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Structural consistency and strategic independence in extensive games

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Summary

In this note I provide a formulation of the joint principle of structural consistency and strategic independence, which is used to model players' expectations in finite extensive games. I compare updating systems of conjectures and conditional probability systems, showing that they represent equivalent formalizations of structural consistency. The notion of strategic independence cannot be adequately formalized by properties of updating systems of conjectures. However, it can be naturally translated in an intuitive stochastic independence property for conditional probability systems.

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1. Introduction

In this note I analyse the relationship between two principles which are often used to model the expectations of strategically interacting agents: strategic independence and structural consistency. They can be informally stated as follows:

Strategic Independence (SI): each player i does not change his or her expectations about a group of opponents J, if he or she gets new information exclusively concerning the complementary group of opponents K.

Structural Consistency (SC): each player i has a probabilistic conjecture about the strategic choices of his or her opponents, which he or she uses to assess conditional probabilities according to Bayes' rule whenever possible; if the player receives a new

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piece of information h with zero prior probability according to his or her former conjecture, he or she looks for an alternative conjecture which assigns positive prior probability to h and uses it to assess conditional probabilities according to Bayes' rule whenever possible.

Both principles concern conditional expectations and are relevant for extensive games, where new information may actually be received before some players choose among alternatives. SI is also relevant for simultaneous move games, where SI is equivalent to saying that each player regards the strategic choices of different opponents as stochastically independent, i.e. the joint probability measure is the product of the marginal measures. However, this "multiplicative" stochastic independence is a weaker property than SI in extensive games, because it allows player i to change his or her expectations about group J, if he or she gets completely unexpected information exclusively concerning the complementary group K.

SI is a cornerstone of normal-form game theory, though Aumann (1974, 1987) has quite convincingly argued that it may be reasonable to allow for spurious correlation, induced for example by the possibility of preplay communication. Correlated conjectures are even more plausible if it is not assumed that players have common expectations, as is the case in the theory of rationalizability (see, e.g., Hammond, 1994). In the context of extensive games it can be argued that even if priors must have the independence property, there may be unexpected events which signal an unforeseen degree of correlation between opponents' strategies (see, e.g., Kreps & Ramey, 1987; Fudenberg, Kreps & Levine, 1988). Although such arguments are plausible, there are certainly contexts where players' expectations should conform to SI. This principle is closely related to the assumption that the types of different players in a game of incomplete information are independent, which is common in economic analysis (see, e.g., Fudenberg & Tirole, 1991*b*: chapter 8, and references therein). Furthermore, SI can be regarded as a default assumption for those cases where there is no *positive* reason to expect any specific kind of correlation. Therefore a correct formulation of SI in an extensive game theory of strategic rationality is worth pursuing.

The first explicit reference to SC is in Kreps and Wilson (1982). In this paper Kreps and Wilson take for granted that conjectures correspond to behavioural strategies of the opponents. Since they restrict their analysis to games of perfect recall, this is equivalent to saying that conjectures are uncorrelated probability measures on the opponents' strategies. Kreps and Wilson regard SC as an appealing property of players' assessments and claim that it is implied by a topological notion of consistency, which is used to refine the Nash equilibrium concept into the sequential equilibrium concept. Later a similar notion of SC was used by Pearce (1984) in the definition of a non-equilibrium notion of strategic rationality: extensive form rationalizability. In Pearce's paper it is also assumed that conjectures are uncorrelated. In both approaches the idea is to use SC to endow the players with reasonable conditional expectations and make them maximize conditional expected utility. Since it is also assumed that initial conjectures and new "alternative conjectures" satisfy stochastic independence, both approaches can be considered as tentative formulations of the joint SI–SC principle in order to define an extensive game notion of rationality.

It turns out that these first attempts to formulate the joint SI–SC principle in a theory of strategic rationality are not satisfactory. Kreps and Ramey (1987) show that SC (as defined by Kreps and Wilson, 1982) is not implied by Kreps and Wilson's topological notion of consistency and that there are games where no sequential equilibrium assessment is structurally consistent. Battigalli (1993*b*) shows that Pearce's notion of extensive form rationalizability may fail to replicate intuitive solutions relying on SI–SC.

Section 2 of this paper offers a coherent formulation of the joint SI–SC principle for finite extensive games without chance moves.

Section 2.1 provides the game-theoretic framework. Section 2.2 presents (a generalization of) Pearce's (1984) formulation of SC using updating systems of conjectures and shows that updating systems of uncorrelated conjectures are inadequate to represent SI. Then Section 2.3 shows that it is useful to characterize the pattern of conditional expectations of a player even with respect to virtual information which actually cannot be acquired in a given game. This motivates using conditional probability systems (Myerson, 1986) on the opponents' strategies. Section 2.4 is the core of the paper. Here the relationship between updating systems of conjectures and conditional probability systems is studied, showing that they are equivalent representations of SC. Then an intuitive formalization of the SI principle is provided: the SI-property for conditional systems. It is argued that the SI-property yields all the relevant restrictions implied by the SI–SC principle.

Some concluding remarks in Section 3 indicate how to extend the analysis.

2. Conditional expectations in finite extensive games

2.1. THE GAME-THEORETIC FRAMEWORK

I shall restrict my analysis to finite extensive games with perfect recall. I also assume for simplicity that there are no chance moves, but this last assumption can easily be removed (see Section 3). Since the formal description of extensive games is by now standard, I list below with very terse explanations the symbols I need, and I only define some primitive and derived elements of an extensive form.

Notation	*Terminology*
N	players' set (finite)
H^k ($k \in N$)	collection of k's information sets
$A(h)$ ($h \in H^k$)	available actions at h
$s^k \in S^k$ (S^k finite)	a pure strategy for player k
$S = \Pi_{k \in I} S^k$	the joint strategy set
$-i = N \setminus \{i\}$	player i's opponents
$\{J, K\}$ ($J \bigcup K = -i, J \bigcap K = \emptyset$)	a bipartition of i's opponents
$s^J \in S^J = \Pi_{j \in J} S^j$ ($J \subseteq -i$)	a $\#J$-tuple of strategies of players in J
$S^J(h)$ ($h \in H^k$)	the set of s^J reaching h
$\Delta(X)$	the set of probability measures on X
$c^{-i} \in \Delta(S^{-i})$	a conjecture about i's opponents
$c^J = \text{marg}\ [c^{-i}] \in \Delta(S^J)$	a conjecture about group J

I say that a strategy profile s^J reaches a node t of the game tree, if for every information set $h \in (\bigcup_{j \in J} H^j)$ along the path $P(t)$ to t, profile s^J selects the edge in $P(t)$. A strategy profile s^J *reaches an information set* $h \in H = (\bigcup_{k \in N} H^k)$, written $s^J \in S^J(h)$, if s^J reaches t for some $t \in h$. A conjecture c^J reaches h if its support contains a joint strategy s^J reaching h, i.e. $\text{Supp}(c^J) \bigcap S^J(h) \neq \emptyset$.

