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NOTES AND COMMENTS

A NOTE ON COMPARATIVE AMBIGUITY AVERSION AND JUSTIFIABILITY

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We consider a decision maker who ranks actions according to the smooth ambiguity criterion of Klibanoff, Marinacci, and Mukerji (2005). An action is justifiable if it is a best reply to some belief over probabilistic models. We show that higher ambiguity aversion expands the set of justifiable actions. A similar result holds for risk aversion. Our results follow from a generalization of the duality lemma of Wald (1949) and Pearce (1984).

KEYWORDS: Comparative ambiguity aversion, justifiability, rationalizability.

1. INTRODUCTION

IN THIS NOTE, we consider a decision maker (DM) who ranks alternatives under uncertainty. The DM holds subjective beliefs over a set of probabilistic models $\Sigma \subseteq \Delta(S)$, where S is a set of states of nature. We assume that the DM ranks actions according to the smooth ambiguity criterion of Klibanoff, Marinacci, and Mukerji (2005). With this, we show that higher ambiguity aversion expands the set of actions that are best replies to at least one belief; for brevity, we call such actions "justifiable." Empirically, they are the actions that an outside observer can infer as possible from the knowledge of the DM attitudes toward uncertainty. Our result shows that such an inference becomes coarser as ambiguity aversion increases. We derive our result from a generalization of the duality lemma of Wald (1949) and Pearce (1984) which should be of independent interest.

Another consequence of this duality lemma is that, under ambiguity neutrality, higher risk aversion expands the set of justifiable actions. This risk version of our result was independently obtained by Weinstein (2016) for subjective expected utility maximizers in finite games (see Section 4 for a detailed discussion).[2] For expositional purposes and to exploit economies of scope, we present the results regarding comparative risk aversion and comparative ambiguity aversion jointly.

At first sight, the result might be counterintuitive. Indeed, if the DM deems possible very different probabilistic models, then higher ambiguity aversion

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increases the attractiveness of "safe" actions whose objective expected utility is somewhat low for each model, but does not change much with the model. Given the *same* belief over probabilistic models, actions that give high expected utility for some models and low expected utility for other models become instead *less* attractive. Yet, an increase in ambiguity aversion cannot make such actions unjustifiable, because—regardless of ambiguity attitudes—they can always be justified by extreme beliefs assigning high probability to models under which they yield high objective expected utility.

This comparative statics result is analogous to another result of ours, which also relies on the smooth ambiguity criterion: higher ambiguity aversion expands the set of self-confirming equilibria (Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2015)).[3] However, as argued in Section 4, the similarity between these results is only superficial, because they rely on different assumptions about the decision or game problem and have very different explanations.

The rest of the note is structured as follows. Section 2 presents the decision criterion we use. Section 3 contains our main results. Our findings are discussed in Section 4 where we also briefly discuss alternative decision models. All proofs are relegated to Appendix 4, where we state and prove the abstract version of the duality lemma of Wald (1949) and Pearce (1984) which underlies our analysis.

2. CRITERION

We consider a standard decision problem under uncertainty with action space A, state space S, and payoff function $r : A \times S \to \mathbb{R}$. We assume that A and S are separable metric spaces and r is continuous in each component and bounded. The payoff function may be interpreted as the composition of a consequence function, or game form, $g : A \times S \to C$, where C is the consequence space, and a von Neumann–Morgenstern utility function $u : C \to \mathbb{R}$, that is, $r = u \circ g$. For interpretational and expositional purposes, we assume that consequences are monetary, that is, $C \subseteq \mathbb{R}$.

Let Σ be a nonempty closed subset of the collection $\Delta(S)$ of all Borel probability measures σ on the state space, each σ being interpreted as a possible stochastic model for states.[4] Actions are ranked by the smooth ambiguity criterion $V_{\phi,r} : A \times \Delta(\Sigma) \to \mathbb{R}$ given by

$$V_{\phi,r}(a, \mu) = \phi^{-1}\left(\int_{\Sigma} \phi\left(\int_S r(a, s)\sigma(\mathrm{d}s)\right)\mu(\mathrm{d}\sigma)\right),$$

[3]In the working paper version of this manuscript, we also show that higher ambiguity aversion expands the set of rationalizable actions of a game, when the rationalizability concept is modified to take into account ambiguity attitudes.

