Econ Theory
DOI 10.1007/s00199-016-0972-5

RESEARCH ARTICLE

Mixed extensions of decision problems under uncertainty

Pierpaolo Battigalli[1] · Simone Cerreia-Vioglio[1] · Fabio Maccheroni[1] · Massimo Marinacci[1]

Received: 8 June 2015 / Accepted: 5 April 2016

Abstract In a decision problem under uncertainty, a decision maker considers a set of alternative actions whose consequences depend on uncertain factors beyond his control. Following Luce and Raiffa (Games and decisions: introduction and critical survey. Wiley, New York, 1957), we adopt a natural representation of such a situation which takes as primitives a set of conceivable actions A, a set of states S and a consequence function $\rho: A \times S \to C$. Each action induces a map from states to consequences, a Savage act, and each mixed action induces a map from states to probability distributions over consequences, an Anscombe–Aumann act. Under an axiom of consequentialism, preferences over pure or mixed actions yield corresponding preferences over the induced acts. This observation allows us to relate the Luce–Raiffa description of a decision problem to the most common framework of modern decision theory which directly takes as primitive a preference relation over the set of all Anscombe–Aumann acts. The key advantage of the latter framework is the possibility of applying powerful convex analysis techniques as in the seminal work of Schmeidler (Econometrica 57:571–587, 1989) and the vast literature that followed. This paper shows that we can maintain the mathematical convenience of the Anscombe–Aumann framework within a description of decision problems which is closer to many applications and

We thank Veronica Cappelli, Karin Hendrix Blissitt, Giacomo Lanzani, Sujoy Mukerji, Kelly Gail Strada, and two anonymous referees, and the participants of the NES 20th anniversary conference (Moscow, December 2012) and D-TEA and RUD workshops (Paris, May 2013) for useful comments and suggestions. Simone Cerreia-Vioglio acknowledges the financial support of the European Research Council (advanced grant BRSCDP-TEA), Fabio Maccheroni acknowledges the financial support of the Italian Ministry of Education, Universities, and Research (Grant PRIN 20103S5RN3_005), and Massimo Marinacci acknowledges the financial support of the AXA Research Fund.

✉ Fabio Maccheroni
fabio.maccheroni@unibocconi.it

1 Università Bocconi and IGIER, Milan, Italy

experiments. We argue that our framework is more expressive as it allows us to be both explicit and parsimonious about the assumed richness of the set of conceivable actions, and to directly capture preference for randomization as an expression of uncertainty aversion.

1 Introduction

In the modern development of economic theory, the role of uncertainty has been fundamental, and the past two decades have witnessed a number of studies which have extended classical risk theory results to better deal with new dimensions of uncertainty, in particular with Knightian uncertainty (often called ambiguity). These theoretical studies have found applications in a variety of fields, from asset pricing to market participation, from contract theory to risk management.[1] The framework which these studies have relied upon remains the one adopted in the seminal paper of Schmeidler (1989), the so-called *Anscombe–Aumann framework*. In the literature, this term does not usually refer to the original framework of Anscombe and Aumann (1963), but to its simplified version proposed by Fishburn (1970). In that version, the objects of choice are functions $f : S \to \Delta(C)$,[2] where $S = \{s_1, \ldots, s_n\}$ is a set of *states* of the world, C is a set of deterministic *consequences*, and $\Delta(C)$ is the set of *random consequences*.[3] In particular, as Fishburn (1970 p. 176) writes:

> …We adopt the following pseudo-operational interpretation for $f \in \Delta(C)^S$. If f is 'selected' and $s \in S$ obtains, then $f(s) \in \Delta(C)$ is used to determine a resultant consequence in C…

This framework endows the set of conceivable alternatives with a tractable and mathematically familiar convex space structure, which has been fundamental for the development of axiomatic models of choice under uncertainty. However, the translation of economic choice situations in this framework is not always intuitive. For this reason, in this paper we study the connection between the mathematically convenient framework of Anscombe and Aumann (1963) and that of Luce and Raiffa (1957), which better portrays the natural decision-making process. For Luce and Raiffa, a decision problem under uncertainty is described by a table

$$
\begin{array}{c|c|c|c|c|}
 & s_1 & s_2 & \ldots & s_n \\
\hline
a_1 & c_{11} & c_{12} & \ldots & c_{1n} \\
\hline
a_2 & c_{21} & c_{22} & \ldots & c_{2n} \\
\hline
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
\tag{1}
$$

in which $c_{ij} \in C$ is the consequence of action $a_i \in A$ in state $s_j \in S$ and the decision maker can choose a *pure action* $a \in A$ or a *mixed action* $\alpha \in \Delta(A)$.[4] Mixed actions are

---

[1] See Gilboa and Marinacci (2013) for a recent survey.

[2] Referred to as horse lotteries by Anscombe and Aumann (1963) and *Anscombe–Aumann acts* in the subsequent literature.

[3] Formally, the set of all finitely supported probability distributions on C.

[4] This is the framework they describe on p. 276. They later replace consequences c_{ij} with their utilities u_{ij}.

interpreted as in usual game theoretic analysis as mixed strategies of a player choosing rows in the table above. The objects of choice in the Luce–Raiffa framework, the mixed actions $\alpha \in \Delta(A)$, are thus common in economics, statistics, and operations research, unlike those of the Anscombe–Aumann framework, the acts $f \in \Delta(C)^S$, which are specific to decision theory.

In this paper we show that the Luce–Raiffa framework is actually as tractable as the Anscombe–Aumann one, and can be embedded into it provided decision makers are *consequentialist*, that is, indifferent toward actions that generate the same distribution of consequences in every state. Our analysis thus presents a lexicon that allows us to translate all decision theoretic results that have been expressed in the language of Anscombe–Aumann acts into the language of mixed actions. Such a translation facilitates the understanding of uncertainty models for those familiar with mixed strategies, making it possible to use these models 'straight off the shelf' in any application framed in the language of game theory (such as auction theory, matching, mechanism design, moral hazard, etc.). Moreover, mixed actions are easier to implement in the controlled setup of an experiment than Anscombe–Aumann acts. Therefore, the framework we consider here facilitates the dialogue between theory and experiments, in particular the experimental analysis of choice models derived within the Anscombe–Aumann framework.

The paper is organized as follows: After a preliminary analysis of decision problems and frameworks (Sect. 2), we first study the properties of randomization of pure actions and relate it to randomization of consequences in the Anscombe–Aumann framework (Sect. 3). The novelty lies not in the introduction of mixed actions, already present in Luce and Raiffa (1957), but in the relation between two kinds of randomization.[5] Subsequently, we study the basic properties of preferences and choice correspondences in the Luce–Raiffa framework (Sect. 4), and we illustrate them by establishing behavioral characterizations of two classical choice criteria (Sect. 5). Finally, we relate uncertainty aversion to preference for randomization, and we show how it provides a natural explanation for commonly observed random choice behavior (Sect. 6). Closing remarks and comments can be found in Sect. 7.

All proofs are in "Appendix 2".

2 Decision problems under uncertainty

2.1 Setup

In a decision problem under uncertainty, a decision maker considers a set of alternative actions whose consequences depend on uncertain factors beyond his control. Formally, there is a set A of *conceivable* pure actions a that can result in different material consequences c, within a set C, depending on which state s of the world (or of the environment) in a space $S = \{s_1, \ldots, s_n\}$ obtains. The dependence of consequences on actions and states is described by a *consequence function*

[5] This relation is hinted at by Kreps (1988) in Chapter 7, where Savage acts (functions $f : S \to C$) are randomized. See the concluding Sect. 7 for details.

$$\begin{array}{rccc} \rho\colon & A \times S & \to & C \\ & (a,s) & \mapsto & \rho(a,s) \end{array}$$

which specifies the consequence $c = \rho(a,s)$ for each action a in each state s.[6] A *decision problem* (*under uncertainty*) is a 4-tuple $(\mathcal{A}, S, C, \rho)$, where $\mathcal{A} \subseteq A$ is a nonempty subset of *feasible* pure or mixed actions.[7] We call *decision framework* the 4-tuple (A, S, C, ρ).

In such a framework, a *bet* on an *event* $E \subseteq S$ is an action $cEd \in A$ that delivers consequence c if E is true and d otherwise, that is,

$$\rho(cEd, s) = \begin{cases} c & s \in E \\ d & s \notin E. \end{cases} \tag{2}$$

In particular, if we let $E = S$, cSd yields c in every state. For this reason such a bet is called a *sure action* and is denoted by cS. We say that *all bets are conceivable* if and only if for every $c, d \in C$ and $E \subseteq S$ there exists a (possibly nonunique) action $cEd \in A$ such that (2) holds. Analogously, *all sure actions are conceivable* if and only if for every $c \in C$ there exists a (possibly nonunique) action $cS \in A$ for which $\rho(cS, \cdot) \equiv c$.

Although the next section presents many examples, the reader may want to skip to decision framework (13) and observe that Ellsberg's decision problem is represented by (12): in that case, an action yielding \$100 with certainty is conceivable but not feasible.

