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Notes, Comments, and Letters to the Editor

A Note on Stochastic Independence without Savage-Null Events*

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We provide two axiomatic characterizations of a stochastic independence property for conditional probability systems, previously proposed by Hammond. One characterization relies on the theory of lexicographic expected utility due to L. Blume *et al.* [*Econometrica* **59** (1991), 61–79]; the other relies on the theory of conditional expected utility maximization due to Myerson ["Game Theory: Analysis of Conflict," Harvard Univ. Press, Cambridge, MA, 1991]. *Journal of Economic Literature* Classification Number: D81.

Given a finite, product *space of states of the world* $\Omega = \Omega_1 \times \cdots \times \Omega_n$ and a probability measure $\mu: 2^{\Omega} \to [0, 1]$, the usual condition of stochastic independence states that for every $\omega = (\omega_1, ..., \omega_n) \in \Omega$, $\mu(\omega) = \prod_{i=1}^{n} \mu_i(\omega_i)$, where μ_i is the *marginal* probability measure of μ on Ω_i, $i = 1, ..., n$. This multiplicative condition is equivalent to the following: $\forall i$, $\forall \omega_i \in \Omega_i$, $\forall E_{-i} \subset \Omega_{-i} = \prod_{j \neq i} \Omega_j$, if $\mu(\Omega_i \times E_{-i}) > 0$, then $\mu(\{\omega_i\} \times E_{-i} \mid \Omega_i \times E_{-i}) = \mu_i(\omega_i)$. This means that, in order to evaluate the probabilities on Ω_i, the decision maker (DM henceforth) does not take into account any information regarding Ω_{-i}. This condition can easily be obtained from the axiomatic system of Anscombe and Aumann [1] by adding a very intuitive axiom, as

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μ	σ	τ
s	1/3	0
t	2/3	0

FIG. 1. A product measure on $\Omega = \{s, t\} \times \{\sigma, \tau\}$.

suggested by Blume *et al.* [5]. Nevertheless, this way of formulating the stochastic independence condition turns out to be inappropriate in some dynamic contexts. In fact, it is sometimes necessary to take into account events with 0-*prior* probability (called Savage-null events) and to have well-defined conditional probabilities on them. In the theory of extensive games, for instance, there may be some nodes in the game tree which are reached with 0-*prior* probability according to the "equilibrium path" and at which the beliefs of the moving player are important to sustain the equilibrium itself.

The following example illustrates the problem: let $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 = \{s, t\}$ and $\Omega_2 = \{\sigma, \tau\}$ and let μ be a probability measure on Ω defined as in Fig. 1. Observe that μ can be obtained as a product of two marginal probability measures on Ω_1 and Ω_2. Consider the following two conditional probabilities: $\mu(\{s\} \times \{\sigma\} \mid \Omega_1 \times \{\sigma\})$ and $\mu(\{s\} \times \{\tau\} \mid \Omega_1 \times \{\tau\})$. These probabilities can be regarded as the marginal probabilities on Ω_1 conditional on the occurrence of $\{\sigma\}$ and $\{\tau\}$ respectively. If $\{s\}$ is assumed independent of events in Ω_2, it seems reasonable that these probabilities coincide, by extension of the property that characterizes the independence condition stated above. However, this need not be the case. In fact, since the event $\{\Omega_1 \times \{\tau\}\}$ has 0-*prior* probability, we can choose any version of conditional probability on it. Consider, for instance, the conditional probability measure in Fig. 2. Since $\mu(\cdot \mid \Omega_1 \times \{\sigma\})$ is identical

$\mu(. \mid \Omega_1 \times \{\tau\})$	σ	τ
s	0	1/2
t	0	1/2

FIG. 2. A version of the conditional probability $\mu(\cdot \mid \Omega_1 \times \{\tau\})$.

to the *prior* $\mu(\cdot)$, we have that a change in information concerning Ω_2 can possibly lead to a change in the probability evaluations on Ω_1, i.e., violate the independence condition (notice also that the probability measure in Fig. 2 can be obtained as a product of two marginal probability measures).

One simple way to solve the problem is to assume away Savage-null events, so that all conditional probability measures are well defined. This approach, however, poses other difficulties. In game theory, for instance, it would also rule out pure-strategy Nash equilibria. Alternative approaches have been put forward in the literature.

Blume *et al.* [5] propose some modifications of Anscombe and Aumann's system of axioms which on the one hand rule out Savage-null events and on the other still preserve the idea that some events may be "infinitely less likely" than others. The main feature of their approach is the weakening of the Archimedean axiom. They develop a consistent non-Archimedean theory of subjective expected utility, in which a preference relation between acts is represented *lexicographically* by means of what they call "lexicographic probability systems."

Another author who deals with this kind of problems is Myerson [10]. He proposes an axiomatic system directly in terms of *conditional* preferences: a DM is supposed to have well-defined preferences conditional on each event $A \subseteq \Omega$. He then provides a numerical representation (as opposed to a lexicographic one as in Blume *et al.* [5]) of the conditional preference relations, using the so-called "conditional probability systems" (CPSs henceforth).

In this paper we prove that both the above-mentioned approaches, which are briefly summarized in Section 1, and the axiom of stochastic independence proposed by Blume *et al.* [5] (or some strengthening of it) lead to a condition of stochastic independence in terms of CPSs, which has been first proposed by Hammond [6,7]. In addition to being very intuitive, this condition enables us to deal easily with 0-*prior* probability events. In particular, within the theory of extensive games it plays a major role in Battigalli [4]. (See also Battigalli [2,3].)