[4]In this presentation of the results, to fix ideas the reader can think of all sets defining the decision problem as finite. In Appendix 4, we prove our results in the general setting.

where $\phi : \overline{\text{co}}\,\text{Im}\, r \to \mathbb{R}$ is strictly increasing and continuous, and μ is a subjective probability measure on the posited set of stochastic models Σ.[5] Function ϕ is also known as the *second-order utility* because it can be interpreted as the "utility of objective expected utility." When ϕ is the identity, the criterion reduces to standard subjective expected utility, that is,

$$V_{\text{Id},r}(a,\mu) = \int_\Sigma \int_S r(a,s)\sigma(\mathrm{d}s)\mu(\mathrm{d}\sigma) = \int_S r(a,s)\sigma_\mu(\mathrm{d}s),$$

where, for any (measurable) event $E \subseteq S$, $\sigma_\mu(E) = \int_\Sigma \sigma(E)\mu(\mathrm{d}\sigma)$ is the predictive probability of E induced by μ. Function ϕ captures the DM's attitudes toward ambiguity, whereas $r = u \circ g$ captures attitudes toward risk.

3. MAIN RESULTS

3.1. *Justifiability*

DEFINITION 1: The collection of *justifiable actions* for ambiguity attitudes ϕ and risk attitudes r given Σ is

$$\mathcal{J}_{\phi,r}(\Sigma) = \{a \in A : \exists \mu \in \Delta(\Sigma), \forall a' \in A, V_{\phi,r}(a,\mu) \geq V_{\phi,r}(a',\mu)\}.$$

In words, $\mathcal{J}_{\phi,r}(\Sigma)$ is the collection of all actions that are best replies, according to $V_{\phi,r}$, to some belief μ over Σ.[6]

3.2. *Risk Attitudes*

We first consider higher risk aversion in the subjective expected utility case. In our monetary setup, $r' = \psi \circ r = (\psi \circ u) \circ g$, with ψ concave, continuous, and strictly increasing, is the payoff function of a more risk averse DM. The following proposition says that, assuming ambiguity neutrality ($\phi = \text{Id}$), a more risk averse DM has more justifiable actions. We denote by δ_s the Dirac probability measure supported by state s.

PROPOSITION 1: *Let S be compact and $\{\delta_s\}_{s\in S} \subseteq \Sigma$. If $r' = \psi \circ r$ for some concave, continuous, and strictly increasing function $\psi : \overline{\text{co}}\,\text{Im}\, r \to \mathbb{R}$, then $\mathcal{J}_{\text{Id},r}(\Sigma) \subseteq \mathcal{J}_{\text{Id},r'}(\Sigma)$.*

[5] Here $\overline{\text{co}}\,\text{Im}\, r$ is the smallest closed interval that contains the image $\text{Im}\, r$ of the payoff function r.

[6] The terminology is inspired by Milgrom and Roberts (1991). Lehrer and Teper (2011) have introduced a new class of "justifiable preferences" under uncertainty. The connection with our notion of justifiability is, however, limited: ours is just the old best-reply-to-some-belief concept, here applied to the smooth ambiguity model.

EXAMPLE 1: Consider the following game form with monetary consequences:

(1)

g:	s'	s''
t	0	1
m	$\frac{1}{3}$	$\frac{1}{3}$
b	1	0

Suppose the DM is a subjective expected utility maximizer ($\phi = \mathrm{Id}$). If the DM is risk neutral ($r = g$), action m is unjustifiable: for every belief $\mu \in \Delta(\Sigma)$,

$$\begin{aligned} V_{\mathrm{Id},g}(m,\mu) &= \frac{1}{3} < \frac{1}{2} \leq \max\{\sigma_\mu(s'), 1 - \sigma_\mu(s')\} \\ &= \max\{V_{\mathrm{Id},g}(b,\mu), V_{\mathrm{Id},g}(t,\mu)\}. \end{aligned}$$

If Σ contains the two Dirac measures $\delta_{s'}$ and $\delta_{s''}$, that is, S is embedded in Σ, then $\mathcal{J}_{\mathrm{Id},g}(\Sigma) = \{t, b\}$. In particular b (resp., t) is a best reply to μ if and only if $\sigma_\mu(s') \geq 1/2$ (resp., $\sigma_\mu(s') \leq 1/2$). Now suppose that the DM is risk averse, with a power utility function $u_\theta(c) = c^{1/\theta}$ (where $\theta \geq 1$ parameterizes risk aversion). Then, the payoff function is $r_\theta = u_\theta \circ g$ and

$$\mathcal{J}_{\mathrm{Id},r_\theta}(\Sigma) = \begin{cases} \{t, b\}, & \theta < \bar{\theta}, \\ \{t, m, b\}, & \theta \geq \bar{\theta}, \end{cases}$$

where $\bar{\theta} = \log_2 3$ solves $u_\theta(g(m)) = 1/2$. The collection of justifiable actions thus expands as θ increases. Note, however, that the sets of beliefs justifying the risky actions b and t shrink as soon as θ increases above the threshold.[7] Also note that the assumption $\{\delta_s\}_{s\in S} \subseteq \Sigma$ is needed. Otherwise, if $\delta_s \notin \Sigma$ (compact) for some s, either $V_{\mathrm{Id},r_\theta}(b,\mu) = \sigma_\mu(s')$ or $V_{\mathrm{Id},r_\theta}(t,\mu) = \sigma_\mu(s'')$ is below 1 whereas $V_{\mathrm{Id},r_\theta}(m,\mu) = (1/3)^{1/\theta} \to 1$ as $\theta \to +\infty$.