2.2 Examples

Example 1 (portfolio selection) An investor in a frictionless financial market with $J \in \mathbb{N}$ primary assets chooses a portfolio $h \in \mathbb{R}^J$ at time 0 when he is uncertain about the vector $r \in \{r_1, r_2, \ldots, r_n\} \subseteq \mathbb{R}^J$ of gross returns at time 1 (h_j is the amount of money invested in asset j). Denoting by S the set $\{1, 2, \ldots, n\}$, the investor's monetary payoff at time 1 is

$$\nu(h,s) = r_s \cdot h \quad \forall (h,s) \in \mathbb{R}^J \times S.$$

The decision framework is thus $\left(\mathbb{R}^J, S, \mathbb{R}, \nu\right)$. Depending on the investor's wealth w, his budget set is $B_w = \left\{h \in \mathbb{R}^J \colon \sum_{j \in J} h_j = w\right\}$ and the corresponding decision problem is $(B_w, S, \mathbb{R}, \nu)$.

Note that in this case all bets are conceivable if and only if the market is complete, while all sure actions are conceivable if and only if a risk-free portfolio exists. □

In the previous example the distinction between the decision framework $\left(\mathbb{R}^J, S, \mathbb{R}, \nu\right)$ and the family of relevant decision problems $\{(B_w, S, \mathbb{R}, \nu) : w \in \mathbb{R}\}$ is

6 In Eq. (1), $\rho(a_i, s_j) = c_{ij}$ for each action $a_i \in A$ and each state $s_j \in S$.

7 In other words, a decision problem is characterized by the set $\mathcal{A}$ of actions that are available to the decision maker in a given choice situation.

clear. Instead, in the next example, the focus is on a single decision problem, whereas the appropriate decision framework depends on the goal of the analysis.[8]

Example 2 (normal game-forms) An agent interacting with other agents chooses his action when uncertain about the choices of the other players. The consequences of the interaction are determined by the profile of actions chosen by all agents. *Normal game-forms* $\langle I, (S_i)_{i\in I}, C, g\rangle$ are used to model players' strategic interaction in a game (see e.g., Glazer and Rubinstein 1996). They consist of a finite set I of players, a set S_i of available strategies for each player $i \in I$, a set C of consequences, and a function $g\colon \Pi_{i\in I} S_i \to C$ that associates consequences with strategy profiles.[9]

Depending on whether player j can commit his actions to a random device or not, that is, whether mixed strategies are available or not, here the decision problem is either $\left(\Delta\left(S_j\right), S_{-j}, C, g\right)$ or $\left(S_j, S_{-j}, C, g\right)$ itself (unless there is an explicit description of the available randomizations). □

We conclude with the famous example of Savage (1954), which we report verbatim as it will be used at the beginning of the next section to clarify the relation between pure actions and Savage acts.

Example 3 (the omelet)…Your wife has just broken five good eggs into a bowl when you come in and volunteer to finish making the omelet. A sixth egg, which for some reason must either be used for the omelet or be wasted altogether, lies unbroken beside the bowl. You must decide what to do with this unbroken egg. Perhaps it is not too great an oversimplification to say that you must decide among three acts only, namely, to break it into the bowl containing the other five, to break it into a saucer for inspection, or to throw it away without inspection. Depending on the state of the egg, each of these three acts will have some consequences of concern to you, say that indicated by…

Table 1	Good	Rotten
Break into bowl	Six-egg omelet	No omelet, and five good eggs destroyed
Break into saucer	Six-egg omelet, and a saucer to wash	Five-egg omelet, and a saucer to wash
Throw away	Five-egg omelet, and one good egg destroyed	Five-egg omelet

…If two different acts had the same consequences in every state of the world, there would from the present point of view be no point in considering them two different acts at all…Or, more formally, an act is a function attaching a consequence to each state… □

[8] This makes the conceptual status of subjective beliefs in games problematic from a decision theoretic perspective (Mariotti 1995).

[9] By contrast, *normal-form games* also include a profile $(u_i)_{i\in I}$ of von Neumann–Morgenstern utility functions on C.

2.3 Reduced decision problems and consequentialism

Savage's final sentence in Example 3 refers to the identification of each pure action $a \in A$ with the section $\rho_a \in C^S$, which is referred to as the *Savage act* induced by a, that is,

$$\begin{array}{rccc} \rho_a\colon & S & \to & C \\ & s & \mapsto & \rho(a, s) \end{array}$$

For example, the action 'break into bowl' is identified by Savage with the act $f = \rho_{\text{break into bowl}}$ given by

$$f\,(\text{Good}) = \text{Six-egg omelet}; \quad f\,(\text{Rotten}) = \text{No omelet, and five good eggs destroyed}$$

Two actions a and b are thus identified if

$$\rho(a, s) = \rho(b, s) \quad \forall s \in S$$

that is, if $\rho_a = \rho_b$. Two such actions generate the same state-contingent consequences and are called *realization equivalent*,[10] denoted as $a \approx b$.

The identification of realization equivalent actions is ubiquitous in the modelling of individual and interactive decisions. For example, when $\left\langle I, (S_i)_{i \in I}, Z, g \right\rangle$ is the normal-form representation of an extensive game-form (Example 2), the set of consequences Z is the set of terminal nodes, the consequence function g is the outcome function, and our definition of realization equivalence corresponds to that of Kuhn (1953). His Theorem 1 shows that two strategies are realization equivalent if and only if they induce the same decision plan (see Rubinstein 1991, p. 911). In this case, the identification of realization equivalent strategies leads to the quasi-reduced normal game-form.[11]

Definition 1 A decision framework (A, S, C, ρ) is *reduced* if and only if $a \approx b$ implies $a = b$.

Non-reduced decision frameworks can always be reduced by identifying all elements of each realization equivalence class. Sometimes the equivalence classes obtained by reduction maintain a clear interpretation with regard to the original problem. For example, when the normal-form representation of an extensive game-form is considered, realization equivalence classes of strategies correspond to decision plans. In other cases, their interpretation is less immediate. In Example 1, the $S \times J$ matrix $R = [r_s]_{s=1}^n$, known as *Arrow–Debreu tableau*, groups the state-contingent returns of the marketed assets. Therefore, the section $v_h = Rh$ of v at portfolio h, referred to as the contingent claim replicated by h, describes the state-contingent payoffs generated

[10] Marschak and Radner (1972, Ch. 1) call them *essentially equivalent*.

[11] See Swinkels (1989) for a detailed distinction among the various reduced forms of a game.

by h.[12] The usual interpretation of a realization equivalence class $\{h \in \mathbb{R}^J : Rh = x\}$ as an object of choice consists in identifying it with the contingent claim x itself.[13] However, while a portfolio is a collection of primary assets, a contingent claim $x \in \mathbb{R}^S$ is a contract that pays $x(s)$ if r_s is the true vector of gross returns, namely a derivative asset.[14]

Be that as it may, from the decision maker's perspective the reduction of a nonreduced decision framework is an innocuous simplification as long as he is indifferent toward pure actions that yield the same consequences in every state. Denoting the decision maker's preferences on A by $\succsim = \succsim_A$ and its symmetric (resp. asymmetric) part by $\sim$ (resp. $\succ$), this amounts to:

Consequentialism *For all $a, b \in A$, $a \approx b$ implies $a \sim b$.*

This assumption is reasonable when the decision maker only cares about consequences, and actions are just a means to an end (to obtain 'desirable' consequences). Paraphrasing Marschak and Radner (1972, p. 12), if all consequences were directly available, the 'most desirable' would be chosen. Yet, it is actions, not consequences, that are available for choice.

For any given decision framework (A, S, C, ρ), consequentialism allows us to elicit both the 'desirability' of consequences and the 'plausibility' of events, provided $\succsim$ is transitive.[15]

Proposition 1 *Let (A, S, C, ρ) be a decision framework and $\succsim$ a transitive binary relation on A.*

1. *If $\succsim$ satisfies consequentialism, then $\succsim$ is a preorder; the converse is true provided (A, S, C, ρ) is reduced.*
2. *If $\succsim$ satisfies consequentialism and all sure actions are conceivable, then*

$$c \succsim_C d \overset{\text{def}}{\iff} cS \succsim dS \tag{3}$$

is a well-defined preorder on C.

3. *If $\succsim$ satisfies consequentialism and all bets are conceivable, then*

$$E \succsim^* F \overset{\text{def}}{\iff} cEd \succsim cFd \quad \forall c \succ_C d$$

is a well-defined preorder on the power set of S.

The simple proof of this proposition shows that consequentialism is crucial for the well-posedness of the preorders $\succsim_C$ and $\succsim^*$. These preorders are interpreted as

[12] In particular, the decision framework is reduced if and only if there are no redundant assets, that is, the rank of the Arrow–Debreu tableau is equal to the number of marketed assets. See e.g., LeRoy and Werner (2000, Ch. 1) or Cerny (2009, Ch. 1).

[13] This equivalence class can be seen geometrically as a hyperplane.

[14] According to the Office of the Comptroller of the Currency (1999) a *derivative* is 'a financial contract whose value is derived from the performance of assets, interest rates, currency exchange rates, or indexes.'

[15] A transitive binary relation which is also reflexive is called *preorder*.

qualitative descriptions of the decision maker's tastes and beliefs on C and S, respectively. For later reference, we remark that to elicit tastes and beliefs all bets must be conceivable (even if not necessarily feasible). This assumption, although strong, is weaker than that of Savage which requires all acts to be conceivable.[16] In Savage's framework the role of bets cEd is played by binary acts $c_E d$ taking value c on E and d on E^c, and sure actions cS correspond to constant acts c_S.

