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Strategic Rationality Orderings and the Best Rationalization Principle*

Pierpaolo Battigalli

Department of Economics, Princeton University, Princeton, New Jersey 08544-1021

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In a finite game fix a space of extended probabilities over strategies and a profile of best response correspondences. A profile of *rationality orderings* is then given by an ordered partition of the set of strategies of each player, representing different degrees of rationality, where at-least $k + 1$-rational strategies are best responses against extended probabilities reflecting at least k degrees of rationality. This solution can be constructed inductively, providing a Bayesian foundation for controversial deletion procedures such as extensive form rationalizability and iterated weak dominance. Focusing on extensive games, this approach formalizes the best rationalization principle. *Journal of Economic Literature* Classification Number: C72.

1. INTRODUCTION AND GENERAL CONCEPTS

"What constitutes rational behavior and rational beliefs in a noncooperative strategic situation?" Most game-theoretic solution concepts rely on the assumption that there is no intermediate level between pure strategic rationality and pure strategic irrationality. Thus they can be defined by some sort of fixed-point condition with the following feature: rational beliefs must assign zero probability to irrational strategies and rational strategies must be optimal against rational beliefs. Fixed point conditions of this sort characterize standard equilibrium concepts as well as nonequilibrium solution concepts such as rationalizability (see Bernheim (1984) and Pearce (1984)).

In this paper we adopt a different game-theoretic approach, whereby there may be more than two degrees of strategic rationality/irrationality. This approach

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provides a unified decision-theoretic (Bayesian) foundation for known iterative deletion procedures such as extensive form rationalizability and the iterative deletion of weakly dominated strategies. Since these procedures are related to the notion of forward induction in extensive games (Pearce (1984); Battigalli (1990); Sobel *et al*. (1990); Börgers (1991); Ben-Porath and Dekel (1992); Gul (1991); Hammond (1994)), we think that our approach also sheds some light on this somewhat elusive principle. For this reason we focus our analysis on extensive games.

Many forward induction arguments found in the literature rely on the following:

RATIONALIZATION PRINCIPLE. *A player should always try to interpret her information about the behavior of her opponents assuming that they are not implementing "irrational" strategies*.[1]

In order to provide a precise statement of the rationalization principle we have to say what "irrational" means. Clearly a strategy is irrational if it is not a best response against any belief. But we are tempted to argue that a strategic notion of irrationality should go further and deem a strategy irrational if it is not optimal against any "rational" belief. This extension, however, suffers from logical difficulties similar to those concerning the backward induction solution (see, e.g., Basu (1990) and Reny (1993)). In this introductory section we illustrate the problem using a well-known example due to Van Damme (1989) and Ben-Porath and Dekel (1992). Then we describe our general solution concept for the case of two players. Finally we briefly discuss the related literature.

1.1. *Dissipative Moves and the Best Rationalization Principle*

A player, say the Female, is given the opportunity to burn an amount x of money before playing the Battle of the Sexes, where x is strictly larger than the Female's payoff in her least preferred equilibrium (T, L), but strictly smaller than the difference between her payoff in her most preferred equilibrium (B, R) and the payoff in (T, L) (see Fig. 1, where $1 < x < 2$). The Male observes this move (either x or 0) before choosing between T and B. The forward induction argument is as follows: the Female can signal that she is going to play R by burning the money, since xR is a best response to the Male strategy (B if x; T if 0), while xL is irrational. Therefore the Male has to assess $\text{Prob}(xR \mid x) = 1$. This implies that ($T$ if x) is an irrational move. Hence the Female is sure to get $(3 - x)$ if she plays xR and it is irrational for her to play $0L$, which gives a payoff of $1 < (3 - x)$. On the other hand $0R$ gives payoff $3 > (3 - x)$ if the

[1] For different statements of the forward induction principle in the context of equilibrium analysis see Kohlberg and Mertens (1986), Kohlberg (1990), and Van Damme (1989).

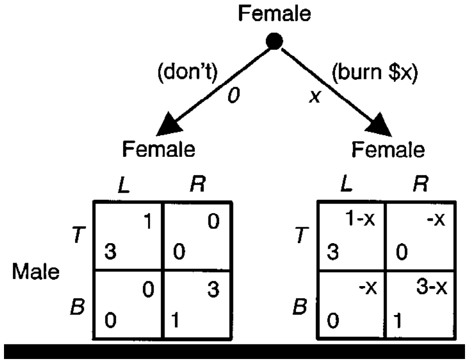

FIG. 1. Battle of the Sexes with a dissipative move $1 < x < 2$.

Male plays (B if 0). Knowing this, the Male must assess $\text{Prob}(0R \mid 0) = 1$ and play B if he observes 0. Therefore the Female must play $0R$.

This is the same outcome selected by two iterative deletion procedures: extensive form rationalizability (Pearce (1984)) and the iterated deletion of weakly dominated strategies. But something is puzzling about this story: we have concluded that xR is also a strategically irrational strategy; why should the Male think that xR is much more likely than xL, when both are strategically irrational?[2] A possible answer is that in a strategic situation the players may have a coherent and common view about different degrees of rationality and irrationality. Given this common view, in an extensive game the players conform to the following

BEST RATIONALIZATION PRINCIPLE. *A player should always believe that her opponents are implementing one of the "most rational" (or "least irrational") strategy profiles which are consistent with her information.*

In the present example we may argue that xR is more rational (or less irrational) than xL, because xR is a best response against some belief while xL is not. Therefore the Male should think that xR is much (or infinitely) more likely than xL if he observes x, and his rational response to x is B. This leads us to the concept of strategic rationality orderings.

1.2. *Strategic Rationality Orderings*: *An Abstract Formulation*

Consider a two-person finite game with strategy space $S^1 \times S^2$. For each i, fix an ordered partition of S^i into subsets $R_0^i, R_1^i, \ldots, R_M^i$. We tentatively interpret R_0^i as the class of completely irrational strategies, R_M^i as the class of most rational

[2] Börgers and Samuelson (1992) and Samuelson (1992) point out corresponding difficulties concerning iterated weak dominance.

strategies and R_k^i as the class of strategies with k degrees of rationality, or *k-rational* strategies. Of course, if $k < m$ the strategies in R_k^i are supposed to be less rational than the strategies in R_m^i. The assumption that the number M is the same for both players is innocuous if we allow for some classes to be empty. We maintain, however, that R_M^i is not empty. According to our intended interpretation $R_k^i \cup \cdots \cup R_M^i$ is the set of player i's *at-least-k-rational* strategies. Assume that player i is able to assess coherently all conditional probabilities of the form $p(s^j \mid \{s^j, t^j\})$. Such an assignment rule $p(\cdot \mid \cdot)$ can be derived from different kinds of extended probability systems, such as the conditional probability systems (Myerson, 1986), the lexicographic probability systems with full support, or some kind of non-Archimedean probabilities (see Blume *et al.*, 1991a).[3] Let us also assume that for each probability system p assessed by player i about player j we can sensibly define a set $B^i(p) \subseteq S^i$ of best responses against p.

We emphasize that the relevant space of probability systems and the relevant best response correspondences $B^i(p)$ are *primitives* of our analysis. Therefore, what we actually propose is a "meta-solution concept." Focusing on extensive games, we will use Myerson's conditional systems and the notion of sequential rationality. But other sensible definitions could be used, leading to different solution concepts.

Our fundamental assumption is that if a rational player knew that one out of two strategies s and t is played and s is less rational than t, then this player should think that s is infinitely less likely than t. Therefore we say that a probability system p assessed by player i is *most rational* if $s^j \in R_m^j$, $t^j \in R_n^j$ and $m < n$ implies $p(s^j \mid \{s^j, t^j\}) = 0$. Suppose that, for some k, p is such that $s^j \in R_m^j$ and $t^j \in R_n^j$ implies $p(s^j \mid \{s^j, t^j\}) = 0$ whenever $m < \min\{k, n\}$. Then we only say that p is *at-least-k-rational*, since such a probability system may "pool" distinct degrees of rationality which are (weakly) higher than k.