3.3. *Ambiguity Attitudes*

Next we consider a change in ambiguity attitudes. The following proposition says that a more ambiguity averse DM has a larger set of justifiable actions. As argued in the Introduction, the result is not intuitively obvious. Note that the hypothesis of Σ being compact is weaker than the hypothesis, made in the previous proposition, of S being compact.

PROPOSITION 2: *Let Σ be compact. If $\phi' = \varphi \circ \phi$ for some concave, continuous, and strictly increasing function $\varphi : \overline{\mathrm{co}}\,\mathrm{Im}\,\phi \to \mathbb{R}$, then $\mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\phi',r}(\Sigma)$.*

[7] We comment on this matter in more detail in the next example, which deals with the analogous case of an increase in ambiguity aversion.

EXAMPLE 2: Consider again the game form (1) and suppose, just for simplicity, that the DM is risk neutral, that is, $r = g$, and $\Sigma = \{\delta_{s'}, \delta_{s''}\}$. Let $\phi_\theta(x) = x^{1/\theta}$, where $\theta \geq 1$ parameterizes ambiguity aversion. Then, it can be shown that the belief μ that maximizes $V_{\phi_\theta,g}(m, \mu) - \max\{V_{\phi_\theta,g}(t, \mu), V_{\phi_\theta,g}(b, \mu)\}$ satisfies $\mu(\delta_{s'}) = \mu(\delta_{s''}) = 1/2$ (cf. Battigalli et al. (2015, Lemma 6)). With this, calculations similar to those of Example 1 yield

$$\mathcal{J}_{\phi_\theta,r}(\Sigma) = \begin{cases} \{t, b\}, & \theta < \bar{\theta}, \\ \{t, m, b\}, & \theta \geq \bar{\theta}, \end{cases}$$

where $\bar{\theta} = \log_2 3$ solves $\phi_\theta(g(m)) = 1/2$. The collection of justifiable actions thus expands as θ increases. Note, however, that the sets of beliefs justifying ambiguous actions b and t shrink: In fact, $V_{\phi_\theta,g}(m, \mu) = 1/3$ regardless of θ, whereas $V_{\phi_\theta,g}(b, \mu) = (\mu(\delta_{s'}))^\theta$ and $V_{\phi_\theta,g}(t, \mu) = (\mu(\delta_{s''}))^\theta$ are strictly decreasing in θ if $0 < \mu(\delta_{s'}), \mu(\delta_{s''}) < 1$; as θ increases above the threshold $\bar{\theta}$, the probability $\mu(\delta_{s'})$ must increase to make b a best reply, and similarly for t. See Figure 1: On the horizontal (resp., vertical) axis we report the second-order utility of objective expected utility given model $\sigma^1 = \delta_{s'}$ (resp., $\sigma^2 = \delta_{s''}$). As ambiguity aversion increases, the expected utility vector corresponding to action m shifts north-eastward.

To sum up, higher aversion to either ambiguity or risk (under ambiguity neutrality) expands the collection of justifiable actions. As for the set of beliefs justifying any action, we can only say that, if it is not empty, an increase in risk or ambiguity aversion cannot make it empty. Propositions 1 and 2 are purely comparative results which do not require either risk or ambiguity aversion, that is, the functions u and ϕ are not assumed to be concave.

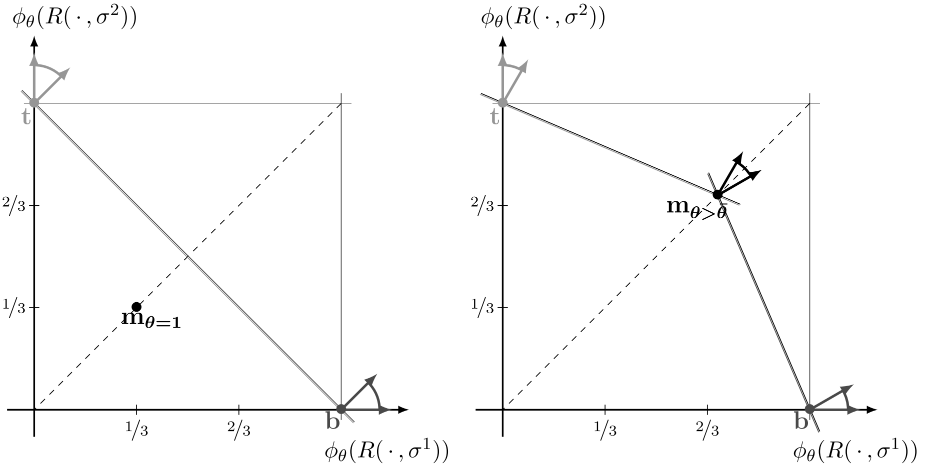

FIGURE 1.—As θ increases, the sets of beliefs justifying b and t shrink.