3 Mixed actions and immersion

Following the common practice of game theory and statistics, we consider a decision maker who contemplates the possibility of delegating the choice of his actions to some random device. As a result, the set of conceivable actions is the set $\Delta(A)$ of all mixed actions. Let us reiterate that this should not be interpreted as an assumption concerning the feasibility of all mixed actions, but rather as an assumption as to the ability of the decision maker to consider hypothetical alternatives. We will not further discuss the difference between feasibility and conceivability; however, referring to Savage's omelet example one last time, tossing a coin before deciding where to break the sixth egg seems easier to conceive than an act that results in a 'six-egg omelet' if the sixth egg is rotten and 'no omelet, and five good eggs destroyed' if the sixth egg is good.

Formally, for any set X,

$$\Delta(X) = \left\{ \xi \in \mathbb{R}_+^X : \xi(x) > 0 \text{ for finitely many } x\text{'s in } X \text{ and } \sum_{x \in X} \xi(x) = 1 \right\}$$

is the set of all probability distributions on X with finite support.[17] In particular, mixed actions are the elements of $\Delta(A)$ and pure actions can be viewed as special mixed actions via the embedding $a \hookrightarrow \delta_a$ of points into point-masses. Conceptually, the elements $\alpha \in \Delta(A)$ should be interpreted as chance distributions, namely objective probabilities such as those inherent in random devices, and not as epistemic distributions, that is, subjective probabilities describing beliefs.[18] Mixed actions correspond to mixed strategies in game theory and to randomized decision rules in statistics.[19]

The relevance of mixed actions is well illustrated by the decision problem:

[16] That is, for every $f : S \to C$ there exists $a \in A$ such that $\rho_a = f$. Formally, Savage assumes $(A, S, C, \rho) = \left(C^S, S, C, \varrho\right)$ where $\varrho(f, s) = f(s)$ is the evaluation pairing. In this decision framework, the number of bets is $|C| + |C|(|C| - 1)\left(2^{|S|-1} - 1\right)$ while the number of acts is $|C|^{|S|}$. With 10 states and 10 consequences there are forty-six thousand bets and ten billion acts.

[17] $\mathbb{R}_+^X$ is the set of functions from X to $\mathbb{R}_+$. With the usual abuse of notation we denote a probability distribution on X and the probability measure it induces on the set of all parts of X by the same Greek letter.

[18] We refer to Luce and Raiffa (1957) and Anscombe and Aumann (1963) for a discussion of this distinction.

[19] One should distinguish between mixed decision rules and behavioral decision rules. But, since we are considering finite state spaces, the distinction is immaterial because a version of Kuhn (1953) equivalence theorem holds.

	s_1	s_2
a_1	0	1
a_2	1	0

(4)

with action set $A = \{a_1, a_2\}$, state space $S = \{s_1, s_2\}$, consequence space $C = \{0, 1\}$, and consequence function $\rho(a_1, s_1) = \rho(a_2, s_2) = 0$ and $\rho(a_1, s_2) = \rho(a_2, s_1) = 1$. As Luce and Raiffa (1957, p. 279) observe, the mixed action

$$\alpha = \frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$$

guarantees $1/2$ in expectation regardless of which state obtains, while the minimum guaranteed by both pure actions is 0. Randomization may thus hedge uncertainty, an obviously important feature in analyzing these decision problems (see also Debreu 1959, p. 101).

Keep in mind that $\Delta(C)$ is the collection of random consequences, that is to say the set of all chance distributions on C with finite support. Each mixed action (a chance distribution of pure actions) induces a random consequence (a chance distributions of deterministic consequences) in every state: If the decision maker takes the mixed action α, the chance of obtaining consequence c in state s is

$$\alpha(\{a \in A\colon \rho(a, s) = c\})$$

which we denote by $\rho_\alpha(c \mid s)$. Note that

$$\rho_\alpha(\cdot \mid s) = \left(\alpha \circ \rho_s^{-1}\right)(\cdot)$$

is an element of $\Delta(C)$ for all $s \in S$, that is, each mixed action $\alpha \in \Delta(A)$ determines an Anscombe–Aumann act

$$\begin{aligned} \rho_\alpha\colon S &\to \Delta(C) \\ s &\mapsto \alpha \circ \rho_s^{-1} \end{aligned}$$

which associates to each $s \in S$ the distribution of consequences resulting from the choice of α in state s. As anticipated in the Introduction, Anscombe–Aumann acts are functions $f \in \Delta(C)^S$ from states to random consequences, and they are the usual objects of choice in the axiomatic literature on decision making under uncertainty.[20] The next proposition describes some properties of the relation between mixed actions and Anscombe–Aumann acts.

Proposition 2 *Let (A, S, C, ρ) be a decision framework. The map*

$$\begin{array}{cccc} F\colon & \Delta(A) & \to & \Delta(C)^S \\ & \alpha & \mapsto & \rho_\alpha \end{array}$$

[20] See again, Gilboa and Marinacci (2013).

has the following properties:

1. *For every $a \in A$ and every $s \in S$, $\rho_{\delta_a}(s) = \delta_{\rho(a,s)}$; that is, $\digamma(a) = \rho_a$ under the identification of points and point-masses.*
2. *For every $\alpha, \beta \in \Delta(A)$ and every $q \in [0, 1]$, $\rho_{q\alpha+(1-q)\beta} = q\rho_\alpha + (1-q)\rho_\beta$; that is, $\digamma$ is affine.*
3. *$\{\rho_\alpha\}_{\alpha\in\Delta(A)} = \Delta(C)^S$ if and only if $\{\rho_a\}_{a\in A} = C^S$; that is, $\digamma$ is onto if and only if all Savage acts are conceivable.*

The first point of this proposition shows that the relation between a mixed action α and the corresponding Anscombe–Aumann act ρ_α extends the relation between a pure action a and the corresponding Savage act ρ_a. Specifically, the Anscombe–Aumann act ρ_{δ_a} induced by the pure action a coincides with the Anscombe–Aumann act induced by the Savage act ρ_a. This observation allows us to consistently extend the definition of realization equivalence given for pure actions to mixed actions.

Definition 2 Let (A, S, C, ρ) be a decision framework. Two mixed actions $\alpha, \beta \in \Delta(A)$ are *realization equivalent* if and only if they generate the same distribution of consequences in every state of the world, that is, $\alpha \approx \beta \overset{\text{def}}{\iff} \rho_\alpha = \rho_\beta$.

By the first point of Proposition 2, given two pure actions $a, b \in A$ and $s \in S$,

$$\rho(a, s) = \rho(b, s) \iff \delta_{\rho(a,s)} = \delta_{\rho(b,s)} \iff \rho_{\delta_a}(s) = \rho_{\delta_b}(s)\,. \tag{5}$$

That is to say a and b are realization equivalent in the sense of the previous section (i.e., $\rho_a = \rho_b$) if and only if they are realization equivalent as mixed actions (i.e., $\rho_{\delta_a} = \rho_{\delta_b}$). In turn, Definition 2 allows us to extend the notion of consequentialism to preferences $\succsim = \succsim_{\Delta(A)}$ over mixed actions. A decision maker is now consequentialist if and only if he is indifferent toward mixed actions that generate the same distribution of consequences in every state of the world. Formally,

Mixed consequentialism *For all $\alpha, \beta \in \Delta(A)$, $\alpha \approx \beta$ implies $\alpha \sim \beta$.*

By (5), consequentialism is the restriction to pure actions of mixed consequentialism. As consequentialism allows us to embed any decision problem with pure actions in the Savage framework ($a \overset{\digamma}{\mapsto} \rho_a$), mixed consequentialism allows us to embed any decision problem with mixed actions in the Anscombe–Aumann framework ($\alpha \overset{\digamma}{\mapsto} \rho_\alpha$).[21] Indeed, mixed consequentialism makes it possible to define a binary relation $\succsim_\digamma$ by

$$\rho_\alpha \succsim_\digamma \rho_\beta \overset{\text{def}}{\iff} \alpha \succsim \beta \tag{6}$$

on the collection of Anscombe–Aumann acts $\{\rho_\alpha\}_{\alpha\in\Delta(A)} \subseteq \Delta(C)^S$. The fact that $\digamma$ is affine allows us to easily translate behavioral assumptions of $\succsim$ on $\Delta(A)$ into corresponding properties of $\succsim_\digamma$ on $\{\rho_\alpha\}_{\alpha\in\Delta(A)}$, as detailed in Proposition 3 below and in Lemma 2 in "Appendix 2". Using the 'transfer principle' between frameworks established in (6), it is still possible to take advantage of the analytical tractability of

[21] Formally, the embedding is $[\alpha] \mapsto \rho_\alpha$, where $[\alpha]$ is the $\approx$-equivalence class of α.

the Anscombe–Aumann setup in the mathematical derivation of choice models; in this way we avoid the interpretational difficulties of the Anscombe–Aumann structure. To put it simply, the transfer principle makes it possible to conduct the behavioral analysis in terms of mixed actions in the main text of a decision theory paper, while taking advantage of the Anscombe–Aumann techniques in its appendix.

The last point of Proposition 2 shows in which sense our framework is never more demanding than that of Anscombe–Aumann in terms of hypothetical comparisons, and equally demanding only when all Savage acts are conceivable. In other words, the Anscombe–Aumann framework $\Delta(C)^S$ corresponds, via the transfer principle (6), to the mixed extension $\Delta\left(C^S\right)$ of the Savage framework.

Finally, note that also in a reduced decision framework, mixed actions that are realization equivalent may well differ.[22] Nonetheless, Proposition 2 implies that the framework cannot be reduced any further by eliminating a pure action a which is realization equivalent to a mixed action β with support in $A \setminus \{a\}$.