Finally, we say that $[(R_0^1, \ldots, R_M^1), (R_0^2, \ldots, R_M^2)]$ is a *pair of rationality orderings* if our intended interpretation about different degrees of rationality is consistent, i.e., if the following is satisfied:

$$\text{for } i \in \{1, 2\}, k \in \{0, \ldots, M-1\}$$

(a) s^i is at-least-$(k+1)$-rational (i.e., $s^i \in R_{k+1}^i \cup \cdots \cup R_M^i$) if and only if s^i is a best response against a probability system p, where p is at-least-k-rational;

(b) s^i is most rational (i.e., $s^i \in R_M^i$) if and only if s^i is a best response against a most rational probability system p.

Note that this actually is a fixed-point condition for an operator mapping the space of pairs of orderings into itself.

[3] Hammond (1992) discusses the relationship between different kinds of extended probability systems.

To illustrate part (a) of this definition let us go back to the example. Assume that each player chooses an action which maximizes his/her expected payoff conditional on the observed behavior of the opponent. Since the Female does not receive any information about her opponent, she just maximizes her expected payoff given a prior $p(\cdot \mid S^{\text{Male}})$, while the Male chooses a best response to the conditional distribution $p(\cdot \mid \{0L, 0R\})$ or $p(\cdot \mid \{xL, xR\})$ according to his information. A belief is at-least-0-rational if no rationality constraint is imposed at all. The only strategy of the Female which is not a best response against some belief is xL; all the other strategies are at-least-1-rational. Therefore $R_0^{\text{Fem}} = \{xL\}$. Every strategy of the Male is at-least-1-rational because it can be justified by some belief. Therefore $R_0^{\text{Male}} = \emptyset$. This means that all beliefs about the Male are at-least-1-rational. Hence every at-least-1-rational strategy of the Female is also at-least-2-rational and no strategy has exactly one degree of rationality: $R_1^{\text{Fem}} = \emptyset$. A conditional system p about the Female is at-least-1-rational if and only if $p(xL \mid \{xL, xR\}) = 0$. A strategy of the Male is at-least-2-rational if and only if it is a best response against an at-least-1-rational conditional system. This means that such a strategy must choose action B when x is observed. Thus the at-least-2-rational strategies are TB and BB (the letter on the right denotes the move when x is observed). Taking this into account, it is easy to show that the modified Battle of the Sexes has a unique pair of rationality orderings, which supports the forward induction solution:

$$\begin{array}{ll} \text{Female:} & (\{xL\}, \emptyset, \{0L\}, \emptyset, \{xR\}, \{0R\}), \\ \text{Male:} & (\emptyset, \{TT, BT\}, \emptyset, \{TB\}, \emptyset, \{BB\}). \end{array}$$

This pair of orderings can be computed iteratively, exploiting the inductive structure of condition (a) and the fact that a player's completely irrational strategies can be determined without reference to the rationality ordering of the opponent. We will show that such an iterative procedure always yields the unique profile of rationality orderings.[4]

Part (b) should be clear. Optimality against a rational belief is a necessary condition for strategic rationality in any sensible solution concept. We also require that (b) is a sufficient condition: i.e., we require that the set of rational strategy profiles is "closed under rational behavior." Strict Nash equilibria and the set of rationalizable strategies have this property (see Basu and Weibull (1991) for a discussion).

1.3. *Related Literature*

Not surprisingly, the notion of rationality orderings is related to part of the existing literature. The idea that "if s is less rational than t, the probability of s

[4] In the first version of this paper we give a more general definition where most rational strategies may have a "bootstrap" feature and multiple solutions result.

must be infinitesimal with respect to the probability of t" is present in Myerson's (1978) concept of proper equilibrium, a refinement of Selten's (1975) trembling hand perfect equilibrium (see also Blume *et al.* (1991b)). McLennan (1989b) gives a modified definition of this concept, the sequentially proper equilibrium, which uses conditional systems in the context of extensive games. There are three main differences between such equilibrium concepts and rationality orderings. First, in proper and sequentially proper equilibria the degree of rationality of strategies is evaluated with respect to a unique "focal" equilibrium belief, while we allow for multiple nonequilibrium rational beliefs. Second, this equilibrium belief may assign zero probability to strategies which are best responses against it, while we require that every rational strategy is in the support of some rational belief, i.e., the solution must be closed under rational behavior. Third, in such solution concepts "s is less rational than t" means that s yields a smaller expected payoff than t (given the equilibrium belief), while we use this phrase to say that the beliefs which justify s (if they exist at all) are less rational than the beliefs which justify t. The modified Battle of the Sexes illustrates this point too. The pair of strategies $(0L, TT)$ is a Nash equilibrium where the Male insists on his preferred outcome, ignoring the Female's signal. It is also a proper and sequentially proper equilibrium. Strategy xL yields a higher expected payoff than xR against TT. Therefore we have $p(xR \mid \{xL, xR\}) = 0$ when using Myerson's notion of "less rational than." Of course, this justifies the conditional choice (T if x).

Another strand in the literature proposes solution concepts which can be obtained with iterative procedures. Backward induction, iterated weak dominance, and rationalizability are well-known examples. Let R^i_k be the set of player i's strategies deleted at step k of the procedure (say that the first step is $k = 0$) and let R^i_M bet the set of surviving strategies. Then we obtain candidate rationality orderings, which satisfy our fixed-point condition in some circumstances. An important step toward this interpretation is taken in Gul (1991), Reny (1992), and Stahl (1991).

Gul (1991) considers two kinds of consistent restrictions about the beliefs of strategically rational players: (i) restrictions on beliefs about the behavior of "nonrational" types and (ii) restrictions on beliefs about the behavior of "rational" types. Condition (a)–(b) can be regarded as an example of such restrictions.

Reny (1992) defines the notion of "explicable equilibrium" by means of increasingly strong restrictions on equilibrium beliefs in extensive games (see also McLennan (1985)). Stahl (1991) provides a justification of iterated weak dominance which relies on lexicographic utility maximization. Both solutions can be obtained within the present approach by appropriately specifiying the primitives (for more on this see Section 4).

The rest of the paper is organized as follows. Section 2 defines the notion of correlated sequential rationality orderings, which is equivalent to extensive form rationalizability (Pearce (1984)) either in two-person games or when correlation in beliefs is allowed. Section 3 modifies the notion of rationality orderings in order to incorporate the assumption that each player always regards the strategic choices of different opponents as stochastically independent events. It is shown that Pearce's formalization of this assumption is flawed and may lead to counter-intuitive results. For this reason "independent" sequential rationality orderings do not correspond to Pearce's original solution concept. Section 4 concludes the paper, presenting some alternative definitions of extended probability systems and best response correspondences, which yield different solution concepts such as the iterated deletion of weakly dominated strategies. The Appendix contains some proofs and additional results.

2. CORRELATED SEQUENTIAL RATIONALITY ORDERINGS

In this and the following section we provide alternative formalizations of the best rationalization principle for extensive games. We assume that the players cannot make binding commitments and always maximize their conditional expected utility given a coherent system of subjective conditional probabilities concerning their opponents' strategic choices. Hence knowledge of the extensive form is important, because actions are freely chosen, taking into account information about the past play as encoded in information sets. However, our solution concept does not depend on all the details of the extensive form. Here we list only the primitive and derived elements of an extensive form which are necessary in the analysis, taking for granted a basic understanding of the notion of an extensive game and its strategic representation.

Notation	*Terminology*
$N = \{1, \dots, n\}$	players set
H^i $(i \in N)$	collection of i's information sets
$s^i \in S^i$ (S^i finite)	pure strategies for player i
$S = \times_{i \in N} S^i$	set of strategy profiles
$-i = N \backslash \{i\}$	player i's opponents
$S^{-i} = \times_{j \neq i} S^j$	set of i's opponents' strategy profiles
$U^i \colon S \to \mathbb{R}$	normal-form payoff function for i
$S(h)$, $h \in H^i$, $i \in N$	set of strategy profiles which reach information set h
$S^i(h)$, $S^{-i}(h)$	projection of $S(h)$ on S^i and S^{-i}

Only the last two lines need some explanation. A strategy profile s *reaches* an information set h, written $s \in S(h)$, if the path induced by s contains a node $x \in h$. $S^i(h)$ and $S^{-i}(h)$ are the sets of strategies of i and $-i$ respectively, which do not prevent h from being reached.