The proof is based on an abstract version of the duality lemma of Pearce (cf. Pearce (1984, Lemma 3)) presented in Appendix 4, which is a version of the classic Complete Class Theorem of Wald (see, e.g., Wald (1949, Theorem 2.2)).

4. DISCUSSION

We conclude by discussing the related literature and some aspects of our work.

A Superficial Analogy. First we compare our result with Battigalli et al. (2015) and explain the difference. In that paper, we proved that higher ambiguity aversion expands the set of self-confirming equilibria, a steady-state phenomenon resulting from the strong discipline on beliefs that the notion of self-confirming equilibrium imposes by requiring their consistency with the long-run data that agents observe in recurrent interaction. Specifically, assuming that each agent observes at least his realized payoff in each play, self-confirming equilibrium actions are perceived as unambiguous best replies by players, whereas unused alternatives are typically perceived as ambiguous. Therefore, holding beliefs fixed, an increase in ambiguity aversion leaves the value of self-confirming actions unaltered, but decreases the value of unused alternatives. This implies that, for each equilibrium action, the set of confirmed beliefs justifying it *expands* as ambiguity aversion increases. All this stands in sharp contrast with the justifiability result of the present work: Here we are not trying to characterize steady-state actions; hence, feedback is irrelevant and beliefs are not restricted by experience (although, in games, rationalizable beliefs are restricted by strategic thinking). Therefore, a justifiable action may well be perceived as ambiguous. In this case, as ambiguity aversion increases, the set of beliefs justifying this action typically *shrinks*, as demonstrated by our examples.

Criterion. As explained above, our work builds on the choice model of Klibanoff, Marinacci, and Mukerji (2005) and the duality lemma of Wald (1949) reintroduced into game theory by Pearce (1984). We use the smooth ambiguity model for two reasons. (i) It is portable, that is, it parameterizes personality traits that agents are supposed to exhibit in any decision problem: risk attitudes given by the von Neumann–Morgenstern utility function u and ambiguity attitudes given by the second-order utility function ϕ. Such personality traits can be assumed to be constant across decision, or game, situations; state spaces and beliefs, on the other hand, change according to the situation. (ii) Under this model, an increase in ambiguity aversion is represented by a concave strictly increasing transformation of ϕ, which by a fortuitous coincidence (see Remark 1 in Appendix 4) allows us to rely on the general version of the duality lemma.

As for (i), we could think of alternative models sharing the same properties of portability (see, e.g., eq. (13) of Battigalli et al. (2015)). However, we feel that the smooth ambiguity model—among the known models of decision making under ambiguity—is the one where these features are most evident. As for the possibility to extend our comparative statics result (point ii), it may be natural to consider the class of preferences that can be represented by quasiconcave utility functionals on $\mathbb{R}^S$. Yet, such an extension does not hold. We can see this by comparing justifiability under the smooth ambiguity criterion with justifiability under the maxmin criterion. Indeed, as ambiguity aversion becomes higher and higher, that is, as $-\phi''/\phi' \uparrow +\infty$, we have larger and larger collections $\mathcal{J}_{\phi,r}(\Sigma)$ of justifiable actions and it is natural to wonder how this property relates to the fact that the $V_{\phi,r}(a,\mu)$ criterion tends to the maxmin criterion $V_{\infty,r}(a,\mu) = \min_{\sigma\in\operatorname{supp}\mu}\int_S r(a,s)\sigma(\mathrm{d}s)$, which is a version of the classic criterion of Gilboa and Schmeidler (1989).[8] At the same time, the $V_{\infty,r}$ criterion belongs to the aforementioned class of "quasiconcave" preferences and is more ambiguity averse than $V_{\phi,r}$ for every concave ϕ. Nevertheless, we can construct an example where an action (h below) is justifiable for every criterion $V_{\phi,r}$, but not for the more ambiguity averse criterion $V_{\infty,r}$.

EXAMPLE 3: Consider the payoff function:

r:	s'	s''
t	0	1
h	c'	c''
m	ε	ε
b	1	0

Assume $\Sigma = \{\delta_{s'}, \delta_{s''}\}$ and set $c' = 2/7$, $\varepsilon = 1/3$, and $c'' = 6/7$. On the one hand, if the DM is ambiguity neutral, $\phi = \mathrm{Id}$, action h is justifiable (according to the uniform belief). Hence, $h \in \mathcal{J}_{\mathrm{Id},r}(\Sigma)$ and, by Proposition 2, $h \in \mathcal{J}_{\phi,r}(\Sigma)$ for all concave $\phi \in \Phi$, where Φ is the collection of continuous and strictly increasing real-valued functions on $\overline{\mathrm{co}}\,\mathrm{Im}\,r$. On the other hand, one can show that $h \notin \mathcal{J}_{\infty,r}(\Sigma)$.[9] Alternatively, if $c' = \varepsilon = 1/3 < c'' < 1$, it is easy to check that $m \in \mathcal{J}_{\infty,r}(\Sigma)$, but $m \notin \bigcup_{\phi\in\Phi}\mathcal{J}_{\phi,r}(\Sigma)$.