Corollary 1 *Let (A, S, C, ρ) be a reduced decision framework. For all $a \in A$ and $\beta \in \Delta(A)$, $\delta_a \approx \beta$ if and only if $\beta = \delta_a$.*

The language developed so far also allows us to connect decision analysis in the Anscombe–Aumann framework to statistical decision theory, as detailed in the working paper version.

4 Rationality and dominance

Up to this point, we have been studying two alternative decision theoretic frameworks and their relationship. We now focus our attention on the decision maker's preferences. Let (A, S, C, ρ) be a decision framework and $\succsim = \succsim_{\Delta(A)}$ be a binary relation on $\Delta(A)$ representing the decision maker's preferences.

Axiom A.1 (Weak order) *$\succsim$ is complete and transitive.*

Although Karni et al. (2014) show that completeness is not crucial for our analysis, we maintain it for the sake of simplicity. Inspired again by Luce and Raiffa (1957, p. 276), we also assume that

> …our subject's preferences among …outcomes, and among hypothetical lotteries with this outcomes as prizes, are consistent in the sense that they may be summarized by means of a utility function …

Since the objects of choice are elements of $\Delta(A)$, not of C or $\Delta(C)$, this amounts to assuming that all sure actions are conceivable and that the decision maker's preferences restricted to lotteries of sure actions (the mixed actions with support in the set of all sure actions) satisfy the axioms of von Neumann–Morgenstern's expected utility. For this reason, and for others that soon will become clear, reduced decision frameworks in which all sure actions are conceivable deserve a special name.

[22] For example, (13) below is reduced, and the mixed actions $\alpha = \frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$ and $\beta = \frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2}$ are clearly distinct (they have disjoint support), but $\rho_\alpha = \rho_\beta$, so that they are realization equivalent.

Definition 3 A Luce–Raiffa framework (A, S, C, ρ) is a reduced decision framework in which all sure actions are conceivable.

In these frameworks it becomes possible to elicit both the decision maker's preferences over consequences (Proposition 1.2) and his attitudes toward risk. Indeed, the map

$$\begin{array}{cccc} \epsilon\colon & \Delta(C) & \to & \Delta(A) \\ & \sum_{c\in C}\gamma(c)\,\delta_c & \mapsto & \sum_{c\in C}\gamma(c)\,\delta_{cS} \end{array}$$

is an affine embedding of the set $\Delta(C)$ of random consequences onto the set $\Delta_\ell(A)$ of all mixed actions with support in the set $\{cS\}_{c\in C}$ of all sure actions.[23] For each $\gamma \in \Delta(C)$, by choosing $\epsilon(\gamma)$ the decision maker receives consequence c with objective probability $\gamma(c)$ regardless of which state obtains,[24] both γ and $\epsilon(\gamma)$ are formal descriptions of *lotteries* with consequences as prizes. Since ϵ is an affine bijection from $\Delta(C)$ to $\Delta_\ell(A)$, we can set

$$\gamma \succsim_{\Delta(C)} \zeta \overset{\text{def}}{\iff} \epsilon(\gamma) \succsim \epsilon(\zeta)$$

and infer the decision maker's preferences $\succsim_{\Delta(C)}$ over random consequences from his preferences $\succsim$ over mixed actions. In this way, ϵ becomes an isotone isomorphism between $\left(\Delta(C), \succsim_{\Delta(C)}\right)$ and $\left(\Delta_\ell(A), \succsim\right)$; for this reason, we often write γ instead of $\epsilon(\gamma)$, and $\gamma \succsim \zeta$ instead of $\gamma \succsim_{\Delta(C)} \zeta$.[25]

In this perspective, the utility function adopted by Luce and Raiffa is delivered by the following axiom.

Axiom A.2 (von Neumann–Morgenstern payoffs) *For all* $\gamma, \zeta, \xi \in \Delta_\ell(A)$,

1. $\left\{q \in [0,1] : q\gamma + (1-q)\zeta \succsim \xi\right\}$ *and* $\left\{q \in [0,1] : \xi \succsim q\gamma + (1-q)\zeta\right\}$ *are closed sets;*
2. $\gamma \sim \zeta$ *implies* $\frac{1}{2}\gamma + \frac{1}{2}\xi \sim \frac{1}{2}\zeta + \frac{1}{2}\xi$.

The preference $\succsim$ is a weak order on $\Delta_\ell(A) \simeq \Delta(C)$, and the two above requirements consist of the continuity and independence axioms of Herstein and Milnor (1953) restricted to $\Delta_\ell(A)$. Hence, Axioms A.1 and A.2 imply the existence of a cardinally unique *payoff function* $u : C \to \mathbb{R}$ such that

$$\gamma \succsim \zeta \iff \sum_{c\in C}\gamma(c)\,u(c) \geq \sum_{c\in C}\zeta(c)\,u(c). \tag{7}$$

[23] Note, reduction guarantees that ϵ is well defined. Without reduction one should consider realization equivalence classes of mixed actions with support in the set of all sure actions. This is clearly possible, but leads to a notational cost which is not justified by the conceptual gain since mixed consequentialism will be maintained (see Proposition 3).

[24] In fact, $\rho_{\epsilon(\gamma)} \equiv \gamma$ as shown in Lemma 1 of "Appendix 2".

[25] The same happens in the Anscombe–Aumann framework where one writes γ not only to denote $\gamma \in \Delta(C)$, but also the constant act $\gamma_S \equiv \gamma$ (and $\gamma \succsim \zeta$ means $\gamma_S \succsim \zeta_S$ since the primitive preferences $\succsim$ are defined on $\Delta(C)^S$). In this case, the role of ϵ is played by the embedding $\gamma \hookrightarrow \gamma_S$ of random consequences onto constant Anscombe–Aumann acts.

These two assumptions are very common in the Anscombe–Aumann setting, where decision makers are typically assumed to be expected payoff maximizers as far as lotteries are concerned.[26] With respect to this payoff function, the (very weak) *dominance* relation of game theory

$$\alpha \geqslant_u \beta \overset{\text{def}}{\iff} \sum_{a \in A} \alpha(a) u(\rho(a,s)) \geq \sum_{a \in A} \beta(a) u(\rho(a,s)) \quad \forall s \in S$$

has a natural decision theoretic counterpart

$$\alpha \succsim_S \beta \overset{\text{def}}{\iff} \rho_\alpha(s) \succsim \rho_\beta(s) \quad \forall s \in S$$

also called *dominance*. According to both definitions, α dominates β if and only if the decision maker prefers the lottery generated by α to the lottery generated by β in every state. Axioms A.1 and A.2 guarantee that the two definitions of dominance are equivalent.

The next axiom requires dominant actions to be actually preferred.

Axiom A.3 (Dominance) *For all* $\alpha, \beta \in \Delta(A)$, $\alpha \succsim_S \beta$ *implies* $\alpha \succsim \beta$.

Under this additional axiom, the decision framework (A, S, C, ρ) can be identified with the zero-sum game (A, S, C, ρ, u) between the decision maker and nature.[27] Indeed, Axioms A.1–A.3 imply that two payoff equivalent actions are indifferent to the decision maker,[28] and the consequences c_{ij} in Eq. (1) can be replaced with their utilities $u_{ij} = u(c_{ij})$ (cf. footnote 4).

In terms of choice behavior, the assumptions we have made so far amount to saying that the decision maker is rational in the sense of Arrow (1959) and dominant actions are maximal within the feasible set, whenever they are available. To see why this is the case, denote by $\wp(\Delta(A))$ the collection of all nonempty finite subsets of $\Delta(A)$. A *rational choice correspondence* $\Gamma : \wp(\Delta(A)) \to \wp(\Delta(A))$ maps every set $\mathcal{A} \in \wp(\Delta(A))$ into one of its subsets, $\Gamma(\mathcal{A}) \subseteq \mathcal{A}$, and satisfies the weak axiom of revealed preference:

WARP *For all* $\mathcal{A}, \mathcal{B} \in \wp(\Delta(A))$, *if* $\mathcal{A} \subseteq \mathcal{B}$ *and* $\mathcal{A} \cap \Gamma(\mathcal{B}) \neq \varnothing$, *then* $\Gamma(\mathcal{A}) = \mathcal{A} \cap \Gamma(\mathcal{B})$.

A binary relation $\succsim$ is said to be generated by a rational choice correspondence Γ if and only if $\alpha \succsim \beta$ is equivalent to $\alpha \in \Gamma(\{\alpha, \beta\})$. By Theorem 3 of Arrow (1959), $\succsim$ is a weak order and

$$\Gamma(\mathcal{A}) = \left\{\alpha \in \mathcal{A} : \alpha \succsim \beta \ \forall \beta \in \mathcal{A}\right\} \quad \forall \mathcal{A} \in \wp(\Delta(A)).$$

[26] Nevertheless, we remark that this is not a requirement of the Anscombe–Aumann framework, but rather a preferential assumption; in fact, Proposition 2 holds without any assumption on preferences.

[27] As Luce and Raiffa (1957, p. 279) discuss, the choice of $-u$ as a payoff function for nature is best seen as purely formal.

[28] Formally, $\alpha \geqslant_u \beta$ and $\beta \geqslant_u \alpha$ imply $\alpha \sim \beta$.