By perfect recall, if $h \in H^i$, then $S(h) = S^i(h) \times S^{-i}(h)$, i.e. player i can decompose the information content of h into information about him or herself and information about his or her opponents.

Let $P(h; c^J, s^{N \setminus J})$ be the probability of reaching information set $h \in H$ according to conjecture c^J when $s^{N \setminus J}$ is played. Then it is straightforward to check that for all $h \in H^i$, $P(h; c^{-i}, s^i) > 0$ if and only if c^{-i} and s^i reach h (the "if" part is due to perfect recall). Furthermore, if s^i reaches $h, g \in H^i$ and $S^{-i}(h) = S^{-i}(g)$, then $P(h; c^{-i}, s^i) = P(g; c^{-i}, s^i)$.

The precedence relation of the game tree can be used to define a binary relation $<^i \subseteq (\{\emptyset\} \bigcup H^i) \times (\{\emptyset\} \bigcup H^i)$ as follows: for all $g, h \in H^i$, $g <^i h$ if and only if every node $t \in h$ comes after a node $x \in g$ and no node $x \in g$ comes after a node $t \in h$; for all $h \in H^i$, $\emptyset <^i h$. By perfect recall, $(\{\emptyset\} \bigcup H^i, <^i)$ is a tree. Note that $g <^i h$ implies $S^{-i}(h) \subseteq S^{-i}(g)$ (with the obvious convention $S^{-i}(\emptyset) = S^{-i}$). If g and h are unrelated by $<^i$, then $S^{-i}(h) \bigcap S^{-i}(g) = \emptyset$.

2.2. STRUCTURAL CONSISTENCY AND UPDATING SYSTEMS OF UN/CORRELATED CONJECTURES

In order to formalize SC, Pearce (1984) endows each player i with mappings of the form $c^{-i}[\cdot]:(H^i\bigcup\{\emptyset\})\rightarrow\Delta(S^{-i})$ having the following properties:

$$\forall h\in H^i, c^{-i}[h] \text{ reaches } h, \tag{1}$$

$$\forall g\in H^i\bigcup\{\emptyset\}, \forall h\in H^i, \text{ if } g<^i h \text{ and } c^{-i}[g] \text{ reaches } h, \text{ then } c^{-i}[g]=c^{-i}[h]. \tag{2}$$

I call such mappings *updating systems of conjectures*: $c^{-i}[h]$ is the conjecture used by i to compute all relevant conditional probabilities through Bayes' rule and it is interpreted as the conjecture that i thinks he or she should have had from the beginning, when he or she gets information h. $c^{-i}[\emptyset]$ is i's initial conjecture.† Condition (1) says that i's conjecture is always consistent with his or her current information and condition (2) says that i does not change his or her conjecture unless it becomes inconsistent with his or her new information.

Pearce (1984) uses the additional condition that the conjectures $c^{-i}[h]$ be *uncorrelated*, i.e. for all s^{-i},

$$c^{-i}[h](s^{-i})=\Pi_{j\neq i}c^j(s^j).$$

In this case $c^{-i}[\cdot]$ is an *updating system of uncorrelated conjectures*.

Clearly, (1)–(2) correctly formalize SC, while the use of non-correlated conjectures can only be reasonably justified by SI. However, updating systems of uncorrelated conjectures are not a correct formulation of the joint SI–SC principle. This is shown by some examples.

EXAMPLE 1: Figure A represents a three-player extensive form. Numbers in parentheses near actions and numbers in brackets near nodes are the conditional probabilities (given that the relevant information set is reached) derived by an underlying updating system of conjectures for player III. Note that at information set h, III's opinion about opponent I is the opposite of the initial one, even though III has not observed anything about I's behaviour (III only observes II's move and II does not observe I's move). This

† Pearce (1984) does not use initial conjectures, but it can be shown that in games without chance moves, for every updating system defined on H^i there exists an equivalent updating system defined on $H^i\bigcup\{\emptyset\}$.

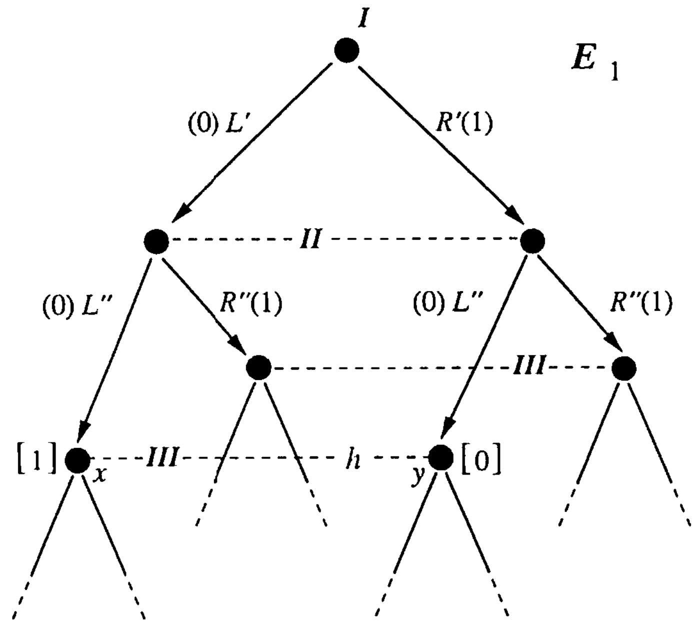

FIGURE A. Player III's conditional expectations are derived from an updating system of uncorrelated conjectures, but strategic independence is violated.

violation of the SI principle may happen because h is not reached by III's initial joint conjecture.

This example suggests a condition which updating systems should satisfy under SI–SC. Its informal statement is: "if i did not expect to reach h, but this is not due to i's former conjecture about group J, this conjecture about J is not modified at h". In such a situation I say that i's previous conjecture about J is "secure" at h.† I propose the following formal definition:

DEFINITION 1: *c^J (where $J \subseteq -i$) is secure at $h \in H^i$ if, for all $s^{N \setminus J} \in S^{N \setminus J}(h)$, $P(h; c^J, s^{N \setminus J}) > 0$.*

Clearly c^J is secure at $h \in H^i$ only if c^J reaches h and—by perfect recall—a conjecture about all the opponents c^{-i} is secure at h if and only if c^{-i} reaches h. But when $J \neq (-i)$ it may happen that c^J is "insecure" at h even if c^J reaches h. This is possible if the information about i's opponents' behaviour given by h cannot be decomposed into information about group J and information about the complementary group K, so that it is impossible to discover which component of (c^J, c^K) must be wrong, if h is reached and $P(h;$

† This terminology has been suggested by an anonymous referee.

$c^J, c^K) = 0$. It is interesting to characterize the games where this cannot happen.