To sum up, the previous example shows that there are actions that are justifiable under all smooth ambiguity criteria with concave $\phi \in \Phi$ but not under the maxmin criterion, as well as actions that are justifiable under the maxmin criterion $V_{\infty,r}$ but not under any smooth ambiguity criterion $V_{\phi,r}$ with $\phi \in \Phi$.[10] This implies that (1) our comparative result does not extend to all "quasiconcave"

[8]See Klibanoff, Marinacci, and Mukerji (2005, p. 1867).

[9]The set $\mathcal{J}_{\infty,r}(\Sigma)$ is defined as $\mathcal{J}_{\phi,r}(\Sigma)$ just by replacing $V_{\phi,r}$ with $V_{\infty,r}$. The failure of the inclusion $\bigcup_{\phi\in\Phi}\mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\infty,r}(\Sigma)$ further exemplifies how feeble is the connection with the comparative result of Battigalli et al. (2015), where the set of smooth self-confirming equilibria is included in the set of maxmin ones.

[10]For a more thorough discussion of the maxmin case, we refer the interested reader to the working paper version of this manuscript.

preferences, and (2) the justifiability correspondence $\phi \mapsto \mathcal{J}_{\phi,r}(\Sigma)$ is neither upper nor lower hemicontinuous at "infinite" ambiguity aversion.

Weinstein. Our Proposition 1 implies, by means of an induction argument, that the set of normal-form rationalizable strategy profiles of a game with ambiguity neutral players expands as risk aversion increases. This is also the content of Proposition 1 in Weinstein (2016), the work most related to ours.[11] On the one hand, our note can be seen as a generalization of this (independent) result about comparative risk aversion and rationalizability. It is a generalization because Weinstein considered finite games, while we allow for a continuum of actions and states. It is also an extension because we allow for non-neutral ambiguity attitudes and study the effect of changes in ambiguity aversion. On the other hand, Weinstein (2016) also studied other solution concepts as well as the limits of rationalizability and other solutions as risk aversion (or love) goes to infinity. His analysis can be adapted to our uncertainty setup to obtain the limit of the set of justifiable actions as ambiguity aversion (or love) goes to infinity, under the assumption that $A \times \Sigma$ is finite.[12]

Other Papers. Other papers in the literature analyzed notions of justifiability or rationalizability with non-neutral attitudes toward ambiguity. Ghirardato and Le Breton (2000) characterized actions that are best replies to some possibly non-additive belief under the Choquet expected utility criterion of Schmeidler (1989). Epstein (1997) analyzed rationalizability under several criteria, including the Choquet criterion of Schmeidler (1989) and the maxmin criterion of Gilboa and Schmeidler (1989). To the best of our knowledge, ours is the first work reporting a result on comparative ambiguity aversion and justifiability.

APPENDIX: PROOFS AND RELATED MATERIAL

A.1. *Abstract Pearce–Wald Lemma*

Fix two nonempty subsets A_1 and A_2 of a Hausdorff locally convex topological vector space. Let $B_i = \operatorname{co} A_i$ and $\bar{B}_i = \overline{\operatorname{co}} A_i$ denote respectively the convex hull of A_i and its closure, for $i = 1, 2$.

Given $F : B_1 \times \bar{B}_2 \to \mathbb{R}$, we say that $a_1^* \in A_1$ is *dominated* if and only if

$$\exists a_1 \in A_1, \forall a_2 \in A_2, \quad F\big(a_1^*, a_2\big) < F(a_1, a_2),$$

otherwise we say that a_1^* is *undominated*; we say that a_1^* is *co-dominated* if and only if

$$\exists b_1 \in B_1, \forall a_2 \in A_2, \quad F\big(a_1^*, a_2\big) < F(b_1, a_2),$$

otherwise we say that a_1^* is *co-undominated*.

[11] Weinstein (2016) expands on an earlier version dated 2013.

[12] See Section 5.2 of the working paper version of our manuscript.

LEMMA 1: *Suppose that*:
(i) A_2 *is closed and* $\bar{B}_2$ *is compact*;
(ii) F *is quasiconcave and upper semicontinuous on* B_1;
(iii) F *is affine and continuous on* $\bar{B}_2$.
An element $a_1^* \in A_1$ *is co-undominated only if there exists some* $b_2 \in \bar{B}_2$ *such that* $a_1^* \in \arg\max_{a_1 \in A_1} F(a_1, b_2)$. *The converse is true if* F *is affine on* B_1.