In Section 2 we introduce the above-mentioned notions of conditional probability system and of stochastic independence. In Section 3 we state our main theorems.

1. Axiomatic Theories of Subjective Expected Utility

Let X be a set of *consequences* and Ω a *finite* set of *states of the world.* Assume that Ω is a product space: $\Omega = \Omega_1 \times \cdots \times \Omega_n$. Let $\mathscr{P}$ denote the set of simple probability measures on X and $\mathscr{P}^\Omega$ the set of acts (as in Anscombe and Aumann's approach, an act is a function $f: \Omega \to \mathscr{P}$). For

$f \in \mathscr{P}^{\Omega}$, f_{ω} denotes the (roulette) lottery associated with $\omega \in \Omega$, $f_A = (f_{\omega})_{\omega \in A}$ and $f_{-A} = f_{\Omega - A}$. We say that $f \in \mathscr{P}^{\Omega}$ is *constant* if $f_{\omega} = f_{\omega'}$ $\forall \omega, \omega' \in \Omega$. Given a partition $\{J, K\}$ of $\{1, ..., n\}$, denote $\Omega_J = \prod_{i \in J} \Omega_i$ and $\Omega_K = \prod_{i \in K} \Omega_i$. An act f is said *constant across* Ω_J if for every $\omega_J, \omega'_J \in \Omega_J$ and all $\omega_K \in \Omega_K$, $f_{(\omega_J, \omega_K)} = f_{(\omega'_J, \omega_K)}$. Notice that $\mathscr{P}^{\Omega}$ is a *mixture set*: for $f, g \in \mathscr{P}^{\Omega}$, $\alpha \in [0, 1]$, $\alpha f + (1-\alpha) g$ is the act giving probability $\alpha f_{\omega}(x) + (1-\alpha) g_{\omega}(x)$ to the consequence x if the state ω obtains. We use "$f \alpha g$" to denote the mixture "$\alpha f + (1-\alpha) g$."

1.1. *Anscombe and Aumann's System of Axioms*

The following are the usual axioms for an expected-utility representation of preferences on $\mathscr{P}^{\Omega}$. Let $\succcurlyeq$ be a binary relation on $\mathscr{P}^{\Omega}$ (preference relation); $\succ$ and $\sim$ are defined in the usual way.

A1 (Order). $\succcurlyeq$ is transitive and complete on $\mathscr{P}^{\Omega}$.

A2 (Objective Independence). For all $f, g, h \in \mathscr{P}^{\Omega}$ and $\alpha \in (0, 1]$, $f \succ$ (resp. $\sim$) $g \Rightarrow f \alpha h \succ$ (resp. $\sim$) $g \alpha h$.

A3 (Non-triviality). There are $f, g \in \mathscr{P}^{\Omega}$ such that $f \succ g$.

A4 (Archimedean Property). $f \succ g \succ h \Rightarrow \exists \alpha, \beta \in (0, 1), \alpha > \beta$, such that $f \alpha h \succ g \succ f \beta h$.

DEFINITION 1.1. Given $A \subseteq \Omega$, $f, g \in \mathscr{P}^{\Omega}$, f is *preferred* to g *conditional* on A, written $f \succcurlyeq_A g$, if $(f_A, h_{-A}) \succcurlyeq (g_A, h_{-A})$ for some $h \in \mathscr{P}^{\Omega}$. (A1, A2 imply that this definition is independent of the choice of h.)[1]

THEOREM 1.1. *Under Axioms* A1 *and* A2, *the following property holds*: (*Strong Subjective Substitution*) $\forall f, g \in \mathscr{P}^{\Omega}$, $\forall A, B \subseteq \Omega$ *with* $A \cap B = \varnothing$, *if* $f \succcurlyeq_A g$ *and* $f \succ_B$ (*resp.* $\succcurlyeq_B$) g *then* $f \succ_{A \cup B}$ (*resp.* $\succcurlyeq_{A \cup B}$) g (*cf. Blume et al.* [5, *Theorem* 4.1]).

"Strong subjective substitution" is often referred to as the sure thing principle. It will be used repeatedly later.

DEFINITION 1.2. An event $A \subseteq \Omega$ is *Savage-null* if $f \sim_A g$ $\forall f, g \in \mathscr{P}^{\Omega}$.

Notice that under A1–A3, there is at least one state ω^* which is not Savage-null (apply strong subjective substitution and A3).

[1] Suppose by way of contradiction that for some acts h and d we have (a) $(f_A, h_{-A}) \succcurlyeq (g_A, h_{-A})$ but (b) $(g_A, d_{-A}) \succ (f_A, d_{-A})$. Then (a) and A2 imply that $(f_A, h_{-A}) \frac{1}{2} (g_A, d_{-A}) \succcurlyeq (g_A, h_{-A}) \frac{1}{2} (g_A, d_{-A})$ and (b) and A2 imply that $(g_A, h_{-A}) \frac{1}{2} (g_A, d_{-A}) \succ (g_A, h_{-A}) \frac{1}{2} (f_A, d_{-A})$. Then A1 implies that $(f_A, h_{-A}) \frac{1}{2} (g_A, d_{-A}) \succ (g_A, h_{-A}) \frac{1}{2} (f_A, d_{-A})$, but this contradicts $(f_A, h_{-A}) \frac{1}{2} (g_A, d_{-A}) = ((f \frac{1}{2} g)_A, (h \frac{1}{2} d)_{-A}) = (g_A, h_{-A}) \frac{1}{2} (f_A, d_{-A})$.