DEFINITION 2: *an extensive form E has observed deviators if, for all $i \in N$ and all $h \in H^i$, $S(h) = \Pi_{k \in N} S^k(h)$.*†

In an extensive form with observed deviators, whenever player i believes that his or her opponents play a given profile s^{-i} and detects a deviation at information set h, he or she is able to identify which opponents caused h to be reached. Another interpretation is the following: each information set h corresponds to a list of messages h^1, h^2 ..., where message h^j says that player j is implementing some strategy in the set $S^j(h)$. Two-person games (of perfect recall), multistage games with observed actions (see Fudenberg & Tirole, 1991*b*: pp. 70–72) and games of perfect information clearly have observed deviators.

LEMMA 1: *the following statements are equivalent:*

(a) E has observed deviators;

(b) for all $i \in N$, $h \in H^i$, $J \subseteq -i$, conjecture c^J reaches h if and only if c^J is secure at h.

PROOF: see the Appendix.

2.3. CONDITIONAL PROBABILITY SYSTEMS

The SI–SC principle may be violated even if secure conjectures persist, as the following example shows.

EXAMPLE 2:‡ consider the conditional probabilities associated with

† In an earlier version of this paper I used the phrase "strategically decomposable". The present terminology is borrowed from Fudenberg and Levine (1993), where an equivalent definition is put forward. They use this concept to analyse the relationship between equilibria with common expectations (e.g. Nash equilibria) and equilibria where the expectations of different players may differ, but are correct along the path of play.

‡ Fudenberg and Tirole (1991*a*) consider a related example.

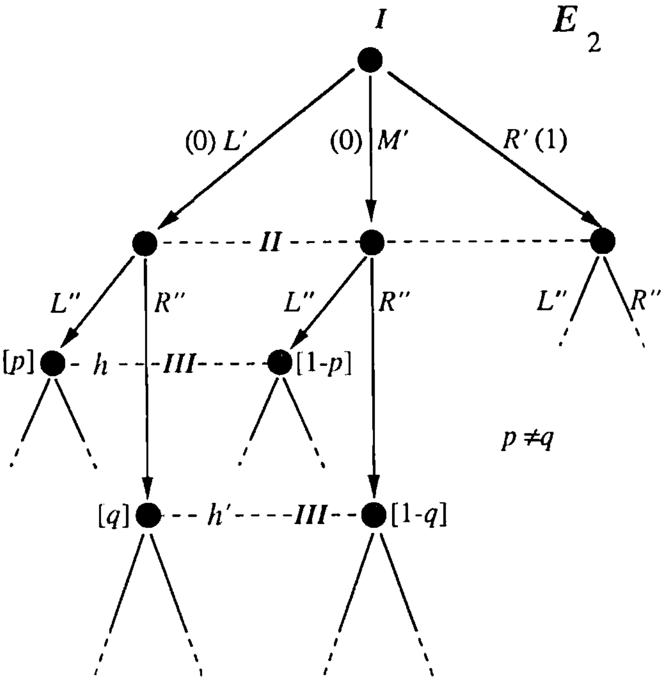

FIGURE B. Player III's initial conjecture about player I is not secure at h and h'. Thus, strategic independence can be violated even if secure conjectures persist.

extensive form E_2 in Figure B. Such probabilities can be derived by an underlying updating system of uncorrelated conjectures for III satisfying the condition that secure conjectures are not changed. Player III has exactly the same information about I at h and h' but his or her expectations about I are different. This may happen because III's initial conjecture about I is not secure at h and h'. But SI is clearly violated, because III's expectations about I conditional on "I does not choose R'" should not be modified when III gets information about II. Now modify E_2 as in E_2', introducing a "dummy" information set h^* for III before h and h'. III has no actual choice to make at h^* and the information content of h^* is just that I did not choose R'. In this modified extensive form, if secure conjectures are not changed, the pattern of III's conditional expectations satisfies SI–SC.

An information set $h \in H^i$ is *dummy* if i has only one available action at h. Intuitively, addition of dummy information sets should not modify the strategic properties of a game, which in turn depend on the properties of the pattern of conditional expectations of each player at non-dummy information sets. Therefore, we should provide conditions for such expectations, which are invariant with

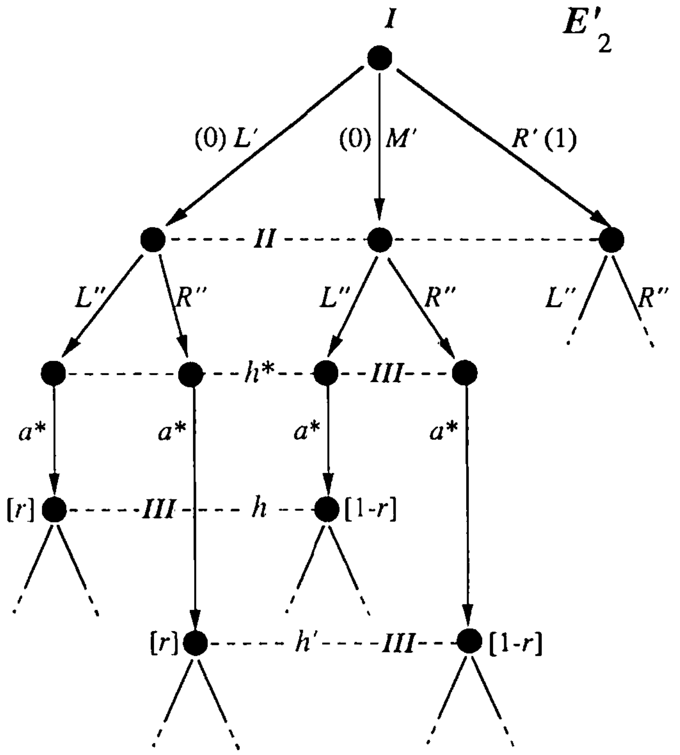

FIGURE C. After the introduction of a dummy information set, persistence of secure conjectures implies strategic independence.

respect to addition or deletion of dummy information sets. A direct way to accomplish this is to provide a player with virtual information in the form of sets of his or her opponents' strategies and look for consistency conditions on conditional probabilities. The strategically relevant conditional probabilities for a player i are then obtained by conditioning on sets of the form $S^{-i}(h)$ where $h \in H$ is non-dummy. I therefore use the notion of "conditional probability system" (cf. Rényi, 1955, 1956; Myerson, 1986; McLennan, 1989a,b). The definition of a conditional probability is given here for an arbitrary finite set of "states" Ω. In the following analysis Ω will be S^J, $J \subseteq N$

DEFINITION 3: *let* $\mathbb{P}(\Omega)$ *denote the power set of* Ω *and let* $\mathbb{P}_0(\Omega) = \mathbb{P}(\Omega)\backslash\{\emptyset\}$. *A conditional probability system is a function* $p(\cdot\,|\,\cdot)$: $\mathbb{P}(\Omega) \times \mathbb{P}_0(\Omega) \rightarrow [0,1]$ *such that for all* $X \in \mathbb{P}(\Omega)$ *and all* $Y, Z \in \mathbb{P}_0(\Omega)$, $p(\cdot\,|\,Z) \in \Delta(Z)$ *and*

$$X \subseteq Y \subseteq Z \text{ implies } p(X\,|\,Z) = p(X\,|\,Y)p(Y\,|\,Z). \tag{3}$$

$\Delta^*(\Omega)$ *denotes the set of conditional probability systems on* Ω.