Of course, condition (i) implies that A_2 (a subset of $\bar{B}_2$) is compact. In many examples, condition (i) is equivalent to the compactness of A_2.[13] Condition (ii) is satisfied if F is *concave* and upper semicontinuous on B_1.

PROOF OF LEMMA 1: First note that, since A_2 is compact, there exists a function $\bar{\iota}_2 : \bar{B}_2 \to \Delta(A_2)$ (the set $\Delta(A_2)$ here denotes the set of all regular Borel probability measures) such that

$$\phi(b_2) = \int_{A_2} \phi(a_2)\bar{\iota}_2(b_2)(\mathrm{d}a_2)$$

for all continuous and affine $\phi : \bar{B}_2 \to \mathbb{R}$.[14] Moreover, by definition of convex hull, there is a function $\iota_1 : B_1 \to \Delta_0(A_1)$ (the set $\Delta_0(A_1)$ here denotes the set of convex linear combinations of Dirac measures) such that $b_1 = \sum_{a_1 \in A_1} a_1 \iota_1(b_1)(a_1)$.

(Only if) Suppose that a_1^* is co-undominated. We must show that there exists $b_2^* \in \bar{B}_2$ such that $F(a_1^*, b_2^*) \geq F(a_1, b_2^*)$ for all $a_1 \in A_1$. Define the function $h : B_1 \times \bar{B}_2 \to \mathbb{R}$ by

$$h(b_1, b_2) = F\big(a_1^*, b_2\big) - F(b_1, b_2).$$

Since a_1^* is co-undominated, for each $b_1 \in B_1$ there exists $a_2^{b_1} \in A_2$ such that $F(a_1^*, a_2^{b_1}) \geq F(b_1, a_2^{b_1})$, that is, $h(b_1, a_2^{b_1}) \geq 0$. We can conclude that

$$\forall b_1 \in B_1, \quad \max_{b_2 \in \bar{B}_2} h(b_1, b_2) \geq h\big(b_1, a_2^{b_1}\big) \geq 0.$$

In turn, this yields $\inf_{b_1 \in B_1} \max_{b_2 \in \bar{B}_2} h(b_1, b_2) \geq 0$. Given the properties of F, the function h satisfies all the assumptions of the Sion Minimax Theorem (cf. Sion (1958, Corollary 3.3)), namely, h is quasiconvex and lower semicontinuous on B_1, and it is affine and continuous on $\bar{B}_2$. This implies that

$$\max_{b_2 \in \bar{B}_2} \inf_{b_1 \in B_1} h(b_1, b_2) = \inf_{b_1 \in B_1} \max_{b_2 \in \bar{B}_2} h(b_1, b_2) \geq 0.$$

[13] Consider, for example, quasi-complete locally convex topological vector spaces (see Holmes (1975, p. 61)).

[14] See Propositions 1.2 and 4.5 of Phelps (1966). For later use, we find it convenient to denote this map by $\bar{\iota}_2$.

By choosing $b_2^* \in \arg\max_{b_2 \in \bar{B}_2}(\inf_{b_1 \in B_1} h(b_1, b_2))$, we have that

$$0 \leq \inf_{b_1 \in B_1} h(b_1, b_2^*) = \inf_{b_1 \in B_1} \big(F(a_1^*, b_2^*) - F(b_1, b_2^*)\big),$$

thus, $F(a_1^*, b_2^*) \geq F(a_1, b_2^*)$ for each $a_1 \in A_1$.

(If) Suppose that F is also affine on B_1. By way of contraposition, suppose that a_1^* is co-dominated, that is, there exists $b_1 \in B_1$ such that

$$\forall a_2 \in A_2, \quad F(a_1^*, a_2) < F(b_1, a_2) = \sum_{a_1 \in A_1} F(a_1, a_2)\iota_1(b_1)(a_1).$$

We must show that for each $b_2 \in \bar{B}_2$ there exists some $a_1^{b_2} \in A_1$ such that $F(a_1^{b_2}, b_2) > F(a_1^*, b_2)$. Fix $b_2 \in \bar{B}_2$ arbitrarily. Since a_1^* is co-dominated by b_1, integrating over A_2 and by using the maps ι_1 and $\bar{\iota}_2$, we obtain that

$$\begin{aligned}
F(a_1^*, b_2) &= \int_{A_2} F(a_1^*, a_2)\bar{\iota}_2(b_2)(\mathrm{d}a_2) \\
&< \int_{A_2} \Big(\sum_{a_1 \in A_1} F(a_1, a_2)\iota_1(b_1)(a_1)\Big)\bar{\iota}_2(b_2)(\mathrm{d}a_2) \\
&= \sum_{a_1 \in A_1} \Big(\int_{A_2} F(a_1, a_2)\bar{\iota}_2(b_2)(\mathrm{d}a_2)\Big)\iota_1(b_1)(a_1) \\
&= \sum_{a_1 \in A_1} F(a_1, b_2)\iota_1(b_1)(a_1).
\end{aligned}$$

If $a_1^{b_2} \in \arg\max_{a_1 \in \operatorname{supp} \iota_1(b_1)} F(a_1, b_2)$, then

$$F(a_1^{b_2}, b_2) \geq \sum_{a_1 \in A_1} F(a_1, b_2)\iota_1(b_1)(a_1) > F(a_1^*, b_2).$$

Q.E.D.