We assume for simplicity that the game has *no chance moves*[5] and, more importantly, *perfect recall*. A consequence of perfect recall is that if $h \in H^i$, then $S(h) = S^i(h) \times S^{-i}(h)$. Since i knows her past moves, information set h corresponds to the event that i's opponents are implementing some strategy profile $s^{-i} \in S^{-i}(h)$.

The next step is the formal definition of players' beliefs. Consider a finite set of "states" Ω. In the following analysis Ω will be S^i or S^{-i}. $\Delta(\Omega)$ denotes the set of probability measures on Ω. For every nonempty subset $E \subseteq \Omega$, $\Delta(E)$ denotes the set of probability measures on Ω with support in E. A *conditional probability system* (Renyi (1955); Myerson (1986)) is a function $p(\cdot \mid \cdot)$ such that for all $A, B, C \subseteq \Omega$, $B, C \neq \emptyset$, $p(\cdot \mid C) \in \Delta(C)$, and

$$A \subseteq B \subseteq C \qquad \text{implies } p(A \mid C) = p(A \mid B)p(B \mid C). \tag{2.1}$$

This means that conditional probabilities are updated *via* Bayes rule whenever possible. The set of conditional probability systems on Ω is denoted by $\Delta^*(\Omega)$.

We assume that player i's conditional beliefs about her opponents are represented by some $p \in \Delta^*(S^{-i})$. In this section we do not assume that beliefs satisfy stochastic independence, because a player may think that there is spurious correlation between the strategies of different opponents (Aumann (1987)). This is even more plausible within a nonequilibrium solution concept, where beliefs are subjective and at least partially arbitrary (Hammond (1994); Stalnaker (1993)). Nevertheless we think that stochastic independence is a meaningful and worth studying game-theoretic assumption. Therefore we will consider it in Section 3.

We now turn to the definition of the best response correspondences. Assume that player i is given the move at information set $h \in H^i$. Before h was reached i must have followed some strategy $s^i \in S^i(h)$ if she played at all. Assume that player i is endowed with a conditional system $p \in \Delta^*(S^{-i})$. Then at each information set $h \in H^i$, she can compute the expected utility of any $s^i \in S^i(h)$ conditional on h using the formula

$$U^i(s^i, p \mid S^{-i}(h)) = \sum_{s^{-i} \in S^{-i}(h)} U^i(s^i, s^{-i}) p(s^{-i} \mid S^{-i}(h)). \tag{2.2}$$

By perfect recall, $U^i(s^i, p \mid S^{-i}(h))$ depends only on the choices prescribed by s^i on the collection of information sets containing h and its followers in H^i, provided that $s^i \in S^i(h)$. We assume that player i maximizes her conditional expected utility whenever she has to play.

DEFINITION 2.1. Let $h \in H^i$. A strategy $s^i \in S^i(h)$ is a best response against $p \in \Delta^*(S^{-i})$ at h if and only if

$$U^i(s^i, p \mid S^{-i}(h)) \geq U^i(t^i, p \mid S^{-i}(h)) \qquad \text{for all } t^i \in S^i(h).$$

[5] The introduction of chance moves and incomplete information is quite straightforward (see Battigalli (1993a, Section 6)).

We say that s^i is a *sequential best response against* $p \in \Delta^*(S^{-i})$, written $s^i \in SB^i(p)$, if and only if s^i is a best response against p at every information set h such that $s^i \in S^i(h)$.[6]

Note that $SB^i(p)$ is included in the set of best responses against the prior $p(\cdot \mid S^{-i})$ and these sets coincide in simultaneous games.

LEMMA 2.1. *For every $p \in \Delta^*(S^{-i})$ there is at least one sequential best response against p, i.e., $SB^i(p) \neq \emptyset$.*

Proof. See the Appendix.

Conditional systems enable us to formalize the notion that "state s is infinitely less likely than state t" without using limits: we just have $p(s \mid \{s, t\}) = 0$. Of course, this is meaningful if the relation "infinitely less likely than" is transitive. But formula (2.1) ensures that this is the case (McLennan (1989a), Lemma 2.3)). Therefore each conditional system $p \in \Delta^*(\Omega)$ induces a total preorder on Ω, which can be represented by an ordered partition $\mathbf{Q}(p) = (Q_0, \ldots, Q_{M(p)})$, where $p(s \mid \{s, t\}) = 0$ if and only if $s \in Q_\ell$, $t \in Q_m$, $\ell < m$. (This implies that $s, t \in Q_m$ for some m if and only if $0 < p(s \mid \{s, t\}) < 1$.)

An *ordering* of Ω is an ordered tuple $\mathbf{Q} = (Q_0, Q_1, \ldots, Q_M)$, where $\{Q_0, Q_1, \ldots, Q_M\}$ is a partition of Ω. For notational convenience we allow some cells Q_k to be empty.[7] We say that $\mathbf{Q} = (Q_0, Q_1, \ldots, Q_M)$ is a *refinement* of $\mathbf{R} = (R_0, \ldots, R_K)$ written $\mathbf{Q} \geq \mathbf{R}$, if $\{Q_0, Q_1, \ldots, Q_M\}$ is a refinement of $\{R_0, R_1, \ldots, R_K\}$ and $\mathbf{Q}$ preserves the order of $\mathbf{R}$, i.e., $\emptyset \neq Q_\ell \subseteq R_j$, $\emptyset \neq Q_m \subseteq R_k$ and $j < k$ imply $\ell < m$.

Remark 2.1. The refinement relation $\geq$ is transitive.

Remark 2.2. For every ordering $\mathbf{R}$ of Ω there is at least one $p \in \Delta^*(\Omega)$ such that $\mathbf{Q}(p) \geq \mathbf{R}$: take any linear order $<$ of Ω that refines $\mathbf{R}$ and let $p(s \mid \{s, t\}) = 0$ if $s < t$; by Lemma A.1 in the Appendix this determines a whole conditional system p such that $\mathbf{Q}(p) \geq \mathbf{R}$.

The refinement relation provides a convenient characterization of "rational" conditional systems. Let $\Omega = S^{-i}$ and let $\mathbf{R}^{-i} = (R_0^{-i}, \ldots, R_M^{-i})$ be a "rationality ordering" of S^{-i}. Then i's belief $p \in \Delta^*(S^{-i})$ is "most rational" if and only if $\mathbf{Q}(p) \geq \mathbf{R}^{-i}$ and it is "at-least-k-rational" if and only if $\mathbf{Q}(p) \geq (R_0^{-i}, \ldots, R_{k-1}^{-i}, (R_k^{-i} \cup \cdots \cup R_M^{-i}))$. Since $\mathbf{R}^{-i} \geq (R_0^{-i}, \ldots, R_{k-1}^{-i}, (R_k^{-i} \cup \cdots \cup R_M^{-i}))$ and the refinement relation is transitive, a most rational conditional system is also at-least-k-rational.

[6] The fact that we consider only information sets reached by s^i means that we actually check the rationality of plans of action (cf. Reny (1992)), but this is not essential for the argument.

[7] Note the McLennan (1989a) uses a different notation, in which Q_0 is the set of maximal elements and Q_M is the set of minimal elements.

To see how the refinement condition formalizes the best rationalization principle, assume that player i is endowed with a conditional system p such that $\mathbf{Q}(p) \geq \mathbf{R}^j$. Then, at the beginning of the game player i believes with probability one that her opponents' strategy profile belongs to R_M^{-i}. When she gets at information set h, she believes with probability one that her opponents' strategy profile belongs to $R_{m(h)}^{-i} \cap S^{-i}(h)$, where $m(h)$ is the highest index m such that $R_m^{-i} \cap S^{-i}(h) \neq \emptyset$ (see Lemma A.1 in the Appendix).