A5 (Non-null-state Independence). For all states $\omega, \omega' \in \Omega$ which are not Savage-null and for any two constant acts $f, g \in \mathscr{P}^{\Omega}$, $f \succsim_{\omega} g \Leftrightarrow f \succsim_{\omega'} g$.

1.2. *Blume, Brandenburger, and Dekel's approach*

Let us recall some definitions and introduce the weakening of the Archimedean axiom put forward by Blume *et al.* [5, Axiom 4″].

DEFINITION 1.3. For disjoint events $A, B \subseteq \Omega$ with $A \neq \emptyset$, $A \gg B$ (read "A is infinitely more likely than B") if $f \succ_A g \Rightarrow (f_{-B}, h_B) \succ_{A \cup B} (g_{-B}, d_B)$ for all $h, d \in \mathscr{P}^{\Omega}$.

A4″. There exists a partition $\{\Lambda_1, ..., \Lambda_R\}$ of Ω such that:

(i) $\forall r = 1, ..., R$, $f \succ_{\Lambda_r} g \succ_{\Lambda_r} h \Rightarrow \exists \alpha, \beta \in (0, 1), \alpha > \beta$, such that $f \alpha h \succ_{\Lambda_r} g \succ_{\Lambda_r} f \beta h$.

(ii) $\Lambda_1 \gg \Lambda_2 \cdots \gg \Lambda_{R-1} \gg \Lambda_R$.

A5′ (State Independence). For all states $\omega, \omega' \in \Omega$ and for any two constants acts $f, g \in \mathscr{P}^{\Omega}$, $f \succsim_{\omega} g \Leftrightarrow f \succsim_{\omega'} g$.

Since A1–A3 imply that at least one state ω^* is non-null, the additional axiom A.5′excludes the existence of Savage-null events.[2]

The following is an axiom of stochastic independence (see Blume *et al.* [5, axiom 6]):

A6 (Stochastic Independence). For any $i = 1, ..., n$, every pair of acts $f, g \in \mathscr{P}^{\Omega}$ constant across Ω_i and every $\omega_i, \omega'_i \in \Omega_i$: $f \succsim_{\{\omega_i\} \times \Omega_{-i}} g \Leftrightarrow f \succsim_{\{\omega'_i\} \times \Omega_{-i}} g$.

The following lemma will be useful later:

LEMMA 1.1. *Under axioms* A1 *and* A2, *Axiom* A6 *holds if and only if for all* $i = 1, ..., n$, *for all* $D_{-i} \subseteq \Omega_{-i}$, *for every pair of acts* $f, g \in \mathscr{P}^{\Omega}$ *constant across* Ω_i *and every* $\omega_i, \omega'_i \in \Omega_i$: $f \succsim_{\{\omega_i\} \times D_{-i}} g \Leftrightarrow f \succsim_{\{\omega'_i\} \times D_{-i}} g$.

Proof. (If) Obvious.

(Only if) Fix $\emptyset \neq D_{-i} \subseteq \Omega_{-i}$, $\omega_i, \omega'_i \in \Omega_i$ and suppose $f \succsim_{\{\omega_i\} \times D_{-i}} g$. We show that $f \succsim_{\{\omega'_i\} \times D_{-i}} g$. Let h be any act constant across Ω_i and consider the following two acts: $f^* = (f_{(\Omega_i \times D_{-i})}, h_{(\Omega_i \times (-D_{-i}))})$ and $g^* = (g_{(\Omega_i \times D_{-i})}, h_{(\Omega_i \times (-D_{-i}))})$. By definition of f^* and g^* we have $f^* \succsim_{\{\omega_i\} \times D_{-i}} g^*$ and $f^* \sim_{\{\omega_i\} \times (-D_{-i})} g^*$. Thus, by strong subjective substitution, $f^* \succsim_{\{\omega_i\} \times \Omega_{-i}} g^*$. Now by A6, we have $f^* \succsim_{\{\omega'_i\} \times \Omega_{-i}} g^*$. We show that

[2] By A1–A3 there are f, g, and ω^* such that $f \succ_{\omega^*} g$. Consider the constant acts f^* and g^* such that $f^*_{\omega} = f_{\omega^*}$, $g^*_{\omega} = g_{\omega^*}$ for all $\omega \in \Omega$. By A5′, $f^* \succ_{\omega} g^*$ for all $\omega \in \Omega$. This shows that there are no Savage-null states. Applying strong subjective substitution inductively, the existence of Savage-null events is excluded.

this implies that $f^* \succsim_{\{\omega_i'\} \times D_{-i}} g^*$. In fact, suppose $g^* \succ_{\{\omega_i'\} \times D_{-i}} f^*$. Then, since $f^* \sim_{\{\omega_i'\} \times (-D_{-i})} g^*$, strong subjective substitution implies that $g^* \succ_{\{\omega_i'\} \times \Omega_{-i}} f^*$, which contradicts $f^* \succsim_{\{\omega_i'\} \times \Omega_{-i}} g^*$. Finally, since $f(g)$ coincides with $f^*(g^*)$ over $\Omega_i \times D_{-i}$ we obtain $f \succsim_{\{\omega_i'\} \times D_{-i}} g$ by the definition of conditional preference. ∎

Let us now introduce the following:

Definition 1.4. A lexicographic conditional probability system (hereafter LCPS) is an R-tuple $\rho = (p_1, ..., p_R)$, for some integer R, of probability measures on Ω such that $\text{Supp}(p_r) \cap \text{Supp}(p_s) = \varnothing$, if $r \neq s$. An LCPS has *full support* if $\bigcup_{r=1}^{R} \text{Supp}(p_r) = \Omega$.