A.2. *Randomization*

Now let A_1 and A_2 be two separable metric spaces. Denote by $\Delta_0(A_i)$ the collection of all Borel probability measures with finite support, and by $\Delta(A_i)$ the collection of all Borel probability measures on A_i. Given a function $f : A_1 \times A_2 \to \mathbb{R}$, say that $a_1^* \in A_1$ is *dominated under randomization* if and only if

$$\exists \beta_1 \in \Delta_0(A_1), \forall a_2 \in A_2, \quad f(a_1^*, a_2) < \sum_{a_1 \in A_1} f(a_1, a_2)\beta_1(a_1).$$

Otherwise, we say that a_1^* is *undominated under randomization*.

At this level of generality, the separable metric spaces A_1 and A_2 are *not* required to be subsets of some Hausdorff locally convex topological vector space. Denote by $\mathcal{B}_i$ the Borel sigma-algebra of A_i. In this framework, we can identify each element a of A_i with the Dirac δ_a at a. The set of Dirac probability measures is a subset of the space of all Borel countably additive measures of bounded variation $ca(A_i, \mathcal{B}_i)$ which, when it is endowed with the w^*-topology $\sigma(ca(A_i, \mathcal{B}_i), C_b(A_i))$, is a Hausdorff locally convex topological vector space. Since A_i is a separable metric space, the set of corresponding Dirac probabilities is also closed. Under this identification, B_i corresponds to the set $\Delta_0(A_i)$ of all probability measures on A_i with finite support, while $\bar{B}_i$ corresponds to the set $\Delta(A_i)$ of all Borel probability measures on A_i. The set $\bar{B}_i$ is compact if and only if A_i is compact.[15] Finally, note that if A_i is finite, then it is a separable metric space once it is endowed with the discrete metric; moreover, if both A_1 and A_2 are finite, then f is continuous in each component and bounded.

In what follows, with a small abuse of notation, we will denote by A_i both the original set A_i and the set of corresponding Dirac probability measures. Also, we will denote the elements of B_1 and $\bar{B}_2$ with the letter β rather than b to stress that we interpret them as probability measures.

COROLLARY 1: *Let A_1 and A_2 be two separable metric spaces. If*
(i) *A_2 is compact,*
(ii) *f is continuous on A_1 and A_2 and bounded,*
then, an element $a_1^* \in A_1$ is undominated under randomization if and only if there exists $\beta_2 \in \bar{B}_2 = \Delta(A_2)$ such that $a_1^* \in \arg\max_{a_1 \in A_1} \int_{A_2} f(a_1, a_2)\beta_2(\mathrm{d}a_2)$.

PROOF: Define $F : B_1 \times \bar{B}_2 \to \mathbb{R}$ by

$$F(\beta_1, \beta_2) = \int_{A_2} \left(\sum_{a_1 \in A_1} f(a_1, a_2)\beta_1(a_1) \right) \beta_2(\mathrm{d}a_2).$$

It is routine to check that the function F is affine and continuous in each component. Given our identifications, $a_1^* \in A_1$ is undominated under randomization if and only if $\delta_{a_1^*}$ is co-undominated. By Lemma 1, the statement follows. *Q.E.D.*

Set

$$\mathcal{J}_f = \Big\{ a_1 \in A_1 : \exists \beta_2 \in \Delta(A_2), \forall a_1' \in A_1, \\ \int_{A_2} f(a_1, a_2)\beta_2(\mathrm{d}a_2) \geq \int_{A_2} f(a_1', a_2)\beta_2(\mathrm{d}a_2) \Big\}.$$

[15]See Chapter 15 of Aliprantis and Border (2006) for all the above notions, in particular, Footnote 1 and Theorems 15.8, 15.10, and 15.11.

Say that a function $f : A_1 \times A_2 \to \mathbb{R}$ is *nice* (resp., *semi-nice*) when any $a_1 \in A_1$ is undominated under randomization if and only if (resp., only if) $a_1 \in \mathcal{J}_f$. Corollary 1 establishes conditions for niceness.