We are now ready to state our main definition. Consider a profile of orderings $[(R_0^1, \ldots, R_M^1), \ldots, (R_0^n, \ldots, R_M^n)]$ of $S^1, \ldots, S^n$. Let

$$R_k^{*j} = R_k^j \cup R_{k+1}^j \ldots \cup R_M^j,\ R_k^{*-i} = \times_{j \neq i} R_k^{*j}$$

and

$$\mathbf{R}^{-i} = (S^{-i} \backslash R_1^{*-i}, R_1^{*-i} \backslash R_2^{*-i}, \ldots, R_{M-1}^{*-i} \backslash R_M^{-i}, R_M^{-i}).$$

According to our intended interpretation R_k^{*-i} is the set of at-least-k-rational profiles and $\mathbf{R}^{-i}$ is the rationality ordering of i's opponents regarded as a whole. For example, if the players are I, II, and III, $t^{\mathrm{III}} \in R_M^{\mathrm{III}}$, $s^{\mathrm{III}} \in R_0^{\mathrm{III}}$, and $s^{\mathrm{I}} \in R_0^{\mathrm{I}}$, then $(s^{\mathrm{I}}, t^{\mathrm{III}})$ and $(s^{\mathrm{I}}, s^{\mathrm{III}})$ belong to the same cell of irrational profiles $R_0^{-\mathrm{II}}$. This agrees with our assumption that in general beliefs exhibit correlation: player II may interpret a deviation from rationality by opponent I as evidence that III is also irrational. Therefore $(s^{\mathrm{I}}, t^{\mathrm{III}})$ need not be infinitely more likely than $(s^{\mathrm{I}}, s^{\mathrm{III}})$ (see the game depicted in Fig. 2).

Finally, let $\mathbf{R}^{-i}(k)$ denote the coarsening of $\mathbf{R}^{-i}$, where all the profiles of R_k^{*-i} are put in the same cell, i.e.,

$$\mathbf{R}^{-i}(k) = (S^{-i} \backslash R_1^{*-i}, R_1^{*-i} \backslash R_2^{*-i}, \ldots, R_{k-1}^{*-i} \backslash R_k^{*-i}, R_k^{*-i}).$$

(Note that $\mathbf{R}^{-i}(0) = (S^{-i})$ is the trivial ordering.)

DEFINITION 2.2. A profile of *correlated sequential rationality orderings* of an extensive game is an n-tuple of orderings $[(R_0^1, \ldots, R_M^1), \ldots, (R_0^n, \ldots, R_M^n)]$ of each player's strategies such that for all $i \in N$ and $k = 0, \ldots, M-1$

$$\bigcup_{r=k+1}^{r=M} R_r^i = \{s^i \in S^i \mid \exists p \in \Delta^*(s^{-i})\colon \mathbf{Q}(p) \geq \mathbf{R}^{-i}(k) \text{ and } s^i \in SB^i(p)\}, \quad (2.3)$$

$$R_M^i = \{s^i \in S^i \mid \exists p \in \Delta^*(S^{-i})\colon \mathbf{Q}(p) \geq \mathbf{R}^{-i} \text{ and } s^i \in SB^i(p)\} \quad (2.4)$$

Remark 2.3. Since each $s^i \in SB^i(p)$ is a best response to the prior $p(\cdot \mid S^{-i})$, (2.4) implies that if $R_M^i = \{\hat{s}^i\}$ (a singleton) for all i, then $(\hat{s}^i)_{i \in N}$ is a Nash equilibrium.

As we mentioned in the Introduction, the set of irrational strategies R_0^i is uniquely determined: it is just the set of strategies which are not sequential best responses against any conditional system in $\Delta^*(S^{-i})$. Furthermore, Definition 2.2 has a partly inductive structure given by (2.3). Therefore it is not surprising that a unique profile of rationality orderings can be computed with an inductive procedure.

DEFINITION (INDUCTIVE PROCEDURE) 2.3. For all $i \in N$ let $R_0^{*i} = S^i$. Fix $k \geq 0$. Given $R_0^{*j}, R_1^{*j} \ldots, R_k^{*j}$, $j \in N$, for all $i \in N$ define $\mathbf{Q}^{-i}(k)$ and R_{k+1}^{*i} as

$$R_\ell^{*-i} = \times_{j\neq i} R_\ell^{*j}, \qquad \ell = 0, \ldots, k,$$

$$\mathbf{Q}^{-i}(k) = (R_0^{*-i} \backslash (R_1^{*-i} \cup \cdots \cup R_k^{*-i}),$$

$$R_1^{*-i} \backslash (R_2^{*-i} \cup \cdots \cup R_k^{*-i}), \ldots, R_k^{*-i}),$$

$$R_{k+1}^{*i} = \{s^i \in S^i \mid \exists p \in \Delta^*(S^{-i})\colon\ s^i \in SB^i(p), \mathbf{Q}(p) \geq \mathbf{Q}^{-i}(k)\}.$$

THEOREM 2.1. *Consider Procedure* 2.3. *For all players $i \in N$ and all $m \geq 0$*

(a) $R_{m+1}^{*i} \subseteq R_m^{*i}$ *and* $\mathbf{Q}^{-i}(m) = (R_0^{*-i} \backslash R_1^{*-i}, \ldots, R_{m-1}^{*-i} \backslash R_m^{*-i}, R_m^{*-i})$;

(b) *there is a minimal integer M_i such that for all $k \geq M_i$, $R_k^{*i} = R_{M_i}^{*i} \neq \emptyset$;*

(c) *let* $M = \max\{M_i; i \in N\}$ *and* $R_m^i = R_m^{*i} \backslash R_{m+1}^{*i}$, $m = 0, \ldots, M-1$; *then* $[(R_0^1, \ldots, R_{M-1}^1, R_M^{*1}), \ldots, (R_0^n, \ldots, R_{M-1}^n, R_M^{*n})]$ *is the unique profile of correlated sequential rationality orderings.*

Proof. (a) By inspection of Procedure 2.3 it follows immediately that $R_1^{*i} \subseteq R_0^{*i}$ and $R_1^{*-i} \subseteq R_0^{*-i}$ for all i. Assume that $R_{\ell+1}^{*i} \subseteq R_\ell^{*i}$ and $R_{\ell+1}^{*-i} \subseteq R_\ell^{*-i}$ for all i and $\ell = 0, \ldots, (m-1)$. Then $R_k^{*-i} \cup \cdots \cup R_\ell^{*-i} = R_k^{*-i} = R_k^{*-i} \cup \cdots \cup R_\ell^{*-i} \cup R_{\ell+1}^{*-i}$ for all $k = 0, \ldots, \ell$. Therefore

$$\begin{aligned}\mathbf{Q}^{-i}(m) &= (R_0^{*-i} \backslash R_1^{*-i}, R_1^{*-i} \backslash R_2^{*-i}, \ldots, R_{m-1}^{*-i} \backslash R_m^{*-i}, R_m^{*-i}) \\ &\geq (R_0^{*-i} \backslash R_1^{*-i}, R_1^{*-i} \backslash R_2^{*-i}, \ldots, R_{m-1}^{*-i}) = \mathbf{Q}^{-i}(m-1).\end{aligned}$$

Since the refinement relation $\geq$ is transitive, $\mathbf{Q}(p) \geq \mathbf{Q}^{-i}(m)$ implies $\mathbf{Q}(p) \geq \mathbf{Q}^{-i}(m-1)$. Hence $R_{m+1}^{*i} \subseteq R_m^{*i}$ and $R_{m+1}^{*-i} \subseteq R_m^{*-i}$ for all i.

(b) By Lemma 2.1, $SB^i(p) \neq \emptyset$ for all $p \in \Delta^*(S^{-i})$. Furthermore, for any ordering $\mathbf{Q}^{-i}$ on S^{-i} there is at least one conditional system p such that $\mathbf{Q}(p) \geq \mathbf{Q}^{-i}$ (see Remark 2.2). This implies that $R_k^{*i} \neq \emptyset$ for all k. Since S^i is finite and $\{R_k^{*i}\}_0^\infty$ is a weakly decreasing sequence of subsets of S^i, there must be a first integer M_i such that $R_{M_i}^{*i} = R_k^{*i}$ for all $k \geq Mi$.

