), 125–139.

15. A. McLennan, Consistent conditional systems in noncooperative game theory, *Internat. J. Game Theory* **18** (1989), 141–174.
16. P. Milgrom and R. Weber, Distributional strategies for games with incomplete information, *Math. Oper. Res.* **10** (1985), 619–632.
17. R. Myerson, Multistage games with communication, *Econometrica* **54** (1986), 323–358.
18. R. Myerson, "Game Theory. Analysis of Conflict," Harvard Univ. Press, Cambridge MA, 1991.
19. D. Pearce, Rationalizable strategic behavior and the problem of perfection, *Econometrica* **52** (1984), 1029–1050.
20. R. Selten, Reexamination of the perfectness concept for equilibrium points in extensive games, *Internat. J. Game Theory* **4** (1975), 25–55.
21. J. Swinkels, Independence for conditional probability systems, mimeo, Kellogg Graduate School of Management, Northwestern University, 1993.
22. E. Van Damme, "Stability and Perfection of Nash Equilibria" (2nd ed.), Springer-Verlag, Berlin, 1991.
23. J. Von Neumann and O. Morgenstern, "Theory of Games and Economic Behavior," Princeton Univ. Press, Princeton, 1944.