COROLLARY 2: *Let $f, h : A_1 \times A_2 \to \mathbb{R}$ be, respectively, nice and semi-nice. If $h = \varphi \circ f$, with $\varphi : \mathrm{co}(\mathrm{Im}\, f) \to \mathbb{R}$ concave and strictly increasing, then $\mathcal{J}_f \subseteq \mathcal{J}_h$.*

PROOF: Let $\bar{a}_1 \in \mathcal{J}_f$. Since f is nice, $\bar{a}_1$ is undominated under randomization. Hence, since φ is concave and strictly increasing, this implies that

$$\forall \beta_1 \in \Delta_0(A_1), \exists a_2 \in A_2, \quad f(\bar{a}_1, a_2) \geq \sum_{a_1 \in A_1} f(a_1, a_2)\beta_1(a_1),$$

that is, for each $\beta_1 \in \Delta_0(A_1)$ there exists $a_2 \in A_2$ such that

$$\begin{aligned}(\varphi \circ f)(\bar{a}_1, a_2) &\geq \varphi\left(\sum_{a_1 \in A_1} f(a_1, a_2)\beta_1(a_1)\right) \\ &\geq \sum_{a_1 \in A_1} (\varphi \circ f)(a_1, a_2)\beta_1(a_1).\end{aligned}$$

Since $h = \varphi \circ f$ is semi-nice, this implies $\bar{a}_1 \in \mathcal{J}_h$. *Q.E.D.*

A.3. *Proofs of Propositions 1 and 2*

First observe that, since S is a separable metric space, also $\Delta(S)$ is a separable metric space (once endowed with the Prohorov metric). We denote by $\mathcal{B}$ its Borel sigma-algebra. Given a set $\Sigma \in \mathcal{B}$, we denote by $\mathcal{B}_{|\Sigma}$ the relative Borel sigma-algebra and by $\Delta(\Sigma)$ the collection of all Borel probability measures $\mu : \mathcal{B}_{|\Sigma} \to [0, 1]$. We endow $\Delta(\Sigma)$ with the w^*-topology. $\Delta(\Sigma)$ is compact if and only if Σ is w^*-compact in $\Delta(S)$.

PROOF OF PROPOSITION 1: Let $A_1 = A$, $A_2 = S$, $f = r$, and $h = r' = \psi \circ r$. By Corollary 1 and given the properties of A, S, r, and ψ, it is immediate to see that f and h are nice. By Corollary 2 and since ψ is concave and strictly increasing, we have that $\mathcal{J}_f \subseteq \mathcal{J}_h$. Finally, since $\Sigma \supseteq \{\delta_s\}_{s \in S}$, we can conclude that $\mathcal{J}_{\mathrm{Id},r}(\Sigma) = \mathcal{J}_f$ and $\mathcal{J}_h = \mathcal{J}_{\mathrm{Id},r'}(\Sigma)$. Therefore, $\mathcal{J}_{\mathrm{Id},r}(\Sigma) \subseteq \mathcal{J}_{\mathrm{Id},r'}(\Sigma)$. *Q.E.D.*

PROOF OF PROPOSITION 2: Define $R(a, \sigma) = \int_S r(a, s)\sigma(\mathrm{d}s)$ for all $a \in A$ and $\sigma \in \Sigma$. Note that R is continuous in each component and bounded. Let $A_1 = A$, $A_2 = \Sigma$, $f = \phi \circ R$, and $h = \phi' \circ R = (\varphi \circ \phi) \circ R = \varphi \circ f$. By Corollary 1 and given the properties of A, Σ, R, and ϕ and φ, it is immediate to see that f and h are nice. By Corollary 2 and since φ is concave and strictly increasing, we have that $\mathcal{J}_f \subseteq \mathcal{J}_h$. By construction, we have that $\mathcal{J}_{\phi,r}(\Sigma) = \mathcal{J}_f$ and $\mathcal{J}_h = \mathcal{J}_{\phi',r}(\Sigma)$. Therefore, $\mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\phi',r}(\Sigma)$. *Q.E.D.*

REMARK 1: To invoke the abstract Pearce–Wald lemma, in the proof of Proposition 2, from a decision theoretic viewpoint, we consider randomized actions β_1 on A as ex ante randomizations rather than the more customary ex post randomizations à la Anscombe and Aumann. Such ex ante randomizations are merely ancillary analytical objects. Formally, in our proofs the value of a randomized action β_1 is

$$\phi^{-1}\left(\sum_{a\in A}\int_{\Sigma}\phi\left(\int_S r(a,s)\sigma(\mathrm{d}s)\right)\mu(\mathrm{d}\sigma)\beta_1(a)\right)$$

rather than

$$\phi^{-1}\left(\int_{\Sigma}\phi\left(\sum_{a\in A}\int_S r(a,s)\sigma(\mathrm{d}s)\beta_1(a)\right)\mu(\mathrm{d}\sigma)\right).$$

Conceptually, we reiterate that DM can only choose in A, which encompasses any feasible randomization. It is a fortuitous coincidence that the smooth model permits such a treatment of randomized actions, which in turn allows us to exploit the abstract Pearce–Wald lemma.

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