$L\Delta^*(\Omega)$ denotes the space of all LCPSs on Ω with full support. Given an LCPS $\rho = (p_1, ..., p_R) \in L\Delta^*(\Omega)$, let us denote $\Lambda_r = \text{Supp}(p_r)$ and let $\Lambda(\rho) = (\Lambda_1, ..., \Lambda_R)$ be the ordered partition of Ω induced by ρ.

1.3. *Myerson's Approach*

Myerson's axioms are stated in terms of *conditional preferences*. That is, the conditional preference relations $\succsim_A (\varnothing \neq A \subseteq \Omega)$ are not derived from a unique relation $\succsim$ as in Definition 1.1; they are primitive elements of the analysis (see Myerson [10, Chap. 1]). It is assumed that, for every non-empty $A \subseteq \Omega$, $\succsim_A$ satisfies order, objective independence, non-triviality, and the Archimedean property (see A1*–A4* below). Furthermore, the whole system of conditional preference relations $(\succsim_A)_{\varnothing \neq A \subseteq \Omega}$ satisfies some "glue axioms" relating its elements to each other (see A5*, A7*, A8*):

A1* (Order). For every $\varnothing \neq A \subseteq \Omega$, $\succsim_A$ is transitive and complete on $\mathscr{P}^\Omega$.

A2* (Objective Independence). For every $\varnothing \neq A \subseteq \Omega$, for all $f, g, h \in \mathscr{P}^\Omega$ and $\alpha \in (0, 1]$, $f \succ_A$ (resp. $\sim_A$) $g \Rightarrow f\alpha h \succ_A$ (resp. $\sim_A$) $g\alpha h$.

A3* (Non-triviality). For every $\varnothing \neq A \subseteq \Omega$, there are $f, g \in \mathscr{P}^\Omega$ such that $f \succ_A g$.

A4* (Archimedean Property). For every $\varnothing \neq A \subseteq \Omega$, $f \succ_A g \succ_A h \Rightarrow \exists \alpha, \beta \in (0, 1), \alpha > \beta$, such that $f\alpha h \succ_A g \succ_A f\beta h$.

A5* (State Independence). For all states $\omega, \omega' \in \Omega$ and for any two constant acts $f, g \in \mathscr{P}^\Omega$, $f \succsim_\omega g \Leftrightarrow f \succsim_{\omega'} g$.

A7* (Relevance). For every $\varnothing \neq A \subseteq \Omega$, $(\forall \omega \in A, f_\omega = g_\omega) \Rightarrow f \sim_A g$.

A8* (Subjective Substitution). For every $\varnothing \neq A, B \subseteq \Omega$, $A \cap B = \varnothing$, $(f \succsim_A$ (resp. $\succ_A$) g, $f \succsim_B$ (resp. $\succ_B$) $g) \Rightarrow f \succsim_{A \cup B}$ (resp. $\succ_{A \cup B}$) g.

Axiom A3* excludes the existence of Savage-null events as defined in Definition 1.2. However, in this context another definition in terms of conditional preferences seems more appropriate:

DEFINITION 1.2*. Given $\emptyset \neq B \subseteq C \subseteq \Omega$, B is *Savage-null conditional on* C, if for all $f, g, h \in \mathscr{P}^{\Omega}$, $(f_B, h_{-B}) \sim_C (g_B, h_{-B})$.

For our purposes, Axiom A6 is too weak in this setting. In fact, strong subjective substitution is not implied by A1*–A5*, A7*, A8*, because B may be Savage-null conditional on $A \cup B$. For this reason the equivalence result stated in Lemma 1.1 does not hold (we will provide a counterexample in Section 3 below). Therefore we need the following (stronger) axiom of stochastic independence for conditional preferences:

A6* (Conditional Stochastic Independence). For all $i = 1, ..., n$, for all $D_{-i} \subseteq \Omega_{-i}$, for every pair of acts $f, g \in \mathscr{P}^{\Omega}$ constant across Ω_i and every $\omega_i, \omega'_i \in \Omega_i$: $f \succcurlyeq_{\{\omega_i\} \times D_{-i}} g \Leftrightarrow f \succcurlyeq_{\{\omega'_i\} \times D_{-i}} g$.

Lemma 1.1 could then be stated as follows: in Blume *et al.*'s setting, under A1 and A2, Axioms A6 and A6* are equivalent.

2. Conditional Probability Systems and the Notion of Stochastic Independence

DEFINITION 2.1. A *conditional probability system* is a function $\mu: 2^{\Omega} \times 2^{\Omega} \backslash \{\emptyset\} \to [0, 1]$ such that for all $A \in 2^{\Omega} \backslash \{\emptyset\}$, $\mu(\cdot \mid A)$ is a probability measure concentrated on A and for all $A, B, C \in 2^{\Omega}$, $A \subseteq B \subseteq C$, $B \neq \emptyset$,

$$\mu(A \mid C) = \mu(A \mid B)\, \mu(B \mid C). \tag{1}$$

Let $\Delta^*(\Omega)$ denote the space of conditional probability systems on Ω. Given a CPS $\mu \in \Delta^*(\Omega)$, a complete and transitive ordering $\geqslant \geqslant_{\mu}$ (read "not infinitely less probable than") on Ω can be introduced (see McLennan [9]): given $\omega, \omega' \in \Omega$, $\omega \geqslant \geqslant_{\mu} \omega' \Leftrightarrow \mu(\omega \mid \{\omega, \omega'\}) > 0$. The ordering $\geqslant \geqslant_{\mu}$ induces an ordered partition $\Lambda(\mu) = (\Lambda_1, ..., \Lambda_R)$, for some integer R, as follows: $\omega, \omega' \in \Lambda_r$ for some r if and only if $\mu(\omega \mid \{\omega, \omega'\}) \in (0, 1)$ and $\omega \in \Lambda_k, \omega' \in \Lambda_r, k < r$ if and only if $\mu(\omega \mid \{\omega, \omega'\}) = 1$. Given an event $A \subseteq \Omega$ and an ordered partition $(\Lambda_1, ..., \Lambda_R)$, define $r(A) = \min\{r: A \cap \Lambda_r \neq \emptyset\}$. The following lemma will be useful below:

LEMMA 2.1. (a) *Given a CPS* $\mu \in \Delta^*(\Omega)$ *with* $\Lambda(\mu) = (\Lambda_1, ..., \Lambda_R)$, *for every non-empty* $A \subseteq \Omega$, $\mu(\omega \mid A) = \mu(\omega \mid A \cap \Lambda_{r(A)})$.

(b) *Given a LCPS* $\rho=(p_1, ..., p_R)\in L\Delta^*(\Omega)$ *with* $\Lambda(\rho)=(\Lambda_1, ..., \Lambda_R)$, *the function* $\mu: 2^\Omega\times 2^\Omega\backslash\{\emptyset\}\to[0,1]$ *defined by* $\mu(A\mid B)=p_{r(B)}(A\mid B)=p_{r(B)}(A\cap B)/p_{r(B)}(B)$, *is a CPS and* $\Lambda(\mu)=\Lambda(\rho)$.

(c) *A CPS* $\mu\in\Delta^*(\Omega)$ *is completely determined by the probability values conditional on events containing only two states*: $\mu(\omega\mid\{\omega,\omega'\})$.

Proof. (a) is a straightforward consequence of Lemma 2.4 in McLennan [9].

(b) Let $A, B, C\subseteq\Omega$, $A\subseteq B\subseteq C$, $B\neq\emptyset$. By definition of $r(\cdot)$, we have the following cases:

(i) $r(A)\geqslant r(B)>r(C)$. If $r(A)\geqslant r(B)>r(C)$ then $A\cap\Lambda_{r(C)}=\emptyset=B\cap\Lambda_{r(C)}$. Thus $\mu(A\mid C)=p_{r(C)}(A\mid C)=0=p_{r(B)}(A\mid B)\,p_{r(C)}(B\mid C)=\mu(A\mid B)\,\mu(B\mid C)$.

(ii) $r(A)\geqslant r(B)=r(C)$. If $r(A)\geqslant r(B)=r(C)$ then $p_{r(B)}(A\mid B)=p_{r(C)}(A\mid B)=p_{r(C)}(A)/p_{r(C)}(B)=p_{r(C)}(A\mid C)/p_{r(C)}(B\mid C)$. Thus $\mu(A\mid C)=p_{r(C)}(A\mid C)=p_{r(B)}(A\mid B)p_{r(C)}(B\mid C)=\mu(A\mid B)\mu(B\mid C)$.

Moreover, by construction $\mu(\cdot\mid A)$ is a probability measure concentrated on A and $\Lambda(\mu)=\Lambda(\rho)$.

(c) By definition the ordered partition $\Lambda(\mu)=(\Lambda_1, ..., \Lambda_R)$ is completely determined by probabilities conditional on binary events (i.e., containing just two states). Furthermore, the LCPS defined by $\rho(\mu)=(\mu(\cdot\mid\Lambda_1), ..., \mu(\cdot\mid\Lambda_R))$ univocally determines the CPS μ by (a) and (b). It is then sufficient to show that all the measures $\mu(\cdot\mid\Lambda_r)$, $r=1, ..., R$, are completely determined by the probabilities conditional on binary events. $\forall r=1, ..., R$, $\forall\omega\in\Lambda_r$, $\mu(\omega\mid\Lambda_r)>0$ by definition. By Bayes rule (1) then, for all $\omega,\omega'\in\Lambda_r$,

$$\frac{\mu(\omega\mid\Lambda_r)}{\mu(\omega'\mid\Lambda_r)}=\frac{\mu(\omega\mid\{\omega,\omega'\})}{\mu(\omega'\mid\{\omega,\omega'\})}.$$

As the probabilities sum to 1, every $\mu(\cdot\mid\Lambda_r)$ is determined. ∎

By this lemma, we have an isomorphism between $\Delta^*(\Omega)$ and $L\Delta^*(\Omega)$. This will enable us to state the representation theorem in Blume *et al.*'s approach in terms of CPSs instead of LCPSs.

Let us introduce the notion of stochastic independence (cf. Hammond [6,7]).

DEFINITION 2.2. Given a non-trivial partition $\{J, K\}$ of $\{1, ..., n\}$, the algebras of events 2^{Ω_J}, 2^{Ω_K} are *mutually independent* with respect to the conditional probability system $\mu: 2^{\Omega} \times 2^{\Omega} \backslash \{\emptyset\} \to [0, 1]$ if for all non-empty subsets $A_J, B_J \subseteq \Omega_J, C_K, D_K \subseteq \Omega_K$ the following equality holds:

$$\mu(A_J \times C_K \mid A_J \times D_K) = \mu(B_J \times C_K \mid B_J \times D_K). \tag{2}$$

A CPS μ *satisfies the independence property* if for all non-trivial partitions $\{J, K\}$ of $\{1, ..., n\}$, 2^{Ω_J} and 2^{Ω_K} are mutually independent with respect to μ.