In this case, it is natural to extend Γ to the family of all subsets $\mathcal{A}$ of $\Delta(A)$ for which $\left\{\alpha \in \mathcal{A} : \alpha \succsim \beta\ \forall \beta \in \mathcal{A}\right\} \neq \varnothing$. Finally, each $u : C \to \mathbb{R}$ induces the dominance correspondence

$$\begin{array}{rccc} \Upsilon_u\colon & \wp(\Delta(A)) & \to & \wp(\Delta(A)) \cup \{\varnothing\} \\ & \mathcal{A} & \mapsto & \{\alpha \in \mathcal{A} : \alpha \geqslant_u \beta\ \forall \beta \in \mathcal{A}\} \end{array}$$

that associates to each set $\mathcal{A}$ of available alternatives the (possibly empty) set of dominant alternatives.

Theorem 1 *Let (A, S, C, ρ) be a Luce–Raiffa framework and $\succsim$ a binary relation on $\Delta(A)$. The following conditions are equivalent:*

(i) *$\succsim$ satisfies weak order, von Neumann–Morgenstern payoffs, and dominance;*
(ii) *$\succsim$ is generated by a rational choice correspondence Γ and there exists $u \in \mathbb{R}^C$ such that*

$$\Upsilon_u(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) \quad \forall \mathcal{A} \in \wp(\Delta(A))$$

with equality if $\mathcal{A} \subseteq \Delta_\ell(A)$.

By Theorem 1, a rational decision maker first computes the expected payoffs of lotteries with consequences as prizes, and then ranks mixed actions according to dominance. At this point, he chooses dominant actions when they are available; in any case, his choices do not violate WARP.

The next result shows that such a decision maker is consequentialist, so that the binary relation $\succsim_F$ on Anscombe–Aumann acts given by (6) is a well-defined weak order satisfying risk independence and monotonicity;[29] hence, it is rational in the sense of Cerreia-Vioglio et al. (2011a).

Proposition 3 *Let (A, S, C, ρ) be a Luce–Raiffa framework and $\succsim$ a binary relation on $\Delta(A)$ that satisfies Axioms A.1–A.3, and $\mathcal{F} = F(\Delta(A))$. Then:*

1. *$\succsim$ satisfies mixed consequentialism;*
2. *$\mathcal{F}$ is a convex subset of $\Delta(C)^S$ containing all constant Anscombe–Aumann acts;*
3. *$\succsim_F$ on $\mathcal{F}$ is a weak order satisfying risk independence and monotonicity.*

All this motivates the following definition.

Definition 4 Let (A, S, C, ρ) be a Luce–Raiffa framework. A binary relation $\succsim$ on $\Delta(A)$ is a *rational preference* (*under uncertainty*) if and only if it satisfies Axioms A.1–A.3.

5 Two classical criteria

We illustrate the notions introduced so far by establishing a behavioral characterization of two important rational choice criteria: classical maxminimization, due to

29 See "Appendix 1" for a list of the most common axioms in the Anscombe–Aumann framework.

Wald (1950), and subjective expected utility maximization, due to Savage (1954). Both criteria rely on a numerical representation of preferences. As for von Neumann–Morgenstern's expected utility, the existence of such a representation is ensured by the continuity axiom of Herstein and Milnor, now assumed on the entire set $\Delta(A)$ rather than on $\Delta_\ell(A)$.

Axiom A.4 (Continuity) *For all* $\alpha, \beta, \eta \in \Delta(A)$, $\{q \in [0,1] : q\alpha + (1-q)\beta \succsim \eta\}$ *and* $\{q \in [0,1] : \eta \succsim q\alpha + (1-q)\beta\}$ *are closed sets.*

Given a Luce–Raiffa framework (A, S, C, ρ) and a rational preference $\succsim$ on $\Delta(A)$, continuity implies that for every $\alpha \in \Delta(A)$ there exists $\alpha_\ell \in \Delta(C)$ such that $\alpha \sim \alpha_\ell$. Therefore, the functional

$$\begin{array}{rccc} V: & \Delta(A) & \to & \mathbb{R} \\ & \alpha & \mapsto & \sum_{c \in C} \alpha_\ell(c)\, u(c) \end{array}$$

that associates to each mixed action its *equivalent expected payoff* $V(\alpha) = \mathbb{E}_{\alpha_\ell}[u]$ represents $\succsim$. In particular, it allows us to associate to any decision problem $(\mathcal{A}, S, C, \rho)$ an *indirect (equivalent expected) payoff*

$$v(\mathcal{A}) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}_{\alpha_\ell}[u] \tag{8}$$

and to describe the rational choice correspondence associated with $\succsim$ by

$$\Gamma(\mathcal{A}) = \arg\sup_{\alpha \in \mathcal{A}} \mathbb{E}_{\alpha_\ell}[u]$$

provided the supremum is attained.[30]

5.1 Classical maxminimization

A conservative criterion that a decision maker might adopt, when confronted with a decision problem $(\mathcal{A}, S, C, \rho)$, consists in choosing an action whose lowest expected payoff is largest, that is, an element of

$$\Gamma(\mathcal{A}) = \arg\max_{\alpha \in \mathcal{A}} \min_{s \in S} \sum_{a \in A} \alpha(a)\, u(\rho(a, s)). \tag{9}$$

We call this decision-making procedure *classical maxminimization*. It may arise when the decision maker is ignorant about the relative likelihood of the various states (see Milnor 1954, p. 49). Behaviorally, the axiom that characterizes (9), along with rationality and continuity, is an extreme version of the 'default to certainty' axiom of Gilboa et al. (2010).

[30] One should write V_u and v_u instead of V and v since these functions obviously depend on u. The subscripts are omitted since u is cardinally unique.

Axiom A.5 (Extreme caution) *For all* $\alpha \in \Delta(A)$ *and* $\gamma \in \Delta_\ell(A)$, *if* $\alpha \not\succsim_S \gamma$ *then* $\gamma \succ \alpha$.

This axiom shows how conservative the classical maxminimization criterion is: if a mixed action does not dominate a lottery, then the lottery is strictly preferred.

Theorem 2 *Let* (A, S, C, ρ) *be a Luce–Raiffa framework and* $\succsim$ *a binary relation on* $\Delta(A)$. *The following conditions are equivalent:*

(i) $\succsim$ *is a continuous rational preference that satisfies extreme caution;*
(ii) *there exists* $u \in \mathbb{R}^C$ *such that, if* $\alpha, \beta \in \Delta(A)$,

$$\alpha \succsim \beta \iff \min_{s \in S} \sum_{a \in A} \alpha(a)\, u(\rho(a, s)) \geq \min_{s \in S} \sum_{a \in A} \beta(a)\, u(\rho(a, s)). \tag{10}$$

Compared to the original axiomatization of Milnor (1954), this axiomatization is simpler and arguably more intuitive.

5.2 Subjective expected utility

In game theory rationality is often identified with the adoption of the choice criterion

$$\Gamma(\mathcal{A}) = \arg\max_{\alpha \in \mathcal{A}} \mathbb{E}_{\alpha \times \mu}[u \circ \rho]$$

where μ is a subjective probability on S that the decision maker uses to assess the *subjective expected utility*

$$\mathbb{E}_{\alpha \times \mu}[u \circ \rho] = \sum_{s \in S} \mu(s) \sum_{a \in A} \alpha(a)\, u(\rho(a, s))$$

of each mixed action $\alpha \in \Delta(A)$. This criterion, which evaluates each action by the expectation of its payoffs $u(\rho(a, s))$ with respect to the hybrid probability $\alpha(a)\mu(s)$,[31] is characterized by the independence axiom, which we now assume on the entire set $\Delta(A)$ rather than on $\Delta_\ell(A)$.

Axiom A.6 (Independence) *For all* $\alpha, \beta, \eta \in \Delta(A)$, $\alpha \sim \beta$ *implies* $\frac{1}{2}\alpha + \frac{1}{2}\eta \sim \frac{1}{2}\beta + \frac{1}{2}\eta$.

Along with Axiom A.4, clearly Axiom A.6 implies Axiom A.2. A variation on the Expected Utility Theorem of Anscombe and Aumann, together with the techniques we have developed so far, then delivers:

[31] We say that $\alpha(a)\mu(s)$ is hybrid because it is the product of a chance $\alpha(a)$ and a belief $\mu(s)$. Likewise, we use the terms *payoff* for $u(\rho(a, s))$, *expected payoff* for the (objective) average $\sum_{a \in A} \alpha(a)\, u(\rho(a, s))$ of payoffs, and *expected utility* for the (subjective) average $\sum_{s \in S} \mu(s) \sum_{a \in A} \alpha(a)\, u(\rho(a, s))$ of expected payoffs.

Theorem 3 *Let* (A, S, C, ρ) *be a Luce–Raiffa framework and* $\succsim$ *a binary relation on* $\Delta(A)$*. The following conditions are equivalent:*

(i) $\succsim$ *is a continuous rational preference that satisfies independence;*
(ii) *there exist* $u \in \mathbb{R}^C$ *and* $\mu \in \Delta(S)$ *such that, for all* $\alpha, \beta \in \Delta(A)$*,*

$$\alpha \succsim \beta \iff \mathbb{E}_{\alpha\times\mu}[u \circ \rho] \geq \mathbb{E}_{\beta\times\mu}[u \circ \rho].$$

In general, μ is not unique. As in Lehrer and Teper (2014), the set of Anscombe–Aumann acts generated by $\Delta(A)$ may be a proper subset of the whole set $\Delta(C)^S$ of all Anscombe–Aumann acts. In the Anscombe–Aumann framework, uniqueness holds under two additional assumptions:

- Preferences are not trivial, so that $\succ_C$ is not empty (see Proposition 1).
- All acts are conceivable, so that $\succsim^*$ is a weak order on the power set of S (see again Proposition 1).