REMARK 1: $\Delta^*(\Omega)$ can be regarded as a subset of a Euclidean space of dimension $\#\Omega \cdot \#\mathbb{P}_0(\Omega)$ ($\#X$ means "cardinality of X") endowed with the relative topology. Clearly, a strictly positive prior on Ω generates a whole conditional system, which belongs to $\Delta_0^*(S^{-i})$, the "interior" of $\Delta^*(\Omega)$. Myerson (1986) and McLennan (1989*a*) show that $\Delta^*(S^{-i})$ is precisely the closure of $\Delta_0^*(S^{-i})$ in the topology of $\mathbb{R}^{\Omega \times \mathbf{P}_0(\Omega)}$.

2.4. UPDATING SYSTEMS OF CONJECTURES, CONDITIONAL PROBABILITY SYSTEMS AND INDEPENDENCE

Condition (3) just says that conditional probabilities satisfy Bayes' rule. We are going to see that SC is equivalent to condition (3) as far as strategically relevant conditional probabilities are concerned.

An updating system $c^{-i}[\cdot]$ is said to be *consistent* with a conditional probability system $p(\cdot\,|\,\cdot) \in \Delta^*(S^{-i})$ if and only if for all $h \in H^i$ and all $s^{-i} \in S^{-i}(h)$

$$\{c^{-i}[h](S^{-i}(h))\}^{-1}\{c^{-i}[h](s^{-i})\} = p(s^{-i}\,|\,S^{-i}(h)) \tag{4}$$

(recall that $c^{-i}[h](S^{-i}(\mathrm{h}))>0$ by (1)).

Equation (4) just says that all the strategically relevant probabilities computed by i via the updating system coincide with those given by the conditional probability system. If two updating systems are consistent with the same conditional probability system, then they are obviously equivalent from the decision theoretic point of view.

THEOREM 1: *every updating system $c^{-i}[\cdot]$ is consistent with some conditional system $p(\cdot\,|\,\cdot) \in \Delta^*(S^{-i})$ and every conditional system $p(\cdot\,|\,\cdot) \in \Delta^*(S^{-i})$ is consistent with some updating system $c^{-i}[\cdot]$.*

PROOF: here I give only the main idea of the proof, which is quite simple. Less straightforward details are provided in the Appendix. Use equation (4) to define $p(\cdot\,|\,S^{-i}(h))$ for all $h \in H^i \bigcup \{\emptyset\}$. It is easy to show that condition (3) is not violated. Consider two information sets $g <^i h$ and let $s^{-i} \in S^{-i}(h) \subseteq S^{-i}(g)$. We have to show that $p(s^{-i}\,|\,S^{-i}(g)) = p(s^{-i}\,|\,S^{-i}(h))p(S^{-i}(h)\,|\,S^{-i}(g))$. There are two cases. Either $c^{-i}[g]$ does not reach h or condition (2) implies that $c^{-i}[g] = c^{-i}[h]$. In the first case $p(s^{-i}\,|\,S^{-i}(g)) = p(S^{-i}(h)\,|\,S^{-i}(g)) = 0$. In the second case

$$\begin{aligned} p(s^{-i}\,|\,S^{-i}(g)) &= \{c^{-i}[g](S^{-i}(g))\}^{-1}\{c^{-i}[g](s^{-i})\} \\ &= \frac{c^{-i}[h](s^{-i})}{c^{-i}[h](S^{-i}(h))} \times \frac{c^{-i}[g](S^{-i}(h))}{c^{-i}[g](S^{-i}(g))} \\ &= p(s^{-i}\,|\,S^{-i}(h))p(S^{-i}(h)\,|\,S^{-i}(g)). \end{aligned}$$

Thus $p(A \mid B)$ is defined for $B \in \{S^{-i}(h) \mid h \in \{\emptyset\} \bigcup H^i\}$. It is proved in the Appendix that $p(\cdot \mid \cdot)$ can be extended, obtaining a conditional probability system. The second part of the theorem is also proved in the Appendix. ∎

Theorem 1 shows that conditional probability systems correctly formalize SC. It goes without saying that the second part of this result does not hold for updating systems of uncorrelated conjectures.

An advantage of conditional probability systems is that they make it easy to formulate SI. For any set $Z \subseteq S^{-i}$ of the form $Z = Z^J \times Z^K$ it is possible to define the marginal measures of $p(\cdot \mid Z)$ on Z^J and Z^K: given $X^J \subseteq Z^J$ and $X^K \subseteq Z^K$, $\text{marg}^J(p)(X^J \mid Z) := p(X^J \times Z^K \mid Z)$ and $\text{marg}^K(p)(X^K \mid Z) := p(Z^J \times X^K \mid Z)$. According to SI the conditional probability of X^J should not change, if i is told that group K played a strategy in X^K.

DEFINITION 4: *a conditional probability system for i has the SI-property if and only if for all $Z = Z^J \times Z^K \in \mathbb{P}_0(S^{-i})$, all $X^J \subseteq Z^J$ and all $Y^K \in \mathbb{P}_0(S^K)$:*

$$\text{marg}^J(p)(X^J \mid Z) = p(X^J \times Y^K \mid Z^J \times Y^K). \tag{5}$$

$\Delta^*_{\text{SI}}(S^{-i})$ denotes the set of conditional systems with the SI-property. For each $p \in \Delta^*_{\text{SI}}(S^{-i})$, the marginal of p on S^J is the conditional system $p^J \in \Delta^*(S^J)$ such that for all $X^J \in \mathbb{P}(S^J)$, $Z^J \in \mathbb{P}_0(S^J)$, $p^J(X^J \mid Z^J) = p(X^J \mid Z^J \times Y^K \mid Z^J \times Y^K)$, where $Y^K \in \mathbb{P}_0(S^K)$ is arbitrarily chosen.

Hammond (1987) puts forward a similar notion of "conditional independence" for conditional probability systems defined on a product space $\Omega = \Omega^1 \times \Omega^2$.† As Theorem 2(i) will show, my definition is just an extension of Hammond's definition for the more general case where $\Omega = \Omega^1 \times \ldots \times \Omega^n$. A related decision theoretic axiom of stochastic independence is proposed by Blume, Brandenburger and Dekel (1991: p. 74). Battigalli and Veronesi (1992) show that this axiom is indeed equivalent to the present definition given other reasonable axioms.

It is easy to check that $\Delta^*_{\text{SI}}(S^{-i}) \neq \emptyset$: it is sufficient to take a strictly positive product measure on S^{-i} and derive $p(\cdot \mid \cdot)$ via Bayes' rule. However, it is normally the case that a game theoretic solution concept prescribes that i's opponents should play according to product measures without full support. Therefore, a more interesting question is whether for any product probability measure

† This has come to my attention quite recently. Definition 5 has been put forward independently, relying on the game theoretic arguments mentioned above.

m on S^{-i} (perhaps suggested by the theory as the solution of the game) there exists a $p(\cdot|\cdot)$ with the SI-property such that $p(\cdot|S^{-i})=m(\cdot)$. The answer is again affirmative. This is an implication of the following theorem.