The intuition behind this condition of stochastic independence is that any change in DM's information on events in Ω_J should not modify his/her probability evaluations on Ω_K.[3]

Given $\mu \in \Delta^*(\Omega)$, we can define a *marginal* CPS μ_K on Ω_K as follows: $\mu_K(C_K \mid D_K) = \mu(\Omega_J \times C_K \mid \Omega_J \times D_K)$. If μ satisfies the independence property, $\mu_K(C_K \mid D_K) = \mu(A_J \times C_K \mid A_J \times D_K)$ for every non-empty $A_J \subseteq \Omega_J$.

Battigalli [4, Proposition 2.1] proves the following result, showing that the independence condition for CPSs can be regarded as an extension of the usual multiplicative condition of stochastic independence.

THEOREM 2.1. *A CPS μ on Ω satisfies the independence property if and only if for every partition $\{J, K, L, ...\}$ of $\{1, 2, ..., n\}$, every possible rectangular event $E = E_J \times E_K \times E_L \times \cdots$ and every state $\omega = (\omega_J, \omega_K, \omega_L, ...) \in E_J \times E_K \times E_L \times \cdots$, the following holds*:

$$\mu(\omega \mid E) = \mu_J(\omega_J \mid E_J) \mu_K(\omega_K \mid E_K) \mu_L(\omega_L \mid E_L) \cdots$$

The following lemma will be useful below:

LEMMA 2.2. *A CPS $\mu \in \Delta^*(\Omega)$ satisfies the independence property if and only if for every non-trivial partition $\{J, K\}$ of $\{1, ..., n\}$, for every pair $\omega_J, \omega'_J \in \Omega_J$ and every pair $\omega_K, \omega'_K \in \Omega_K$ the following holds*:

$$\mu((\omega_J, \omega_K) \mid \{(\omega_J, \omega_K), (\omega_J, \omega'_K)\}) = \mu((\omega'_J, \omega_K) \mid \{(\omega'_J, \omega_K), (\omega'_J, \omega'_K)\}). \tag{2'}$$

Proof. (Only if) Obvious.

(If) For simplicity, we use Latin letters $r, s, t, ...$ to denote elements of Ω_J and Greek letters $\rho, \sigma, \tau, ...$ to denote elements of Ω_K. $r\sigma, r\tau, s\sigma, \tau, ...$ denote the states of the world. We start by showing that (2) holds when $D_K = \{\sigma, \tau\}$. Let $A_J = \{s, t, ...\}$ be a non-empty subset of Ω_J. We show that

[3] This condition of stochastic independence is called "quasi-independence" by Swinkels [11] and is somewhat stronger than the "quasi-independence" condition proposed by Kohlberg and Reny [8]. See Swinkels [11] for a discussion.

$\mu(A_J \times \{\sigma\} \mid A_J \times D_K)$ is independent of A_J. Let r be any element of Ω_J. We show that $\mu(A_J \times \{\sigma\} \mid A_J \times D_K) = \mu(r\sigma \mid \{r\sigma, r\tau\})$

$$\begin{aligned}
\mu(A_J \times \{\sigma\} \mid A_J \times D_K) &= \mu(s\sigma \mid A_J \times D_K) + \mu(t\sigma \mid A_J \times D_K) + \cdots \\
&= \mu(s\sigma \mid \{s\sigma, s\tau\})\, \mu(\{s\sigma, s\tau\} \mid A_J \times D_K) \\
&\quad + \mu(t\sigma \mid \{t\sigma, t\tau\})\, \mu(\{t\sigma, t\tau\} \mid A_J \times D_K) + \cdots \\
&= \mu(s\sigma \mid \{s\sigma, s\tau\})\, \mu(\{s\sigma, s\tau\} \mid A_J \times D_K) \\
&\quad + \mu(s\sigma \mid \{s\sigma, s\tau\})\, \mu(\{t\sigma, t\tau\} \mid A_J \times D_K) + \cdots \\
&= \mu(s\sigma \mid \{s\sigma, s\tau\})[\mu(\{s\sigma, s\tau\} \mid A_J \times D_K) \\
&\quad + \mu(\{t\sigma, t\tau\} \mid A_J \times D_K) + \cdots] \\
&= \mu(s\sigma \mid \{s\sigma, s\tau\}) = \mu(r\sigma \mid \{r\sigma, r\tau\}),
\end{aligned}$$

where the first equality is obvious, the second is implied by Bayes' rule (1), the third is implied by (2′), the fourth and the fifth are obvious, and the sixth follows again by (2′). For every $A_J \subset \Omega_J$, it is easy to see that $\mu(A_J \times \cdot \mid A_J \times \cdot): 2^{\Omega_K} \setminus \{\emptyset\} \to [0, 1]$ is a CPS on Ω_K. By Lemma 2.1 (c), this CPS is completely determined by the probability values conditional on binary events: $\mu(A_J \times \{\sigma\} \mid A_J \times \{\sigma, \tau\})$. As the latter is independent of A_J by the above argument, we have that $\mu(A_J \times \cdot \mid A_J \times \cdot)$ is independent of A_J as well. This means that (2) holds. ∎

3. Main Theorems

Let $\geqslant_L$ denote the lexicographic ordering between vectors of real numbers.[4] The following are the main theorems:

Theorem 3.1. A1–A3, A4″, A5′ *hold if and only if there exist a linear function* $u: \mathscr{P} \to \mathbb{R}$ *and a conditional probability system* $\mu: 2^{\Omega} \times 2^{\Omega} \setminus \{\emptyset\} \to [0, 1]$ *such that for all* $f, g \in \mathscr{P}^{\Omega}$

$$f \succcurlyeq g \Leftrightarrow \left\{ \sum_{\omega \in \Omega} \mu(\omega \mid \Lambda_r)\, u(f_\omega) \right\}_{r=1}^{R} \geqslant_L \left\{ \sum_{\omega \in \Omega} \mu(\omega \mid \Lambda_r)\, u(g_\omega) \right\}_{r=1}^{R},$$

where $(\Lambda_1, ..., \Lambda_R) = \Lambda(\mu)$ *is the ordered partition mentioned in Axiom* A4″. *Furthermore,* A6 *holds if and only if the conditional probability system satisfies the independence property.*

[4] Given two vectors $x = (x_1, ..., x_R)$ and $y = (y_1, ..., y_R)$, $x \geqslant_L y$ if and only if $y_r > x_r \Rightarrow \exists k < r$ such that $x_k > y_k$.

THEOREM 3.2. A1*–A5*, A7*, A8* *hold if and only if there exist a linear function* $u\colon \mathscr{P} \to \mathbb{R}$ *and a conditional probability system* $\mu\colon 2^{\Omega} \times 2^{\Omega} \backslash \{\varnothing\} \to [0, 1]$ *such that for all* $f, g \in \mathscr{P}^{\Omega}$, *for all non-empty* $A \subseteq \Omega$:

$$f \succcurlyeq_A g \Leftrightarrow \sum_{\omega \in A} \mu(\omega \mid A)\, u(f_\omega) \geqslant \sum_{\omega \in A} \mu(\omega \mid A)\, u(g_\omega).$$

Furthermore, A6* *holds if and only if the conditional probability system satisfies the independence property.*

(Note that B is Savage-null conditional on C if and only if $\mu(B \mid C) = 0$ in the representation.)

The proofs of the representation part of both theorems already exist in the literature. Specifically, Blume *et al.* [5] proved Theorem 3.1 by using lexicographic conditional probability systems, which are isomorphic to conditional probability systems by Lemma 2.1. As for Theorem 3.2, the representation part corresponds to Theorems 1.1, 1.2, and 1.3 in Myerson [10, Chap. 1] (who uses a different set of axioms which can be proved to be equivalent to A1*–A5*, A7*, A8*).

The proof of Theorems 3.1 and 3.2 relies on the following lemma:

LEMMA 3.1. *Axiom* A6* *holds if and only if for any non-trivial partition* $\{J, K\}$ *of* $\{1, ..., n\}$, *for every pair of acts* $f, g \in \mathscr{P}^{\Omega}$ *constant across* Ω_J, *for every pair* $\omega_J, \omega'_J \in \Omega_J$ *and every non-empty subset* $D_K \subseteq \Omega_K$:

$$f \succcurlyeq_{\{\omega_J\} \times D_K} g \Leftrightarrow f \succcurlyeq_{\{\omega'_J\} \times D_K} g.$$

Proof. (If) Obvious.

(Only if) Assume A6* holds and let $J = \{j_1, j_2, j_3, ...\}$. If f, g are constant across Ω_J, they are constant across every $\Omega_{j_k}, j_k \in J$. Thus,

$$f \succcurlyeq_{\{\omega_{j1}\} \times \{\omega_{j2}\} \times \{\omega_{j3}\} \times \cdots \times D_K} g \Leftrightarrow f \succcurlyeq_{\{\omega'_{j1}\} \times \{\omega_{j2}\} \times \{\omega_{j3}\} \times \cdots \times D_K} g$$
$$\Leftrightarrow f \succcurlyeq_{\{\omega'_{j1}\} \times \{\omega'_{j2}\} \times \{\omega_{j3}\} \times \cdots \times D_K} g \Leftrightarrow f \succcurlyeq_{\{\omega'_{j1}\} \times \{\omega'_{j2}\} \times \{\omega'_{j3}\} \times \cdots \times D_K} g$$

The thesis follows by induction. ▮

Proof of Theorem 3.2. The fact that the independence condition implies Axiom A6* is an immediate consequence of the representation part of the theorem. Let us see that Axiom A6* implies that the system of conditional preferences $(\succcurlyeq_A)_{\varnothing \neq A \subseteq \Omega}$ is represented by some (u, μ) where μ satisfies the independence property. By Lemma 2.2, we only have to prove that condition (2′) holds. Again Latin letters $s, t, ...$ denote elements of Ω_J and Greek letters $\sigma, \tau, ...$ denote elements of Ω_K. Given $s, t \in \Omega_J$ and $\sigma, \tau \in \Omega_K$, we consider two acts $f, g \in \mathscr{P}^{\Omega}$ which are constant across Ω_J. Act f is such

$\mu(f_\omega)$	σ	τ
s	α	β
t	α	β

FIG. 3. Utility consequences of act f on $\{s, t\} \times \{\sigma, \tau\}$.

that $u(f_\omega) = \alpha$ if $\omega \in \Omega_J \times \{\sigma\}$ and $u(f_\omega) = \beta$ if $\omega \in \Omega_J \times \{\tau\}$ (see Fig. 3), where $\alpha > \beta$. Act g is such that for any $\omega \in \Omega_J \times \{\sigma, \tau\}$:

$$g_\omega = [\mu(s\sigma \mid \{s\sigma, s\tau\})\, f_{s\sigma} + \mu(s\tau \mid \{s\sigma, s\tau\})\, f_{s\tau}].$$