(c) This is a straightforward consequence of (a) and (b). ∎

Note than an analogous result holds for different specifications of the primitives. It is sufficient that the analogs of Remark 2.2 and Lemma 2.1 hold. Procedure 2.3 yields stronger and stronger rationality restrictions on beliefs given by finer and finer orderings. This may be satisfactory from a foundational point of view, but it may also be computationally cumbersome. The traditional deletion procedures of game theory are *reduction procedures*, whereby the deletion step $k+1$ does not take into account the strategies already deleted in the previous k steps. One of these reduction procedures is *extensive form rationalizability* with correlated beliefs (Pearce (1984)). Pearce used this procedure in order to define a solution concept relying on the following assumptions: (i) each player always maximizes her expected payoff and she updates her expectations using Bayes' rule whenever possible; (ii) each player believes that her opponents are rational unless this is contradicted by their observed moves; (iii) assumptions (i) and (ii) are common knowledge at the beginning of the game.

We now define a version of Pearce's reduction procedure using the language of conditional systems and allowing for correlated beliefs.[8]

DEFINITION (INDUCTIVE PROCEDURE) 2.4. *For all $i \in N$ let $P_0^{*i} = S^i$. Fix $k \geq 0$. Assume that P_k^{*i} has been defined for all $i \in N$ and let $P_k^{*-i} = \times_{j \neq i} P_k^{*j}$. Then $s^i \in P_{k+1}^{*i}$ if and only if*

(a) $s^i \in P_k^{*i}$;

(b) *there is $p \in \Delta^*(S^{-i})$ such that $\mathbf{Q}(p) \geq (S^{-i} \backslash P_k^{*-i}, P_k^{*-i})$ and for all $h \in H^i$ such that $s^i \in S^i(h)$ and $S^{-i}(h) \cap P_k^{*-i} \neq \emptyset$ and all $t^i \in S^i(h) \cap P_k^{*i}$, $U^i(s^i, p \mid S^{-i}(h)) \geq U^i(t^i, p \mid S^{-i}(h))$.*

A strategy s^i is called *rationalizable* if and only if $s^i \in \bigcap_{k \geq 1} P_k^{*i}$.

Consider a generic step $k+1$. Condition (a) says that only the strategies $s^i \in P_k^{*i}$ which survived previous deletion steps are "eligible." Condition (b) imposes rationality restrictions on beliefs and conditional expected utility maximization. Conditional systems need only distinguish between "irrational" strategy profiles and "temporarily rational" (i.e., not yet deleted) strategy profiles $s^{-i} \in P_k^{*-i}$. Expected utility maximization is imposed only at information sets which can be reached by some undeleted strategy profile $s \in P_k^*$. Furthermore, a strategy $s^i \in P_k^{*i}$ is only compared with alternative strategies in P_k^{*i}.

The following theorem says that exstensive form rationalizability corresponds to the profile of correlated sequential rationality orderings.[9]

[8] Pearce's original definition (Pearce (1984, Definitions 9, 10)) is formally different from ours: (i) Pearce does not use conditional systems, (ii) he allows for the choice of mixed strategies, and (iii) he assumes a sort of independence property (see Section 3). In two-person games of perfect recall (where issue (iii) does not arise) these differences are irrelevant. On (i) see Battigalli (1994, Theorem 1). As for (ii) the pure "strategy property" (Pearce (1984, Proposition 4)) implies that the set of pure rationalizable strategies does not change if players are allowed to randomize.

[9] A similar result appears in Battigalli (1990).

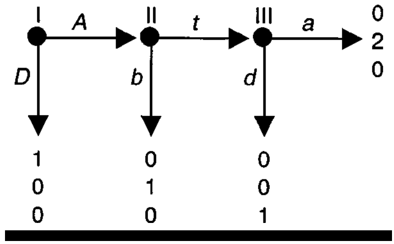

FIG. 2. A perfect information, 3-person game. P-rationalizability does not exclude t.

THEOREM 2.2. *Consider Procedures* 2.3 *and* 2.4. *For all* $i \in N$ *and* $k \geq 0$, $R_k^{*i} = P_k^{*i}$. *Therefore*

$$[(P_0^{*i} \backslash P_1^{*i}, \ldots, P_{M-1}^{*i} \backslash P_M^{*i}, P_M^{*i})]_{i \in N}$$

(*where* M *is defined as in Theorem* 2.1) *is the profile of correlated sequential rationality orderings*.

Proof. See the Appendix.

3. INDEPENDENT SEQUENTIAL RATIONALITY ORDERINGS

The following principle is assumed in many game-theoretic solution concepts: different players choose their strategy independently and this is reflected in each player's probabilistic beliefs about her opponents, which satisfy a stochastic independence condition. We call this principle *strategic independence*. Strategic independence is incorporated into the Nash equilibrium concept and the original definitions of rationalizability due to Bernheim (1984) and Pearce (1984). In his definition of extensive form rationalizability, Pearce (1984) tries to formalize the strategic independence principle by assuming that for each player i and information set $h \in H^i$, the conditional subjective belief $p(\cdot \mid S^{-i}(h))$ can be generated by a product prior distribution on S^{-i}. In the refinements literature this condition is called *structural consistency* (see Kreps and Wilson (1982)). Formally, Pearce's definition of rationalizability can be obtained by adding the structural consistency condition to Procedure 2.4.[10] We will use the terminology *P-rationalizability* to indicate this solution concept ("P" stands for Pearce). The following examples show that the structural consistency condition incorporated into P-rationalizability does not appropriately model strategic independence.

[10] We are still translating Pearce's formalization in our own terms (see Pearce (1984, Definitions 9, 10)).

First consider the perfect information game in Fig. 2. If player II at the beginning of the game believes that her opponents are rational and regards them as stochastically independent, she should predict choice d by player III and play b independently of her information about player I. Thus the solution should be the subgame perfect equilibrium. But according to P-rationalizability player II may interpret the irrational move A by player I as evidence that player III can also behave irrationally. This clearly violates strategic independence. As a consequence, both player II's strategies are P-rationalizable. Note, however, that P-rationalizability and the intuitive solution incorporating strategic independence yield the same terminal node, the one reached by the backward induction path. It can be shown that this is a general property of perfect information games, but the next example shows that different outcomes may result in games with imperfect information.

According to Pearce's definition, player i is allowed to change her belief about opponent j, just because she observed a subjectively unexpected (rational or irrational) behavior by another opponent k, while our intuition suggests that in such a case i's belief about j should not change if strategic independence holds.

We illustrate this point with a story. Benjamin (Ben) and Abigail (Abi) play the following game with Natalie (Nat). Nat chooses X or Y, but she is indifferent with respect to the outcome of the game.[11] Ben has to bet on Nat's choice. After the first stage Ben may choose either the same bet or the opposite one, but if he changes, he has to pay a small fine of $\$\varepsilon$. Abi has to guess Ben's behavior in each period. She gets \$1 for each correct guess and she loses \$1 for each incorrect guess. Abi and Ben observe each other's actions after the first stage and they do not receive any information concerning Nat's choice until the game is over. Ben maximizes his expected final wealth. Abi's attitudes toward risk are immaterial for the argument. Figure 3 shows the game with monetary payoffs.

Abi is smart, and after a moment she understands that by playing properly she cannot lose. If Ben bets on $X(Y)$ in the first stage, it is because this yields a nonnegative subjectively expected payoff. Since Ben is an expected payoff maximizer, he will not change the bet in the second stage, as his probability assessment does not change (if his expected payoff from the bet is zero, he repeats the first stage choice to avoid the fine). Therefore Abi is sure to correctly predict Ben's choice in the second stage and to get a total payoff of either \$2 or \$0, just by replicating Ben's first-stage choice.