Let $\Delta^*_{\times 0}(S^{-i})$ denote the set of conditional systems generated by a strictly positive product measure on S^{-i} and let $\Delta^*_{\mathrm{C}}(S^{-i})$ be the closure of $\Delta^*_{\times 0}(S^{-i})$.

THEOREM 2: *for all* $i\in N$, $\Delta^*_{\mathrm{SI}}(S^{-i})$ *is a compact set with the following properties:*

(*i*) *If* X *and* $Z\neq\emptyset$ *are Cartesian products with respect to a partition* $\{J,K,L,\ldots\}$ *of* $-i$ *and* $X\subseteq Z$, *then for all* $p\in\Delta^*_{\mathrm{SI}}(S^{-i})$

$$p(X^J\times X^K\times X^L\ldots|Z)=p^J(X^J|Z^J)p^K(X^K|Z^K)p^L(X^L|Z^L)\ldots. \quad (6)$$

(*ii*) $\Delta^*_{\mathrm{C}}(S^{-i})\subseteq\Delta^*_{\mathrm{SI}}(S^{-i})$ *and* $\Delta^*_{\times 0}(S^{-i})$ *is the "relative interior" of* $\Delta^*_{\mathrm{SI}}(S^{-i})$.

(*iii*) *If the extensive form has observed deviators, for each* $p\in\Delta^*_{\mathrm{SI}}(S^{-i})$ *there exists* $q\in\Delta^*_{\mathrm{C}}(S^{-i})$ *such that* $(\cdot|S^{-i}(h))=q(\cdot|S^{-i}(h))$ *for all* $h\in H^i$.

PROOF:† since $\Delta^*_{\mathrm{SI}}(S^{-i})$ is defined by equalities between the values of continuous functions of conditional probabilities, it is the intersection of closed sets and it must be closed. Also, $\Delta^*_{\mathrm{SI}}(S^{-i})$ is a subset of $\Delta^*(S^{-i})$, which is compact (see remark 1). A closed subset of a compact set is compact.

Property (*i*) is proved only for the case of a bipartition $\{J,K\}$. The general property follows by an easy induction argument. Fix $\emptyset\neq Z=Z^J\times Z^K\subseteq S^{-i}$ and $X^J\times X^K\subseteq Z$. It must be proved that

$$p(X^J\times X^K|Z)=p^J(X^J|Z^J)p^K(X^K|Z^K). \quad (6')$$

Substitute $X^J\times X^K=X$ and $Z^J\times X^K=Y$ in equation (3):

$$p(X^J\times X^K|Z)=p(X^J\times X^K|Z^J\times X^K)p(Z^J\times X^K|Z). \quad (7)$$

By definition of marginal conditional probability

$$\begin{aligned}\mathrm{marg}^J(p)(X^J|Z^J\times X^K)&=p(X^J\times X^K|Z^J\times X^K)\\ \mathrm{marg}^K(p)(X^K|Z)&=p(Z^J\times X^K|Z).\end{aligned} \quad (8)$$

Substitute (8) into the r.h.s. of (7):

$$p(X^J\times X^K|Z)=\mathrm{marg}^J(p)(X^J|Z^J\times X^K)\,\mathrm{marg}^K(p)(X^K|Z).$$

† Proofs of these results are also offered elsewhere (see, e.g., Battigalli, 1993*a*), but they are included here for completeness and for illustrative purposes.

Equation (5) and the definition of marginal conditional system yield $\text{marg}^J(p)(X^J \mid Z^J \times X^K) = p^J(X^J \mid Z^J)$, $\text{marg}^K(p)(X^K \mid Z) = p^K(X^K \mid Z^K)$. Substitute into (9), to get (6′) as desired.

Properties (ii) and (iii) are easy consequences of property (i). Any point in the interior of $\Delta^*_{\text{SI}}(S^{-i})$ must be generated by a strictly positive prior on S^{-i} and by property (i) this prior must be a product measure. Hence, the interior of $\Delta^*_{\text{SI}}(S^{-i})$ is $\Delta^*_{\times 0}(S^{-i})$. Since $\Delta^*_{\text{SI}}(S^{-i})$ is closed, it follows that $\Delta^*_{\text{C}}(S^{-i}) \subseteq \Delta^*_{\text{SI}}(S^{-i})$, as claimed in (ii). Assume that the extensive form has observed deviators: for all $h \in H^i$, $S^{-i}(h) = \Pi_{j \neq i} S^j(h)$. Consider the marginal conditional systems p^j, $j \neq i$. For each p^j there exists a sequence $\{m^j_n\}^\infty_{n=0}$ of strictly positive measures on S^j such that the associated sequence $\{p^j_n\}^\infty_{n=0}$ of strictly positive conditional systems converges to p^j. Let $\{m_n\}^\infty_{n=0}$ be the corresponding sequence of strictly positive product priors and $\{q_n\}^\infty_{n=0}$ be the associated sequence of strictly positive conditional systems. Choose a limit point q of $\{q_n\}^\infty_{n=0}$. By property (i) and continuity, for all $h \in H^i$, $s^{-i} \in S^{-i}(h)$

$$
\begin{aligned}
p(s^{-i} \mid S^{-i}(h)) &= \Pi_{j \neq i} p^j(s^j \mid S^j(h)) \\
&= \lim_{n \to \infty} \Pi_{j \neq i} p^j_n(s^j \mid S^j(h)) \\
&= \lim_{n \to \infty} \Pi_{j \neq i} m^j_n(s^j)/m^j_n(S^j(h)) \\
&= \lim_{n \to \infty} m_n(s^{-i})/m_n(S^{-i}(h)) \\
&= q(s^{-i} \mid S^{-i}(h)). \qquad \blacksquare
\end{aligned}
$$

It is now easy to answer the initial question. For any product measure m on S^{-i} we can find $p \in \Delta^*_{\text{SI}}(S^{-i})$ such that $p(\cdot \mid S^{-i}) = m(\cdot)$ by just taking a sequence $\{p_n\} \subseteq \Delta^*_{\times 0}(S^{-i})$, where $p_n(\cdot \mid S^{-i})$ converges to $m(\cdot)$: p can be chosen among the limit points of $\{p_n\}$. Clearly, $p \in \Delta^*_{\text{C}}(S^{-i}) \subseteq \Delta^*_{\text{SI}}(S^{-i})$. Such conditional systems are called *consistent*. Consistent conditional systems are essentially equivalent to consistent assessments as defined by Kreps and Wilson (1982) (cf. McLennan, 1989*b*).

Note also that there are extensive forms where the inclusion $\Delta^*_{\text{C}}(S^{-i}) \subseteq \Delta^*_{\text{SI}}(S^{-i})$ is strict and this is strategically relevant (see Kohlberg & Reny, 1992).