The first assumption readily translates into the following axiom.

Axiom A.7 (Non-triviality) *There exist* $\alpha, \beta \in \Delta(A)$ *such that* $\alpha \succ \beta$*.*

To obtain a complete $\succsim^*$, all bets have to be conceivable.

Definition 5 A *Marschak–Radner framework* (A, S, C, ρ) is a reduced decision framework in which all bets are conceivable.

Marschak and Radner (1972) assume that all acts are conceivable, but restrict their attention to pure actions. In the first chapter, they present the counterpart of Savage's Expected Utility Theorem when pure actions, rather than Savage acts, are considered. The next theorem presents the counterpart of Anscombe–Aumann's Expected Utility Theorem, when mixed actions, rather than Anscombe–Aumann acts, are considered.

Theorem 4 *Let* (A, S, C, ρ) *be a Marschak–Radner framework and* $\succsim$ *a binary relation on* $\Delta(A)$*. The following conditions are equivalent:*

(i) $\succsim$ *is a non-trivial and continuous rational preference that satisfies independence;*
(ii) *there exist a nonconstant* $u \in \mathbb{R}^C$ *and* $\mu \in \Delta(S)$ *such that, for all* $\alpha, \beta \in \Delta(A)$*,*

$$\alpha \succsim \beta \iff \mathbb{E}_{\alpha\times\mu}[u \circ \rho] \geq \mathbb{E}_{\beta\times\mu}[u \circ \rho].$$

In this case, μ *is unique.*

In a Bayesian decision theory perspective (with no data, see Berger 1985, Ch. 1), the negative of expected utility

$$r(\alpha, \mu) = -\mathbb{E}_{\alpha\times\mu}[u \circ \rho] = \int_S \mathbb{E}_{\rho_\alpha(s)}[-u]\, d\mu(s)$$

is called *Bayes risk* of (randomized action) α under (prior distribution) μ.[32]

[32] Again one should write r_u instead of r and again the subscript is omitted because of the cardinal uniqueness of u.

6 Uncertainty aversion

6.1 Axiom and behavior

The difference between the two rational preferences described in the previous section, that is to say classical maxminimization and subjective expected utility maximization, is readily seen in the so-called Ellsberg paradox (after Ellsberg 1961). Consider a coin that a decision maker knows to be fair, as well as an urn that he knows to contain 100 black and white balls in an unknown proportion (and so there may be from 0 to 100 black balls).[33] To bet on heads/tails means that the decision maker wins \$100 if the tossed coin lands on heads/tails (and nothing otherwise); similarly, to bet on black/white means that the decision maker wins \$100 if the ball drawn from the urn is black/white (and nothing otherwise).

Ellsberg's thought experiment suggests, and a number of behavioral experiments confirm, that many decision makers are indifferent between betting on either heads or tails and are also indifferent between betting on either black or white, but they strictly prefer to bet on the coin rather than on the urn. We can represent this preference pattern as

$$\text{bet on heads} \sim \text{bet on tails} \succ \text{bet on white} \sim \text{bet on black}. \tag{11}$$

The urn draw is a version of decision problem (4), namely

ρ	B	W
a_1	\$0	\$100
a_2	\$100	\$0

(12)

where a_1 is the bet on white and a_2 is the bet on black. The corresponding Luce–Raiffa framework is obtained by adding the two sure actions b_1 and b_2, corresponding to consequences \$100 and \$0, respectively,

ρ	B	W
a_1	\$0	\$100
a_2	\$100	\$0
b_1	\$100	\$100
b_2	\$0	\$0

. (13)

In turn, the presence of sure actions allows us to express the coin toss as a lottery delivering \$100 and \$0 with even chances, that is, $\frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2} = \frac{1}{2}\delta_{b_2} + \frac{1}{2}\delta_{b_1}$ and (11) becomes

$$\frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2} \sim \frac{1}{2}\delta_{b_2} + \frac{1}{2}\delta_{b_1} \succ a_1 \sim a_2. \tag{14}$$

Consider now the following gamble proposed by Raiffa (1961): toss the coin, then bet on white if the coin lands on heads and bet on black otherwise. Formally, this gamble

[33] A fair coin here is just a random device generating two outcomes with the same 1/2 chance. The original paper of Ellsberg models it as another urn that the decision maker knows to contain 50 white balls and 50 black balls.

is represented by the mixed action $\frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$, which is easily seen to be realization equivalent to $\frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2}$.[34] Under mixed consequentialism, it follows

$$\frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2} \sim \frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2} \succ a_1 \sim a_2 \tag{15}$$

which is consistent with classical maxminimization and inconsistent with subjective expected utility.[35]

Raiffa (1961) also uses the realization equivalence between the 'compound gamble' $\frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$ and the 'coin toss' $\frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2}$ to argue in favor of their indifference. Such indifference leads him to conclude that the decision problem 'mixed urn draw' $\left(\Delta\left(\{a_1, a_2\}\right), \{B, W\}, \left\{\$0, \$100\right\}, \rho\right)$ has the same indirect payoff as the problem 'coin toss' $\left(\left\{\frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2}\right\}, \{B, W\}, \left\{\$0, \$100\right\}, \rho\right)$. Indeed, assuming wlog $u\left(\$100\right) = 1$ and $u\left(\$0\right) = 0$, according to maxminimization, we have

$$\begin{aligned} &\max_{\alpha\in\Delta(\{a_1,a_2\})} \min_{s\in\{B,W\}} \sum_{a\in A} \alpha(a)\, u(\rho(a,s)) = \frac{1}{2} \\ &= \max_{\beta\in\left\{\frac{1}{2}\delta_{b_1}+\frac{1}{2}\delta_{b_2}\right\}} \min_{s\in\{B,W\}} \sum_{a\in A} \beta(a)\, u(\rho(a,s)) \end{aligned}$$

and according to expected utility with uniform beliefs, we have again

$$\begin{aligned} &\max_{\alpha\in\Delta(\{a_1,a_2\})} \sum_{s\in S} \frac{1}{|S|} \sum_{a\in A} \alpha(a)\, u(\rho(a,s)) = \frac{1}{2} \\ &= \max_{\beta\in\left\{\frac{1}{2}\delta_{b_1}+\frac{1}{2}\delta_{b_2}\right\}} \sum_{s\in S} \frac{1}{|S|} \sum_{a\in A} \beta(a)\, u(\rho(a,s)) . \end{aligned}$$

The maxima are attained in both cases on the lhs at $\frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$ and on the rhs at $\frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2}$. But note that this observation, corresponding to the first part $\frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2} \sim \frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2}$ of (15), leaves its second part $\frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2} \succ a_1 \sim a_2$ normatively compelling because of the decision maker's ignorance regarding the relative likelihood of states B and W. Considered together the two parts suggest that the 'mixed urn draw' problem has the same value as the 'coin toss' problem which has a greater value than the 'pure urn draw' problem $\left(\{a_1, a_2\}, \{B, W\}, \left\{\$0, \$100\right\}, \rho\right)$,[36] and this is exactly the normative insight of the Ellsberg paradox. Randomization has value since it eliminates the dependence of the probability of winning on the unknown composition of the urn; randomization, in fact, makes this probability a chance, thus hedging uncertainty. On the other hand, subjective expected utility cannot benefit from randomization due to

[34] If the color of the drawn ball is white (resp. black), then the probability of winning by choosing this gamble is the chance that the coin toss assigns to betting on white (resp. black), and this chance is 1/2 since the coin is fair.

[35] Note that, under the assumptions of Theorem 3, $a_1 \sim a_2$ implies $\mu(B) = \mu(W) = 1/2$.

[36] See, again, Luce and Raiffa (1957, p. 279) and the discussion of Klibanoff (1992, p. 6).

the simple mathematical fact that the expected utility of a mixed action is never greater than the maximum of the expected utilities of all pure actions in its support.

The preference for randomization that might emerge under uncertainty is called *uncertainty aversion*.

Axiom A.8 (Uncertainty aversion) *For all* $\alpha, \beta \in \Delta(A)$, $\alpha \sim \beta$ *implies* $\frac{1}{2}\alpha + \frac{1}{2}\beta \succsim \alpha$.

This axiom, introduced in the Anscombe–Aumann framework by Schmeidler (1989), captures the idea that randomization (here in its simplest fifty-fifty form) may provide a hedge against epistemic uncertainty by substituting objective mixing for subjective mixing. Accordingly, decision makers who dislike uncertainty should (weakly) prefer to randomize. In this perspective, the observation of random choice behavior may be explained by the presence of uncertainty and aversion to it, as predicted by Raiffa in commenting Ellsberg.[37]

Under Axiom A.2, Axiom A.8 can also be regarded as capturing a preference for "smoothing" or averaging (expected) payoff profiles. For example, in (15), randomizing between a_1 and a_2 leads to the same constant lottery in every state, which in turn corresponds to a constant (expected) payoff profile.