Although the solution illustrated above seems pretty trivial, P-rationalizability is unable to replicate it. Even if Ben's conditional beliefs are always given by a product distribution, after an unexpected first-stage action by Abi, Ben's marginal distribution on Nat's choice may be different from the initial one. Consequently,

[11] We like to interpret Nat as the Nature player in a game of incomplete (but symmetric) information with private (completely subjective) beliefs on the state of the world.

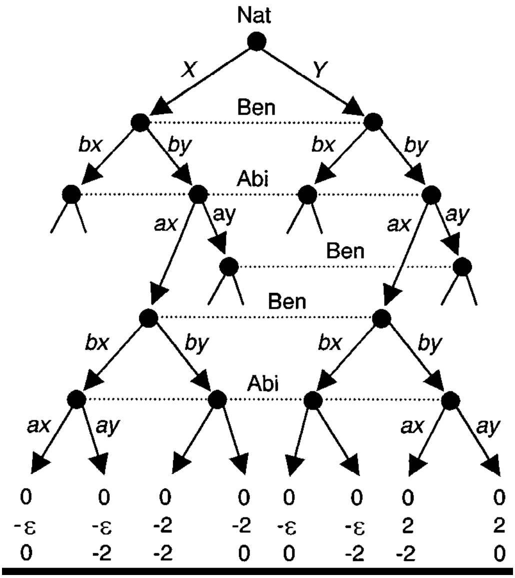

FIG. 3. Part of a game with sequential betting and betting about bets. Every path is consistent with P-rationalizability.

Ben may change his bet in the second stage. This implies that every terminal node of the game tree is reached by some P-rationalizable profile. In this example as well as in the previous one P-rationalizability is equivalent to rationalizabilty with correlated beliefs.[12]

We now provide a formalization of the Strategic Independence principle when players' beliefs are represented by conditional probability systems. Fix a player i and a conditional system $p \in \Delta^*(S^{-i})$. For each subset of opponents $J \subseteq N\backslash\{i\}$ we can derive the *marginal* of p on $S^J := \times_{j \in J} S^j$ as follows: for all A^J, $B^J \subseteq S^J$, $B^J \neq \emptyset$,

$$p^J(A^J \mid B^J) = p(A^J \times S^{N\backslash(J\cup\{i\})} \mid B^J \times S^{N\backslash(J\cup\{i\})}).$$

DEFINITION 3.1. A conditional system $p \in \Delta^*(S^{-i})$ *has the independence property* if for all partitions $\{J_i, \ldots, J_k\}$ of $N\backslash\{i\}$, all rectangular subsets $\emptyset \neq E^{-i} = E^{J_i} \times \cdots E^{J_k} \subseteq S^{-i}$ and all $s^{-i} = (s^{J_i}, \ldots, s^{J_k}) \in S^{-i}$

$$p(s^{-i} \mid E^{-i}) = \prod_{r=1}^{r=k} p^{J_r}(s^{J_r} \mid E^{J_r}). \tag{3.1}$$

[12] It can be proved that P-rationalizability and correlated rationalizability are completely equivalent in games of perfect information (Battigalli (1990, Lemma 1)).

The set of conditional systems for player i satisfying the independence property is denoted $I\Delta^*(S^{-i})$.

Definition 3.1 is an extension of the usual multiplicative independence condition for probability measures.[13] Building on Blume *et al*. (1991a), Battigalli and Varonesi (1992) provide a decision-theoretic rationale for (3.1), showing that it can be derived from a standard stochastic independence axiom for preferences over lotteries. According to this axiom a decision maker does not take into account information concerning the coordinate k of the unknown state (opponent k in the present framework) for decisions whose consequences depend only on other coordinates (the behavior of other opponents).

LEMMA 3.1. *For every profile of orderings* $[\mathbf{R}^j]_{j\neq i}$ *there is at least one conditional system* $p \in I\Delta^*(S^{-i})$ *such that for all* $j \neq i$, $\mathbf{Q}(p^j) \geq \mathbf{R}^j$.

Proof. See the Appendix.

We are now able to state a definition of rationality orderings relying on the strategic independence principle. In agreement with the notation of Section 2, for any ordering $\mathbf{R}^i = (R_0^i, \ldots, R_M^i)$ of S^i we have $\mathbf{R}^i(k) = (R_0^i, \ldots, R_{k-1}^i, \bigcup_{r=k}^{r=M} R_r^i)$.

DEFINITION 3.2. A profile *independent sequential rationality orderings* is an n-tuple of orderings $[(R_0^1, \ldots, R_M^1), \ldots, (R_0^n, \ldots, R_M^n)]$ of each player's strategies such that for all $i \in N$ and $k = 0, \ldots, M-1$

$$\bigcup_{r=k+1}^{r=M} R_r^i = \{s^i \in S^i \mid \exists p \in I\Delta^*(S^{-i}) \colon \forall j \neq i, \mathbf{Q}(p^j) \geq \mathbf{R}^j(k) \text{ and } s^i \in SB^i(p)\}, \tag{3.2}$$

$$R_M^i = \{s^i \in S^i \mid \exists p \in I\Delta^*(S^{-i}) \colon \forall j \neq i, \mathbf{Q}(p^j) \geq \mathbf{R}^j \text{ and } s^i \in SB^i(p)\}. \tag{3.3}$$

Once again we can define an iterative procedure which yields the unique profile of rationality orderings.

DEFINITION (INDUCTIVE PROCEDURE) 3.3. For all $i \in N$ let $R_0^{*i} = S^i$. Fix $k \geq 0$. Given $R_0^{*j}, R_1^{*j}, \ldots, R_k^{*j}$, $j \in N$, define $\mathbf{Q}^j(k)$ and R_{k+1}^{*i}, $j, i \in N$, as

$$\mathbf{Q}^j(k) = (R_0^{*j}\backslash(R_1^{*j} \cup \cdots \cup R_k^{*j}), R_1^{*j}\backslash(R_2^{*j} \cup \cdots \cup R_k^{*j}), \ldots, R_k^{*j}),$$

$$R_{k+1}^{*i} = \{s^i \in S^i \mid \exists p \in I\Delta^*(S^{-i}) \colon \forall j \neq i, \mathbf{Q}(p^j) \geq \mathbf{Q}^j(k) \text{ and } s^i \in SB^i(p)\}.$$

[13] Similar or equivalent independence properties have been put forward and discussed in a game-theoretic context by Hammond (1992), Kohlberg and Reny (1992), Battigalli (1993a, 1994), and Swinkels (1993). Renyi (1955, p. 303) uses a similar (but weaker) condition.

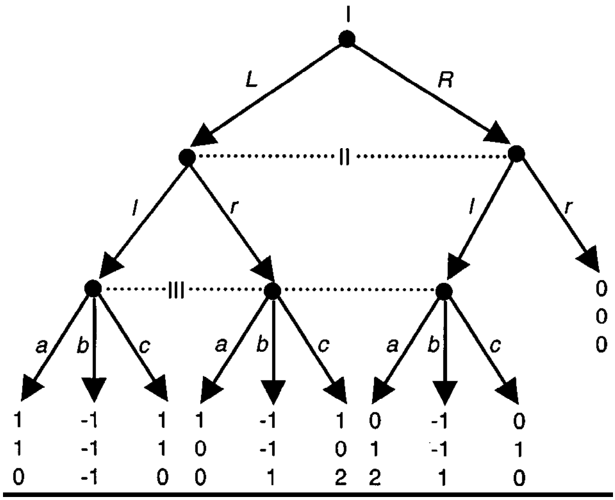

FIG. 4. A 3-person game where independent sequential rationality orderings do not refine P-rationalizability.