I now turn to the relationship between updating systems of uncorrelated conjectures and conditional probability systems. It is reasonable to say that an updating system $c^{-i}[\cdot]$ satisfies the SI-property if it is consistent with some $p(\cdot \mid \cdot) \in \Delta^*_{\text{SI}}(S^{-i})$. The following theorem gives a partial characterization of such updating systems.

THEOREM 3: *let $p(\cdot \mid \cdot)$ be a conditional probability system for i*

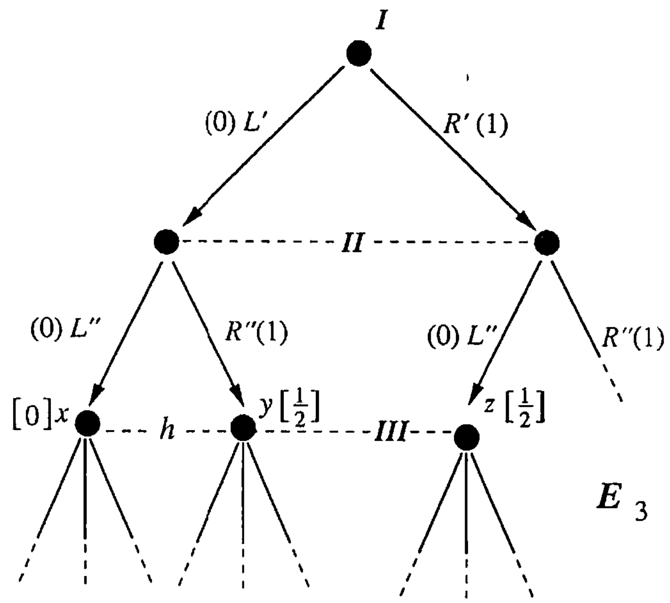

FIGURE D. An extensive form with non-observable deviators. The conditional probabilities at h cannot be derived by an uncorrelated conjecture.

and assume that the extensive form has observed deviators. If $p(\cdot\,|\,\cdot)\in\Delta^{*}_{SI}(S^{-i})$, then there exists an updating system of uncorrelated conjectures $c^{-i}[\cdot]$ consistent with $p(\cdot\,|\,\cdot)$ and such that for all g, $h\in H^{i}$ and all non-trivial partitions $\{J,K\}$ of $-i$:

$$\text{if } g<^{i}h \text{ and } c^{J}[g] \text{ is secure at } h, \text{ then } c^{J}[g]=c^{J}[h], \tag{10}$$

$$\text{if } S^{-i}(h)=X^{J}\times S^{K}(h) \text{ and } S^{-i}(g)=X^{J}\times S^{K}(g), \text{ then } c^{J}[h]=c^{J}[g]. \tag{11}$$

PROOF: see the Appendix.

The proviso in theorem 3 about the information structure is necessary. If an extensive form does not have observed deviators, then an updating system of uncorrelated conjectures consistent with $p(\cdot\,|\,\cdot)$ may not exist, even though $p(\cdot\,|\,\cdot)$ has the SI-property. This is shown in the following example (cf. Kreps & Ramey, 1987: figure 1).

EXAMPLE 3: consider the conditional probabilities in Figure D. They may be obtained taking the limit as $n\rightarrow\infty$ of the following sequence of product measures:

$$m_n(R',R'') = (1 - 1/n)^2,$$
$$m_n(L',R'') = m_n(R',L'') = (1/n)\,(1 - 1/n),$$
$$m_n(L',L'') = (1/n)^2.$$

By theorem 2 such conditional probabilities are part of a conditional probability system with the SI property. But there is no conjecture in $\Delta(S^I) \times \Delta(S^{II})$ which generates these conditional probabilities via Bayes' rule.

Example 3 shows that if one is willing to assume that conditional probabilities are always assessed using suitable uncorrelated conjectures and Bayes' rule, a stronger property for conditional probability systems is needed. But it is not at all clear why such a property is desirable. After all, the SI-property of conditional probability systems seems a natural translation into formal probabilistic language of the intuitive requirement that players' updated expectations should conform to the principle of strategic independence. According to this position, updating systems of uncorrelated conjectures are not well suited to analyse updated expectations. On the one hand, they are not restrictive enough in situations like those depicted in Figures A and B. Theorem 3 suggests that conditions (10) and (11) should be added to (1) and (2) in the definition of an admissible updating system.

On the other hand, example 3 shows that updating systems of uncorrelated conjectures can be too restrictive. Assume that in a game with the extensive form E_3 players I and II are symmetric. Then it is reasonable to allow player III to assign the same conditional probability to a deviation by player I only and to a deviation by player II only. If it is assumed that the conditional system of III has the SI-property and assigns prior probability 1 to (R',R''), then III must also assign zero conditional probability to the event that both players deviated from (R',R''), because the event of two (simultaneous) deviations is "infinitely less likely" than only one deviation (by either player).† A straightforward computation shows that, if the conditional probability is deduced from an uncorrelated prior, at least one of these two requirements, symmetry and SI, must be violated: let $\alpha = \text{Prob}(L') = \text{Prob}(L'') > 0$ according to a symmetric uncorrelated prior, then $\text{Prob}(x \mid h) = \alpha^2/(2\alpha - \alpha^2) > 0$, which violates SI.

† It must be proved that for all $p \in \Delta^*_{SI}(S^I \times S^{II})$.

$$p((R',R'') \mid S^I \times S^{II}) = 1 \Rightarrow p((L',L'') \mid \{(L',L''), (L',R''), (R',L'')\}) = 0.$$

Applying (5) and (3) one obtains

$$p((R',R'') \mid S^I \times S^{II}) = 1 \Rightarrow 0 = p(\{L'\} \times S^{II} \mid S^I \times S^{II}) = p((L',L'') \mid S^I \times L''\})$$
$$\Rightarrow p((L',L'') \mid \{(L',L''),(L',R''),(R',L'')\}) =$$
$$= p((L',L'') \mid S^I \times \{L''\})\, p(S^I \times \{L''\} \mid \{(L',L''),(L',R''),(R',L'')\}) = 0.$$

All this may seem surprising, but on closer inspection, such surprise is unjustified. The SC principle should not be concerned with independence. Therefore, it should be formulated through the definition of systems of conjectures $c^{-i}[h]\in\Delta(S^{-i})$ which need *not* be product measures. Theorem 1 says that there is a precise equivalence between updating systems of possibly correlated conjectures and conditional probability systems. The SI principle is formalized for conditional probability systems by the SI-property (equation (5)). This suggests that the updated expectations satisfying the joint SI–SC principle are exactly those given by conditional systems in $\Delta^*_{SI}(S^{-i})$.

Mathematical considerations go in the same direction. While theorem 2 shows that the set $\Delta^*_{SI}(S^{-i})$ is "well behaved", example 3 shows that the set of conditional probability systems which are consistent with some updating system of uncorrelated conjectures is not closed.