By construction, we have $f \sim_{\{s\} \times \{\sigma, \tau\}} g$, because both acts yield expected utility $\mu(s\sigma \mid \{s\sigma, s\tau\})\, \alpha + \mu(s\tau \mid \{s\sigma, s\tau\})\, \beta$ conditional on $\{s\} \times \{\sigma, \tau\}$. By Lemma 3.1, $f \sim_{\{t\} \times \{\sigma, \tau\}} g$; thus $\mu(t\sigma \mid \{t\sigma, t\tau\})\, \alpha + \mu(t\tau \mid \{t\sigma, t\tau\})\, \beta = \mu(s\sigma \mid \{s\sigma, s\tau\})\, \alpha + \mu(s\tau \mid \{s\sigma, s\tau\})\, \beta$. As $\alpha > \beta$, this equality holds if and only if $\mu(t\sigma \mid \{t\sigma, t\tau\}) = \mu(s\sigma \mid \{s\sigma, s\tau\})$. ∎

Proof of Theorem 3.1. As for the previous case, it is sufficient to show that the preference relation $\succcurlyeq$ is represented by some (u, μ) where μ satisfies (2′). Let f and g be as before, and let h be constant across Ω_J and such that for any $\omega \in \Omega_J \times \{\sigma, \tau\}$:

$$h_\omega = [\mu(t\sigma \mid \{t\sigma, t\tau\})\, f_{s\sigma} + \mu(t\tau \mid \{t\sigma, t\tau\})\, f_{s\tau}].$$

Let d be an act constant across Ω_J, defined as f but with α (β) replaced by β (α), i.e., $u(d_{s\sigma}) = u(d_{t\sigma}) = \beta$ and $u(d_{s\tau}) = u(d_{t\tau}) = \alpha$. Furthermore, we choose $g, h,$ and d so that they coincide with f on $\Omega_J \times (\Omega_K \backslash \{\sigma, \tau\})$. Without loss of generality, we assume that $\mu(s\sigma \mid \{s\sigma, s\tau\}) > 0$. We have at most three possibilities: (i) $\mu(s\sigma \mid \{s\sigma, s\tau\}) \in (0, 1)$; (ii) $\mu(s\sigma \mid \{s\sigma, s\tau\}) = 1$ and $\mu(t\sigma \mid \{t\sigma, t\tau\}) > 0$; (iii) $\mu(s\sigma \mid \{s\sigma, s\tau\}) = 1$ and $\mu(t\sigma \mid \{t\sigma, t\tau\}) = 0$. In case (i), we can argue as in the Proof of Theorem 2 above, because we have $f \sim_{\{s\} \times \Omega_K} g$ and Lemma 3.1 can be applied (recall that under A1–A5, A6, and A6* are equivalent by Lemma 1.1). In case (ii), we have to show that $\mu(t\sigma \mid \{t\sigma, t\tau\}) = 1$. Assume not, i.e., $\mu(s\sigma \mid \{s\sigma, s\tau\}) = 1$ and $0 < \mu(t\sigma \mid \{t\sigma, t\tau\}) < 1$. Then, using the lexicographic representation it can be checked that $f \succ_{\{s\} \times \Omega_K} h$ and $f \sim_{\{t\} \times \Omega_K} h$, which contradicts Lemma 3.1. Hence, $\mu(t\sigma \mid \{t\sigma, t\tau\}) = 1 = \mu(s\sigma \mid \{s\sigma, s\tau\})$. Finally, we show that case (iii) is incompatible with Axiom A6. In fact, we have $f \succ_{\{s\} \times \Omega_K} d$ and $d \succ_{\{t\} \times \Omega_K} f$, which contradicts A6 by Lemma 3.1. ∎

$u(f_\omega)$	ρ	σ	τ
s	α	0	1
t	α	0	1

$u(g_\omega)$	ρ	σ	τ
s	α	1	0
t	α	1	0

FIG. 4. Utility consequences of acts f and g.

Let us show now with an example that Axiom A6 is actually weaker than Axiom A6* within Myerson's framework. Given the state space $\Omega = \{s, t\} \times \{\rho, \sigma, \tau\}$, assume that the DM's conditional-preferences satisfy Axioms A1*–A5*, A7*, A8* and are represented by some utility function and a CPS $\mu \in \Delta^*(\Omega)$ such that $\Lambda(\mu) = (\{s\rho\}, \{s\sigma, s\tau, t\rho\}, \{t\sigma, t\tau\})$. Assume also that the conditional probabilities $(\mu(\cdot \mid \Lambda_r))_{r=1,2,3}$ are as follows: ((1), (1/3, 1/3, 1/3), (1/3, 2/3)). We can see, then, that $\mu(s\rho \mid \{s\rho, s\sigma\}) = \mu(t\rho \mid \{t\rho, t\sigma\}) = 1$, $\mu(s\sigma \mid \{s\sigma, s\tau\}) = 1/2$, $\mu(t\sigma \mid \{t\sigma, t\tau\}) = 1/3$ (notice that condition (2) is violated). While it can be easily verified that Axiom A6 is satisfied, we realise that Axiom A6* is not. In fact, consider two acts f and g as in Fig. 4. Then, we have

$$f \sim_{\{s\} \times \{\sigma, \tau\}} g \qquad \text{and} \qquad f \succ_{\{t\} \times \{\sigma, \tau\}} g.$$

However, notice that within the framework of Blume *et al.* [5] (i.e., assuming that the DM's conditional preferences are derived by a preference relation $\succcurlyeq$ satisfying Axioms A1–A3, A4″, and A5′), also Axiom A6 is not satisfied. In fact, in this case we have

$$f \sim_{\{s\} \times \Omega_2} g \qquad \text{and} \qquad f \succ_{\{t\} \times \Omega_2} g.$$

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