THEOREM 3.1. *Consider Procedure* 3.3. *For all players* $i \in N$ *and all* $m \geq 0$,

(a) $R^{*i}_{m+1} \subseteq R^{*i}_m$ *and* $\mathbf{Q}^i(m) = (R^{*i}_0 \backslash R^{*i}_1, \ldots, R^{*i}_{m-1} \backslash R^{*i}_m, R^{*i}_m)$;

(b) *there is a minimal integer* M_i *such that for all* $k \geq M_i$, $R^{*i}_k = R^{*i}_{M_i}$ *and* $R^{*i}_{M_i} \neq \emptyset$;

(c) *let* $M = \max\{M_i; i \in N\}$ *and* $R^i_m = R^{*i}_{m+1} \backslash R^{*i}_m$, $m = 0, \ldots, M-1$; *then* $[(R^1_0, \ldots, R^1_{M-1}, R^{*1}_M), \ldots, (R^n_0, \ldots, R^n_{M-1}, R^{*n}_M)]$ *is the unique profile of independent sequential rationality orderings.*

Proof. Similar to the proof of Theorem 2.1. In the proof of (b) take into account Lemma 3.1. ∎

It is straightforward to check that the independent rationality orderings of the games in Figs. 2 and 3 correspond to the intuitive solutions outlined above. These examples seem to suggest that independent rationality orderings refine P-rationalizability, but this is not the case.[14] Consider the game of Fig. 4 (see also Kreps and Ramey (1987, Fig. 1)). Action *b* maximizes player III's conditional

[14] Analogously, Battigalli (1993b, Section 6.3) provides an example where correlated rationality orderings are finer than independent rationality orderings.

expected payoff if and only if

$$p((L,r) \mid \{(L,\ell),(L,r),(R,\ell)\}) = 1/2 = p((R,\ell) \mid \{(L,\ell),(L,r),(R,\ell)\}). \tag{3.4}$$

But such conditional beliefs violate structural consistency since they connot be derived from a product prior. Therefore P-rationalizability implies that b is deleted in the first step of the procedure. Hence, in the second step, players I and II are sure to get the highest payoff by playing left and the P-rationalizable solution is $\{L\} \times \{\ell\} \times \{a, c\}$.

On the other hand, there is an independent conditional system $p \in I\Delta^*(S^{\mathrm{I}} \times S^{\mathrm{II}})$ such that $p^{\mathrm{I}}(R \mid S^{\mathrm{I}}) = p^{\mathrm{II}}(r \mid S^{\mathrm{II}}) = 1$, which satisfies (3.4). This implies that the triple of independent rationality orderings is just the trivial one: $[(\emptyset, S^{\mathrm{I}}), (\emptyset, S^{\mathrm{II}}), (\emptyset, S^{\mathrm{III}})]$.

In this game P-rationalizability is more powerful than independent rationality orderings. But we claim that this is due to an implausible restriction imposed by structural consistency on player III's beliefs.[15] Note that the players I and II are symmetric in an obvious sense. Therefore we should allow for independent conditional systems reflecting this symmetry. Furthermore, there is no *a priori* reason to exclude that (R, r) has prior probability one. It can be checked that the only symmetric and independent conditional system such that $p((R,r) \mid S^{\mathrm{I}} \times S^{\mathrm{II}}) = 1$ is the one given above.

4. OTHER DEFINITIONS OF RATIONALITY ORDERINGS

So far we have analyzed only sequential best responses against conditional probability systems, but we pointed out in Section 1 that other sensible definitions of extended probabilities and best response correspondences can be given.

One possibility is to assume that, though each player is a conditional expected utility maximizer, only ordinal preferences over terminal nodes are common knowledge (see Börgers (1993)). In this case the appropriate notion of sequential best response is the following: s^i is a sequential best response against p if there is an increasing transformation m of U^i such that s^i maximizes $(m \circ U^i)(\cdot, p \mid S^{-i}(h))$ for each $h \in H^i$ such that $s^i \in S^i(h)$. It may be easily checked that with this formulation the analog of Theorem 2.1 holds.

Another possiblity is given by lexicographic expected utility maximization. Blume *et al*. (1991a) develop a system of axioms about preferences over lotteries, in which the Archimedean axiom is weakened and the state independence

[15] This may happen because the set of strategies reaching player III's information set is not rectangular, otherwise (3.1) would imply structural consistency.

axiom is strengthened. They show that a decision maker chooses as if she were endowed with a utility function and a hierarchy of probability measures over the (finite) set of states—called *lexicographic probability system*—such that each state is given a positive probability by at least one measure and she maximized lexicographically the resulting vector of expected utilities. A lexicographic probability system $p = (p_0, p_1, \ldots, p_M)$ determines an ordering $\mathbf{Q}(p)$ over the set of states as follows: t is infinitely more likely than s, if and only if $p_{m(s,t)}(s) = 0$, where $m(s, t)$ is the highest index m such that $p_m(\{s, t\}) > 0$. We made clear in the Introduction that these notions of extended probabilities and best responses can also be easily fitted into our framework, yielding the notion of *lexicographic rationality orderings*. It can be shown that the related procedure corresponds to the iterated (maximal) deletion of weakly dominated mixed strategies (see Stahl (1991, Proposition 1) and Rajan (1993)). Veronesi (1994) provides a partial justification in terms of rationality orderings of the iterated deletion of pure strategies which are weakly dominated by other *pure* strategies.

Yet another notion of rationality orderings can be obtained using a procedure put forward by Reny (1992) in order to define a refinement of the Nash equilibrium concept called "explicable equilibrium." Reny's solution can be reformulated in the present framework through a modification of Procedure 3.3 by considering the set of conditional systems satisfying a strong independence property equivalent to Kreps and Wilson's (1982) consistency of assessments (Kohlberg and Reny (1992) and Swinkels (1993) provide a rationale for this property). It turns out that Reny's procedure and Procedure 3.3 coincide on a restricted class of extensive games including two-person games and n-person games with observed actions (for more on this see Battigalli (1993b)).

There is no definite relationship between different notions of rationality orderings in general games. Consider two specifications of the primitives (space of beliefs and best response correspondence) $(\Delta^{-i}, B^i(\cdot))_{i\in N}$ and $(\hat{\Delta}^{-i}, \hat{B}^i(\cdot))_{i\in N}$. Assume that for all $i \in N$ and $p \in \hat{\Delta}^{-i}$, $\hat{\Delta}^{-i} \subseteq \Delta^{-i}$ and $\hat{B}^i(p) \subseteq B^i(p)$, and let R_k^{*i}, $\hat{R}_k^{*i}$ denote the corresponding sets of at-least-k-rational strategies. Intuition might suggest that $\hat{R}_k^{*i} \subseteq R_k^{*i}$, but this need not be the case. It is indeed trivially true that $\hat{R}_1^{*i} \subseteq R_1^{*i}$, but this inclusion relation cannot be extended to higher "degrees of rationality." This is easily understood by considering the structure of the beliefs supporting at-least-2 rational strategies. Assume that for some player $j \neq i$, $\hat{R}_1^{*j} \subset R_1^{*j}$ and $\emptyset \neq R_0^j \subset \hat{R}_0^j$ ($\subset$ denotes strict inclusion). Then $\hat{\mathbf{R}}^{-i}(1) = (\hat{R}_0^{-i}, \hat{R}_1^{*-i})$ is *not* a refinement of $(R_0^{-i}, R_1^{*-i}) = \mathbf{R}^{-i}(1)$. This implies that the set of beliefs supporting strategies in $\hat{R}_2^{*i}$ is *not* a subset of the beliefs supporting strategies in $\hat{R}_2^{*i}$.

Positive results can be obtained by considering restricted classes of games. In particular, applying results due to Battigalli (1990) and Reny (1992), it can be shown that all the solutions considered so far are realization-equivalent to backward induction in generic games of perfect information.

APPENDIX

A.1. Conditional Probability Systems

The set of conditional probability systems $\Delta^*(\Omega)$ can be regarded as a subset of a Euclidean space of dimension $\#\Omega \cdot (2^{\#\Omega} - 1)$ ($\#\Omega$ denotes the cardinality of Ω) endowed with the relative topology. Clearly a strictly positive prior on Ω generates a whole conditional system, which belongs to the "interior" of $\Delta^*(\Omega)$. Myerson (1986) shows that $\Delta^*(\Omega)$ is the closure of the set of conditional systems generated by strictly positive priors on Ω. Since $\Delta^*(\Omega)$ is bounded, it is also compact.