6.2 Uncertainty averse representations

Clearly, both the maxmin preferences and the expected utility preferences we discussed above satisfy the uncertainty aversion axiom,[38] and both admit a representation $V : \Delta(A) \to \mathbb{R}$ of the form

$$V(\alpha) = \inf_{\sigma \in \Sigma} R(\alpha, \sigma) \quad \forall \alpha \in \Delta(A) \tag{16}$$

where $\Sigma \subseteq \Delta(S)$ and $R : \Delta(A) \times \Delta(S) \to (-\infty, \infty]$ is a suitable reward function whose first component is increasing in the expected utility $\mathbb{E}_{\alpha \times \sigma}[u \circ \rho]$ for each fixed σ. Specifically:

- *Classical maxminimization* is characterized by

$$R(\alpha, \sigma) = \mathbb{E}_{\alpha \times \sigma}[u \circ \rho] \quad \forall (\alpha, \sigma) \in \Delta(A) \times \Delta(S)$$

  and $\Sigma = \Delta(S)$, so that

$$V(\alpha) = \min_{\sigma \in \Delta(S)} \mathbb{E}_{\alpha \times \sigma}[u \circ \rho]. \tag{17}$$

[37] See Dominiak and Schnedler (2011) for a recent perspective on uncertainty attitudes and preference for randomization.

[38] Indeed, expected utility preferences satisfy it with indifference $\sim$ instead of weak preference $\succsim$.

- *Subjective expected utility* is characterized by

$$R(\alpha, \sigma) = \mathbb{E}_{\alpha \times \sigma}[u \circ \rho] \quad \forall (\alpha, \sigma) \in \Delta(A) \times \Delta(S)$$

and $\Sigma = \{\mu\}$ is a singleton subset of $\Delta(S)$, so that

$$V(\alpha) = \min_{\sigma \in \{\mu\}} \mathbb{E}_{\alpha \times \sigma}[u \circ \rho] = \mathbb{E}_{\alpha \times \mu}[u \circ \rho]. \tag{18}$$

In both cases, if the decision maker knew the probability of the states, he would maximize expected utility. In the former, insufficient information about the environment, along with the need to take decisions that perform well under different probabilistic scenarios, leads to a (worst-case) robust approach, that is, to (unconstrained) maxminimization. In the latter, complete information about the environment is seen as maxminimization constrained to a singleton set. Both criteria above are intuitively extreme: this leaves room for intermediate criteria of the form (16). The two cases we report below correspond to hard-constrained and soft-constrained maxminimization of expected utility.

- *Maxmin expected utility* (Gilboa and Schmeidler 1989) is characterized by

$$R(\alpha, \sigma) = \mathbb{E}_{\alpha \times \sigma}[u \circ \rho] \quad \forall (\alpha, \sigma) \in \Delta(A) \times \Delta(S)$$

and the requirement that $\Sigma \subseteq \Delta(S)$ is a compact set of probability distributions, so that

$$V(\alpha) = \min_{\sigma \in \Sigma} \mathbb{E}_{\alpha \times \sigma}[u \circ \rho]. \tag{19}$$

- *Variational preferences* (Maccheroni et al. 2006) are characterized by

$$R(\alpha, \sigma) = \mathbb{E}_{\alpha \times \sigma}[u \circ \rho] + c(\sigma) \quad \forall (\alpha, \sigma) \in \Delta(A) \times \Delta(S)$$

where $c : \Delta(S) \to [0, \infty]$ is a lower semicontinuous cost function penalizing the probability distributions and $\Sigma = \Delta(S)$, so that

$$V(\alpha) = \min_{\sigma \in \Delta(S)} \{\mathbb{E}_{\alpha \times \sigma}[u \circ \rho] + c(\sigma)\}. \tag{20}$$

For example, denoting by $\mu \in \Delta(S)$ a reference probability and by $H(\sigma || \mu)$ the relative entropy of $\sigma \in \Delta(S)$ with respect to μ, the *multiplier preferences* of Hansen and Sargent (2001, 2008) correspond to the special case of variational preferences in which $c(\sigma)$ is proportional to $H(\sigma || \mu)$,[39] while their *constraint preferences* are maxmin expected utility preferences with $\Sigma = \{\sigma \in \Delta(S) : H(\sigma || \mu) \leq \varepsilon\}$ for some $\varepsilon > 0$.

As shown by Theorem 5 in "Appendix 2", in a Marschak–Radner framework all continuous rational preferences which are uncertainty averse feature a representation of the form (16). In particular, Gilboa and Schmeidler (1989) show that in order to

[39] See Strzalecki (2011) for an axiomatization.

obtain the maxmin expected utility representation (19) it is necessary and sufficient to add an axiom guaranteeing the cardinal separation of payoffs and beliefs,[40] which we report in the form adopted by Maccheroni et al. (2006).

Axiom A.9 (C-Independence) *For all* $\alpha, \beta \in \Delta(A)$, $\gamma, \zeta \in \Delta_{\ell}(A)$, *and* $p, q \in (0, 1]$,

$$p\alpha + (1-p)\gamma \succsim p\beta + (1-p)\gamma \Longrightarrow q\alpha + (1-q)\zeta \succsim q\beta + (1-q)\zeta.$$

Finally, it is worth mentioning that Axiom A.9 delivers maxmin expected utility in any Luce–Raiffa framework, that is, even if not all bets are conceivable.[41] The weakening of Axiom A.9 obtained by requiring $p = q$ is necessary and sufficient for the variational representation (20), as shown by Maccheroni et al. (2006).

7 Concluding remarks

7.1 Mixed Savage acts

After introducing the Anscombe–Aumann framework $\Delta(C)^S$, Kreps (1988) briefly considers in Chapter 7 the framework $\Delta\left(C^S\right)$ in which the objects of choice are lotteries with Savage acts as prizes. In the language adopted here, $\Delta\left(C^S\right)$ is the set of all mixed actions corresponding to the reduced decision framework $\left(C^S, S, C, \varrho\right)$ of Savage in which all acts are conceivable and $\varrho(f, s) = f(s)$ (see footnote 16 and point 3 of Proposition 2). Note that in this case, for each $\pi \in \Delta\left(C^S\right)$,

$$\varrho_\pi(c \mid s) = \pi\left(\left\{f \in C^S : \varrho(f, s) = c\right\}\right) = \pi\left(\left\{f \in C^S : f(s) = c\right\}\right) = \sum_{\{f \in \mathrm{supp}\pi : f(s) = c\}} \pi(f).$$

Therefore, ϱ_π is the Anscombe–Aumann act that Kreps calls $\phi(\pi)$ in Problem 3 (p. 111), where he also alludes to the role of mixed consequentialism.

His purpose is to discuss the axiomatic foundation of subjective expected utility in the decision framework $\left(C^S, S, C, \varrho\right)$. In this regard our analysis shows (Theorem 3) that it is actually unnecessary to assume that all acts are conceivable in order to obtain subjective expected utility; this exercise can be done in any Luce–Raiffa framework.

More importantly, our focus is also different. Unlike Kreps, we are not seeking any axiomatic representation of preferences. Instead, the purpose of our analysis is to show

40 See Ghirardato et al. (2005).

41 Building on Gilboa et al. (2010), another way to obtain maxmin expected utility is by replacing 'extreme caution' in Theorem 2 with 'default to certainty' with respect to the unambiguous preference

$$\alpha \succsim^* \beta \iff q\alpha + (1-q)\eta \succsim q\beta + (1-q)\eta \quad \forall q \in [0,1] \text{ and } \forall \eta \in \Delta(A).$$

See Ghirardato et al. (2004).

that in applications there is little to lose and much to gain by replacing the framework of Anscombe and Aumann with that of Luce and Raiffa. In fact, tractability is preserved and interpretability is enhanced (as actions are often "few" and not naturally expressed as Savage acts), and the portability to the game theoretic/statistical/optimal control language is immediate.

7.2 Independence and preference for randomization

In a Luce–Raiffa framework, the independence and uncertainty aversion Axioms A.6 and A.8 are expressed for mixed actions, that is, for elements of $\Delta(A)$. Mathematically, these axioms have the same form as the independence and preference for randomization axioms of risk theory, which are expressed for lotteries. In general, we maintain independence on $\Delta_\ell(A) \simeq \Delta(C)$, the set of lotteries with consequences as prizes, and preference for randomization on $\Delta(A)$. This choice is motivated by the consequentialist approach of this paper and by the aim of remaining as close as possible to the uses and conventions of Knightian uncertainty modeling. In particular, independence on $\Delta_\ell(A)$ is required by Axiom A.2.2, which is the analogue in a Luce–Raiffa framework of the standard risk independence axiom in an Anscombe–Aumann framework. The presence of states and the assumption of mixed consequentialism make independence on $\Delta(A)$ a strong requirement since it prevents hedging of payoffs. Its restriction to $\Delta_\ell(A)$ is not vulnerable to this critique since hedging considerations are excluded by the state-independence of the distribution of consequences featured by the elements of $\Delta_\ell(A)$. On the contrary, hedging considerations are exactly what justifies uncertainty aversion on $\Delta(A)$. In this respect, note that preference for randomization as a counterpart of uncertainty aversion is conceptually more compelling under mixed consequentialism, that is, when different mixed actions leading to the same probability distribution of consequences are treated as equivalent. While our perspective here is mainly normative, actual decision makers might treat the same probability distribution over consequences, obtained by different mixed actions, as different outcomes of choice (see e.g., Dominiak and Schnedler 2011).[42] One reason for different treatments may be the induced correlation of consequences generated by different randomizations (see e.g., Epstein and Halevy 2014).[43] Consider the example in (13) again and the mixed actions $\alpha = \frac{1}{2}\delta_{a_1} + \frac{1}{2}\delta_{a_2}$ and $\beta = \frac{1}{2}\delta_{b_1} + \frac{1}{2}\delta_{b_2}$, as we have already observed, they are clearly distinct (they have disjoint support), yet they induce the same distribution of consequences in every state (they are realization equivalent). However, if we look at the induced correlations of consequences, that is at the correlation of the maps $\rho_B : A \to C$ and $\rho_W : A \to C$, we find that this correlation is negative for α and positive for β, thus β can induce "more regret" if the outcome of the randomization is b_2 (and more "elation" in case b_1 obtains). In other words, mixed

[42] As suggested by a referee, even on a purely theoretical level, realization equivalence appears to be very convincing when one considers a framework consisting of unambiguous sets of states, actions, and consequences. If there is ambiguity or a degree of unawareness regarding these primitive sets, it is no longer clear whether realization equivalence can be maintained as a prerequisite of consequentialism (see e.g., Karni and Viero 2014).