3. Concluding remarks

The analysis of Section 2 can be easily extended to games with chance moves. Let "Nature" be player 0. Assume that the probabilities of chance moves are given by a strictly positive behaviour strategy

$$\rho^0\in\Pi_{h\in H^0}\Delta(A(h)).$$

This behaviour strategy ρ^0 is realization-equivalent to a strictly positive mixed strategy $m^0\in\Delta(S^0)$, which generates a conditional probability system $p^0\in\Delta^*(S^0)$. In order to apply the analysis of Section 2 it is sufficient to treat Nature as a non-strategic player, with the additional assumption that the marginal on S^0 of each independent conditional system is precisely p^0.

Games of incomplete information can be represented as games of complete and imperfect information with chance moves. Hence, the previous trick applies to them as well. However, there is a more direct way to deal with games of incomplete information *with independent types*. Let T^i be player i's (finite) type space. Consider an "extended player i" with strategy space $\Sigma^i=(T^i\times S^i)$. Since i observes his or her own type, this fictitious extended player has perfect recall. Hence, the associated extensive form obtained in this way satisfies the hypotheses of Section 2. Consistency, strategic independence and "observed deviators" are defined with respect to the product space $\Pi_{i\in N}\Sigma^i$.

Note that the observed deviators condition is satisfied by all multistage games of incomplete information with observed actions. Most applications of game theory to information economics use games of this kind, which are analysed in depth by Fudenberg and Tirole (1991*a*,*b*: 8.2.3).

This note has provided a coherent formulation of the SI–SC principles. Their implications for the solution of extensive games are studied elsewhere (see Battigalli, 1993*a*; Kohlberg and Reny, 1992, on perfect Bayesian equilibria and Battigalli, 1993*b*; Reny, 1992, on extensive form rationalizability).

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References

Aumann, R. (1974). Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1, 67–96.

Aumann, R. (1987). Correlated equilibrium as an expression of Bayesian rationality. *Econometrica*, 55, 1–18.

Battigalli, P. (1993*a*). *Strategic independence and perfect Bayesian equilibria.* Rapporto interno 93-011, Dipartimento di Economica e Produzione, Politecnico di Milano, Milan, Italy.

Battigalli, P. (1993*b*). *Strategic rationality orderings and the best rationalization principle.* Rapporto interno 93-014, Dipartimento di Economiche e Produzione, Politecnico di Milano, Milan, Italy.

Battigalli, P. & Veronesi, P. (1992). *Stochastic independence without Savage-null events.* Unpublished paper, Istituto di Economia Politica, Università Bocconi, Milan, Italy.

Blume, L., Brandenburger, A. & Dekel, E. (1991). Lexicographic probabilities and choice under uncertainty. *Econometrica*, 59, 61–79.

Fudenberg, D. & Levine, D. (1993). Self-confirming equilibrium. *Econometrica*, 61, 523–545.

Fudenberg, D. & Tirole, J. (1991*a*). Perfect Bayesian equilibrium and sequential equilibrium. *Journal of Economic Theory*, 53, 326–260.

Fudenberg, D. & Tirole, J. (1991*b*). *Game Theory.* Cambridge. MA: MIT Press.

Fudenberg, D., Kreps, D. & Levine, D. (1988). On the robustness of equilibrium refinements. *Journal of Economic Theory*, 44, 354–380.

Hammond, P. (1987). *Extended probabilities for decision theory and games.* Unpublished paper, Department of Economics, Stanford University, CA.

Hammond, P. (1992). *Elementary non Archimedean representations of probability for decision theory and games.* Mimeo, Stanford University.

Hammond, P. (1994). Aspects of rationalizable behavior. In K. Binmore, A. Kirman & P. Tani, Eds. *Frontiers of Game Theory.* Cambridge, MA: MIT Press.

Kohlberg, E. & Reny, P. (1992). *On the rationale of perfect equilibrium.* Mimeo, Harvard University and University of Western Ontario.

Kreps, D. & Ramey, G. (1987). Structural consistency, consistency and sequential rationality. *Econometrica*, 55, 1331–1348.

Kreps, D. & Wilson, R. (1982). Sequential equilibria. *Econometrica*, 50, 863–894.

McLennan, A. (1989*a*). The space of conditional systems is a ball. *International Journal of Game Theory*, 18, 125–139.

McLennan, A. (1989*b*). Consistent conditional systems in noncooperative game theory. *International Journal of Game Theory*, 18, 141–174.

Myerson, R. (1986). Multistage games with communication. *Econometrica*, 54, 323–358.

Pearce, D. (1984). Rationalizable strategic behavior and the problem of perfection. *Econometrica*, 52, 1029–1050.

Reny, P. (1992). Backward induction, normal-form perfection and explicable equilibria. *Econometrica*, 60, 626–649.

Rényi, A. (1955). On a new axiomatic theory of probability. *Acta Mathematica Academiae Scientiarum Hungaricae*, 6, 285–335.

Rényi, A. (1956). On conditional probability spaces generated by a dimensionally ordered set of measures. *Theory of Probability and its Applications*, 1, 61–71.

Appendix

PROOF OF LEMMA 1: assume (a), i.e. that the extensive form has observed deviators. Then, by definition, for all $h\in H$, $S(h)=\Pi_{k\in N} S^k(h)$. Take any $i\in N$, $h\in H^i$, $J\subseteq -i$. It is sufficient to prove that if c^J reaches h, then c^J is secure at h. Let c^J reach h. Then there is some $s^J\in S^J(h)\bigcap \mathrm{Supp}(c^J)$. Decompose $S(h)$ as $S(h)=S^J(h)\times S^{N\setminus J}(h)$. Since for all $s^{N\setminus J}\in S^{N\setminus J}(h)$

$$P(h;c^J,s^{N\setminus J})=\sum_{t^J\in S^J(h)\bigcap \mathrm{Supp}(c^J)} c^J(t^J) \tag{A1}$$

and $s^J\in S^J(h)\bigcap\mathrm{Supp}(c^J)$, it follows that $P(h;c^J,s^{N\setminus J})>0$. Then c^J is secure at h and (b) holds.