Proof of Lemma 3.1. We must show that for every profile of orderings $[\mathbf{R}^j]_{j\neq i}$ we can find some $p \in I\Delta^*(S^{-i})$ such that $\mathbf{Q}(p^j) \geq \mathbf{R}^j$ for all $j \neq i$, where p^j is the marginal of p on S^j. We already know that for all $j \neq i$ there exist a $p^j \in \Delta^*(S^j)$ such that $\mathbf{Q}(p^j) \geq \mathbf{R}^j$ (see Remark 2.2). Furthermore, each p^j is approached by some sequence $\{p_k^j\}_{k=0}^{\infty}$ of conditional systems generated by strictly positive priors. Let $\{p_k\}_{k=0}^{\infty}$ be the sequence generated by the product priors $\prod_{j\neq i} p_k^j(\cdot \mid S^j)$. Clearly $\{p_k\}_{k=0}^{\infty} \subseteq I\Delta^*(S^{-i})$. Since $I\Delta^*(S^{-i})$ is a subset of $\Delta^*(S^{-i})$ defined by equality conditions, $I\Delta^*(S^{-i})$ is compact and $\{p_k\}_{k=0}^{\infty}$ must have a cluster point $p \in I\Delta^*(S^{-i})$. By construction p^j is exactly the marginal of p on S^j. Hence p is a conditional system with the required properties. ∎

LEMMA A.1[16]. *Consider an ordering* $\mathbf{R} = (R_0, \ldots, R_M)$ *of* Ω *and a conditional system* $p \in \Delta^*(\Omega)$ *such that* $\mathbf{Q}(p) \geq \mathbf{R}$. *Suppose that* $s \in E \subseteq \Omega$ *and let* $m(E) = \max\{k \mid R_k \cap E \neq \emptyset\}$. *Then* $p(s \mid E) = p(s \mid E \cap R_{m(E)})$.

Proof. We must show that if $s \in E\backslash R_{m(E)}$, then $p(s \mid E) = 0$. Let $s \in E\backslash R_{m(E)}$. Then, by definition of $m(E)$, we can find $t \in E \cap R_{m(E)}$, and $k < m(E)$ such that $s \in E \cap R_k$. Since $\{s\} \subset \{s, t\} \subseteq E$, condition (2.1) yields $p(s \mid E) = p(s \mid \{s, t\})p(\{s, t\} \mid E)$. Since $\mathbf{Q}(p) \geq \mathbf{R}$, $s \in R_k$, $t \in R_{m(E)}$, and $k < m(E)$, it follows that $p(s \mid \{s, t\}) = 0$. Therefore $p(s \mid E) = 0$. ∎

A.2. Sequential Rationality Orderings

Proof of Lemma 2.1 (Sketch). Note that in games of perfect recall the collection of information sets H^i can be ordered by the precedence relation $<$ of the game tree and it can be regarded as an arborescence. For every $p \in \Delta^*(S^{-i})$, the triple $(H^i, <, p)$ corresponds to a finite decision tree (some random choices may have zero probability, but this is immaterial for the present argument). A strategy $s^i \in SB^i(p)$ can be constructed by backward induction on this decision tree. ∎

[16] This is a slight generalization of McLennan (1989a, Lemma 2.4).

Proof of Theorem 2.2. For notational simplicity and without loss of generality we consider a two-person game.

By inspection of Definitions 2.3 and 2.4 it is clear that $R_0^{*i} = P_0^{*i}$, $i = 1, 2$. Now assume that $R_k^{*i} = P_k^{*i}$, $i = 1, 2$. We have to show that $R_{k+1}^{*i} = P_{k+1}^{*i}$, $i = 1, 2$.

Let $s^i \in R_{k+1}^{*i}$. Since $R_{k+1}^{*i} \subseteq R_k^{*i} = P_k^{*i}$, condition (a) of Definition 2.4 holds. We know that s^i is a sequential best response against some $p \in \Delta^*(S^j)$ such that $\mathbf{Q}(p) \geq \mathbf{R}^j(k)$. By Lemma A 1, $p(\cdot \mid S^j) = p(\cdot \mid P_k^{*j})$ and for all $h \in H^i$ such that $S^j(h) \cap P_k^{*j} \neq \emptyset$ we have $p(\cdot \mid S^j(h)) = p(\cdot \mid S^j(h) \cap P_k^{*j})$. Therefore condition (b) also holds and $s^i \in P_{k+1}^{*i}$.

Let $s^i \in P_{k+1}^{*i}$. Since $P_{k+1}^{*i} \subseteq P_k^{*i} = R_k^{*i}$, we can find two conditional systems $p_{k-1}, p_k \in \Delta^*(S^j)$ such that $s^i \in SB^i(p_{k-1})$, $\mathbf{Q}(p_{k-1}) \geq (R_0^j, \ldots, R_{k-2}^j, R_{k-1}^{*j}) = \mathbf{R}^j(k-1)$, and condition (b) of Definition 2.4 holds for $p = p_k$. Now define a map $p(\cdot \mid \cdot)$ as

$$p(\cdot \mid B^j) = p_k(\cdot \mid B^j), \quad \text{if } B^j \cap P_k^{*j} \neq \emptyset,$$
$$p(\cdot \mid B^j) = p_{k-1}(\cdot \mid B^j), \quad \text{if } B^j \cap P_k^{*j} = \emptyset.$$

Applying Lemma A.1 it is easy to check that p is a conditional system and $\mathbf{Q}(p) \geq \mathbf{R}^j(k)$. Now we show that $s^i \in SB^i(p)$.

We already know that s^i is a best response against p at all $h \in H^i$ (reached by s^i) such that $S^j(h) \cap P_k^{*j} = \emptyset$, since for such information sets $p(\cdot \mid S^j(h)) = p_{k-1}(\cdot \mid S^j(h))$, and $s^i \in SB^i(p_k - 1)$. The remaining collection of i's information sets reached by s^i is $H^i(s^i, k) = \{h \in H^i \mid S^j(h) \cap P_k^{*j} \neq \emptyset, s^i \text{ reaches } h\}$. We now show that s^i is a best response also at all $h \in H^i(s^i, k)$ because the constraint $t^i \in S^i(h) \cap P_k^{*i}$ in the maximization condition (b) of Definition 2.4 is not binding. We rely again on the arborescence structure of H^i (see the proof of Lemma 2.1). We prove that s^i is an unconstrained best response against p at each $h \in H^i(s^i, k)$ by induction on the number of predecessors of h.

Note that each $h \in H^i(s^i, k)$ has exactly the same predecessors in $H^i(s^i, k)$ as in H^i. In what follows we speak of "predecessors" without further qualification. Consider an information set $h \in H^i(s^i, k)$ without predecessors and a strategy $t^i \in SB^i(p)$ computed by backward induction as in the proof of Lemma 2.1. By choice of h, both s^i and t^i reach h. Since $\mathbf{Q}(p) \geq \mathbf{R}^j(k)$, $t^i \in R_{k+1}^{*i} \subseteq R_k^{*i} = P_k^{*i}$. Therefore t^i is an unconstrained best response at h which belongs to the constraint set $S^i(h) \cap P_k^{*i}$ and this implies that s^i also must be an unconstrained best response to p at h. Thus we may assume without loss of generality that t^i and s^i select the same choice at h. By this argument s^i is an unconstrained best response against p at all information sets without predecessors and there is a backward induction strategy $t^i \in SB^i(p)$ such that t^i and s^i reach the same information sets with at most one predecessor.

Now consider an information set $h \in H^i(s^i, k)$ with at most $\ell \geq 1$ predecessors and assume that there is some backward induction strategy $t^i \in SB^i(p)$

which coincides with s^i at all information sets with at most $\ell - 1$ predecessors. Repeating the previous argument, s^i must be an unconstrained best response against p at h and without loss of generality we may assume that t^i and s^i select the same choice at h. Thus s^i is an unconstrained best response against p at all information sets of $H^i(s^i, k)$ with at most ℓ predecessors and there is a strategy $t^i \in SB^i(p)$ such that t^i and s^i reach the same information sets with at most $\ell + 1$ predecessors. ■

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