Now assume that (a) does not hold, i.e. deviators are *not* (always) observed. We must show that (b) does not hold. By assumption, there are $i\in N$, $h\in H^i$, $J\subseteq -i$, $s^J\in S^J(h)$, $s^{N\setminus J}\in S^{N\setminus J}(h)$ such that $(s^J,s^{N\setminus J})\notin S(h)$. Take a c^J such that $c^J(s^J)=\Pi_{j\in J}c^j(s^j)=1$. c^J reaches h (because so does s^J), but obviously $P(h;c^J,s^{N\setminus J})=0$. Hence, c^J is not secure at h and (b) of lemma 1 does not hold. ■

PROOF OF THEOREM 1, SECOND PART: it is well known that, for each finite set Ω with cardinality $\#\Omega=L$, there is a canonical one-to-one correspondence between $\Delta^*(\Omega)$ and the following set of "full support" lexicographic conditional probability systems (LCPS):

$$\bigcup_{\ell=0}^{\ell=L}\{(p_0,p_1,\ldots,p_\ell)\in[\Delta(\Omega)]^{\ell+1}$$

$$\mathrm{Supp}(p_j)\bigcap\mathrm{Supp}(p_k)=\varnothing,\ \text{if } j\neq k;\ \bigcup_{k=0}^{k=\ell}\mathrm{Supp}(p_k)=\Omega\},$$

where the correspondence is given by

$$p(A\mid B)=p_{k(B)}(A\bigcap B)/p_{k(B)}(B),\ k(B)=\min\{k\mid p_k(B)>0\} \tag{A2}$$

(see, e.g., Hammond, 1992). Here it is convenient to use this lexicographic representation of conditional systems.

Fix a player i and an updating system of conjectures $c^{-i}[\cdot]$. Let $\mu(h)$ be the number of predecessors of $h\in\{\emptyset\}\bigcup H^i$ in the tree $(\{\emptyset\}\bigcup H^i,<^i)$, i.e. $\mu(h)=\#\{g\in\{\emptyset\}\bigcup H^i\,|\,g<^i h\}$ (clearly $\mu(\emptyset)=0$). Let $\nu:\{1,\ldots,\#H^i\}\to H^i$ be a one-to-one function such that $\mu(g)<\mu(h)$ implies $\nu^{-1}(g)<\nu^{-1}(h)$, i.e. ν^{-1} is a linear order on H^i refining the partial order given by μ, which in turn refines the partial order given by $<^i$, $\nu(k)$ is the information set with index k. For all $h\in H^i$, let $\pi(h)=\max_{<^i}\{g\,|\,g<^i h\}$ denote the immediate predecessor of h in $H^i\bigcup\{\emptyset\}$.

Define an array of real valued functions p_k with domain S^{-i} as follows:

$$S_0^{-i}=\mathrm{Supp}(c^{-i}[\emptyset]),$$

$$S_k^{-i}=S^{-i}(\nu(k))\bigcap\mathrm{Supp}(c^{-i}[\nu(k)]\backslash\mathrm{Supp}(c^{-i}[\pi(\nu(k)]),\ k=1,\ldots,\#H^i$$

$$S^{-i_i}_{\#H+1}=S^{-i}\backslash\bigcup_{k=0}^{\#H^i}S_k^{-i};$$

delete all the indexes k such that $S_k^{-i}=\emptyset$ and let

$$p_0(\cdot)=c^{-i}[\emptyset](\cdot),$$
$$p_k(s^{-i})=\frac{c^{-i}[\nu(k)](s^{-i})}{c^{-i}[\nu(k)](S_k^{-i})}\text{ for }k=1,\ldots,H^i,S_k^{-i}\neq\emptyset,$$
$$p_{\#H^i+1}=(\#S^{-i_i}_{\#H+1})^{-1}$$

By construction, $p_k\in\Delta(S^{-i})$ for all the undeleted k and $S^{-i}=\bigcup_k S_k$. If $j\neq k$ and the information sets $\nu(j)$ and $\nu(k)$ are unrelated by $<^i$, then $S^{-i}(\nu(j))\bigcap S^{-i}(\nu(k))=\emptyset$, which implies $S_j^{-i}\bigcap S_k^{-i}=\emptyset$. If $j\neq k$ and $\nu(j)=\pi(\nu(k))$ (recall that $\pi(h)$ is the immediate $<^i$-predecessor of h), there are two possible cases:

(a) $\mathrm{Supp}(c^{-i}[\nu(j)])\bigcap S^{-i}(\nu(k))=\emptyset$, or
(b) $\mathrm{Supp}(c^{-i}[\nu(j)])\bigcap S^{-i}(\nu(k))\neq\emptyset$.

It follows immediately that $S_j^{-i}\bigcap S_k^{-i}=\emptyset$ in case (a). In case (b), $c^{-i}[\nu(j)]=c^{-i}[\nu(k)]$ by equation (2). Thus, $S_j^{-i}\bigcap S_k^{-i}=\emptyset$ in case (b) also. This argument can be easily extended to all (j,k) such that $\nu(j)<^i\nu(k)$ by induction on $\mu(k)-\mu(j)$. This shows that $(p_k;\ S_k^{-i}\neq\emptyset)$ is a well-defined LCPS. It can be checked that c^{-i} is consistent with the conditional probability system $p(\cdot\,|\,\cdot)$ derived from $(p_k;\ S_k^{-i}\neq\emptyset)$ according to equation (A2).

Take $p \in \Delta^*(S^{-i})$. It must be shown that p is consistent with some $c^{-i}[\cdot]$. Consider the LCPS $(p_0, \ldots, p_L)$ corresponding to p and construct $c^{-i}[\cdot\ ]$ as follows: for all $h \in \{\emptyset\} \bigcup H^i$, let

$$\ell(h) = \min\{\ell \mid S^{-i}(h) \bigcap \mathrm{Supp}(p_\ell) \neq \emptyset\}.$$

$$c^{-i}[\emptyset] = p_0,$$

$$c^{-i}[h] = p_{\ell(h)}(\cdot), \text{ if } \ell(\pi(h)) < \ell(h),$$

$$c^{-i}[h] = c^{-i}[g], \text{ if } \ell(\pi(h)) = \ell(h).$$

Note that if $g = \pi(h)$, then $\ell(g) \leq \ell(h)$, because $S^{-i}(h) \subseteq S^{-i}(g)$. Therefore, $c^{-i}[h]$ is defined for all information sets of i. Clearly, $c^{-i}[\cdot]$ satisfies condition (1). Condition (2) is satisfied whenever $g <^i h$ is the immediate predecessor of h, i.e. $g = \pi(h)$. An easy induction argument extends this result to the general case $g <^i h$. Therefore, $c^{-i}[\cdot]$ is a well-defined updating system of conjectures. ■

Proof of theorem 3: assume that $p \in \Delta^*_{\S I}(S^{-i})$ and let p^j be the marginal of p on S^j $(j \neq i)$. For each opponent j define the mapping $c^j[\cdot]: H^i \bigcup \{\emptyset\} \to \Delta(S^j)$ in the same way as in the last part of the proof of theorem 1, replacing $-i$ with j. Let $c^{-i}[h](s^{-i}) = \Pi_{j \neq i} c^j[h](s^j)$. By construction, $c^j[h]$ reaches h and whenever $g <^i h$ and $c^j[g]$ reaches h, then $c^j[h] = c^j[g]$. Since the extensive form has observed deviators, if $c^j[h]$ reaches h, then $c^j[h]$ is secure at h (lemma 1). Therefore, $c^{-i}[h]$ reaches h and $c^{-i}[\cdot]$ is a well-defined updating system of correlated conjectures. By theorem 2 $c^{-i}[\cdot]$ is consistent with p. Equations (10) and (11) hold by construction. ■