[43] Again, we thank a referee for this observation.

consequentialism requires a form of indifference to correlation which is needed for the preferential equivalence of the frameworks of Anscombe and Aumann and of Luce and Raiffa. See also Saito (2013) for a menu choice approach aimed at eliciting the extent to which the decision maker believes in the hedging effects of randomization.

In risk theory, preference for randomization may also capture aversion to uncertainty regarding the value of consequences, as discussed by Maccheroni (2002) and Cerreia-Vioglio (2009). This kind of uncertainty can also be viewed as uncertainty about a subjective state space, a concept introduced by Kreps (1979) and Dekel et al. (2001). Along these lines, we could also abandon independence on $\Delta_\ell(A)$, use the techniques of Cerreia-Vioglio (2009), and obtain a representation featuring both states of the world and subjective states.

Finally, observe that, while our analysis relies on the presence of randomization, an altogether different 'utilitarian' perspective on the Anscombe–Aumann framework that dispenses with randomization has been pursued by Ghirardato et al. (2003).

7.3 Timing and commitment

The immersion $\alpha \overset{F}{\mapsto} \rho_\alpha$ of Sect. 3 may seem to associate an ex ante notion of randomization (mixed actions) to an ex post one (Anscombe–Aumann acts). Commitment, however, renders the distinction between ex ante and ex post randomization immaterial. Specifically, in the Anscombe–Aumann framework an act $f : S \to \Delta(C)$ is a non-random object of choice with random consequences $f(s)$. Randomization is usually interpreted to occur *ex post*: the decision maker commits to f, 'observes' the realized state s, then 'observes' the consequence c generated by the random mechanism $f(s)$, and receives c.[44] By contrast, the mixed actions we consider are random objects of choice by definition. Randomization might be interpreted to occur either *ex ante* or *ex post*: in the former case, the decision maker commits to α, 'observes' the realized action a, then 'observes' the realized state s, and receives the consequence $c = \rho(a, s)$; in the latter, the decision maker commits to α, 'observes' the realized state s, then 'observes' the realized action a, and receives the consequence $c = \rho(a, s)$.[45] Clearly the latter interpretation conforms to that of Anscombe–Aumann acts while the former does not. However commitment, that is, the impossibility of changing the action selected by the random device, makes such a distinction immaterial.

Proposition 2.2 helps, *inter alia*, to further clarify this issue. Since each mixed action α can be written as a convex combination $\alpha = \sum_{a \in A} \alpha(a)\delta_a$ of point-masses, since F is affine,

$$\rho_\alpha(s) = \sum_{a \in A} \alpha(a)\rho_{\delta_a}(s) \quad \forall s \in S \tag{21}$$

[44] We say 'interpreted' since this timeline and the timed disclosure of information are unmodeled. In principle, one could think of the decision maker committing to f and receiving the outcome of the resulting process.

[45] Again the timeline and the timed disclosure of information are unmodeled. In principle, one could think that the decision maker commits to α and receives the outcome of the resulting process.

and $\rho_\alpha(s)$ is the chance distribution on C induced by the 'ex ante randomization' of actions a with probabilities $\alpha(a)$ if state s obtains. Consider the act $f_\alpha : S \to \Delta(C)$ given by

$$f_\alpha(s) = \sum_{a \in A} \alpha(a)\, \delta_{\rho(a,s)} \quad \forall s \in S \tag{22}$$

Now $f_\alpha(s)$ is the chance distribution on C induced by the 'ex post randomization' of consequences $\rho(a, s)$ with probabilities $\alpha(a)$ if state s obtains. Proposition 2.1 implies that $\rho_\alpha = f_\alpha$, thus showing that it is impossible to draw the distinction between 'ex ante' and 'ex post' in our abstract setup or in that of Fishburn (1970).[46]

In a richer setup, with two explicit layers of randomization, Anscombe and Aumann (1963) are able to formalize this issue and assume that 'it is immaterial whether the wheel is spun before or after the race'.[47] We do not pursue this matter any further, but we refer to Sarin and Wakker (1997) and Wakker (2010, Ch. 4) for a discussion on single-stage versus multi-stage randomization. See also Eichberger et al. (2013) for a recent explicit dynamic choice model.

Appendix 1: Anscombe–Aumann axioms

Here we report the axioms that are usually stated in the Anscombe–Aumann framework. In this section $\succsim_{\mathcal{F}}$ and $\succsim^{\#}_{\mathcal{F}}$ are binary relations on a convex subset $\mathcal{F}$ of $\Delta(C)^S$ that contains the set $\mathcal{C} \simeq \Delta(C)$ of all constant acts.

Axiom AA.1 (Weak order) *$\succsim_{\mathcal{F}}$ is complete and transitive.*

Axiom AA.2 (Risk independence) *For all $\gamma, \zeta, \xi \in \mathcal{C}$, $\gamma \sim_{\mathcal{F}} \zeta$ implies $\frac{1}{2}\gamma + \frac{1}{2}\xi \sim_{\mathcal{F}} \frac{1}{2}\zeta + \frac{1}{2}\xi$.*

Axiom AA.3 (Monotonicity) *For all $f, g \in \mathcal{F}$, if $f(s) \succsim_{\mathcal{F}} g(s)$ for all $s \in S$, then $f \succsim_{\mathcal{F}} g$.*

Axiom AA.4 (Continuity) *For all $f, g, h \in \mathcal{F}$, $\{q \in [0, 1] : qf + (1-q)g \succsim_{\mathcal{F}} h\}$ and $\{q \in [0, 1] : h \succsim_{\mathcal{F}} qf + (1-q)g\}$ are closed sets.*

Axiom AA.5 (Default to certainty) *For all $f \in \mathcal{F}$ and $\gamma \in \mathcal{C}$, $f \not\succsim^{\#}_{\mathcal{F}} \gamma$ implies $\gamma \succ_{\mathcal{F}} f$.*

Axiom AA.6 (Independence) *For all $f, g, h \in \mathcal{F}$, $f \sim_{\mathcal{F}} g$ implies $\frac{1}{2}f + \frac{1}{2}h \sim_{\mathcal{F}} \frac{1}{2}g + \frac{1}{2}h$.*

Axiom AA.7 (Non-triviality) *There exist $f, g \in \mathcal{F}$ such that $f \succ_{\mathcal{F}} g$.*

46 See Kuzmics (2012) on this issue.

47 Seo (2009) studies the consequences of weakening this assumption.

Axiom AA.8 (Uncertainty aversion) *For all* $f, g \in \mathcal{F}$ *and* $q \in (0, 1)$, $f \sim_{\mathcal{F}} g$ *implies* $qf + (1 - q)\, g \succsim_{\mathcal{F}} f$.

Axiom AA.9 (C-Independence) *For all* $f, g \in \mathcal{F}$, $\gamma \in \mathcal{C}$, *and* $q \in (0, 1]$,

$$f \succsim_{\mathcal{F}} g \iff qf + (1 - q)\gamma \succsim_{\mathcal{F}} qg + (1 - q)\gamma.$$

Appendix 2: Proofs and related material

Throughout this appendix, if $\gamma \in \Delta(C)$ and $u \in \mathbb{R}^C$, we indifferently write

$$\text{either} \sum_{c \in C} \gamma(c)\, u(c) \ \text{ or } \mathbb{E}_\gamma[u] \ \text{ or } u(\gamma).$$

Proof of Proposition 1 1. For each $a \in A$, $\rho_a = \rho_a$ so that $a \approx a$ and by consequentialism $a \sim_A a$. Therefore $\succsim_A$ is reflexive and transitive, that is, a preorder. Conversely, let (A, S, C, ρ) be reduced and $\succsim_A$ be a preorder. For every $a, b \in A$, $a \approx b$ implies (because of reduction) $a = b$, and reflexivity of $\succsim_A$ delivers $a \sim_A b$. Therefore $\succsim_A$ satisfies consequentialism.

2. Let $c, d \in C$ and $a_c, b_c, a_d, b_d \in A$, not necessarily distinct, be such that

$$\rho(a_c, \cdot) = \rho(b_c, \cdot) \equiv c \text{ and } \rho(a_d, \cdot) = \rho(b_d, \cdot) \equiv d. \tag{23}$$

These sure actions exist since all sure actions are conceivable. By consequentialism, $\succsim_A$ is a preorder and (23) implies $a_c \sim_A b_c$ and $a_d \sim_A b_d$. Therefore, by transitivity of $\succsim_A$,

$$a_c \succsim_A a_d \iff b_c \succsim_A b_d$$

and $\succsim_C$ is well defined.[48] Reflexivity and transitivity of $\succsim_C$ follow from reflexivity and transitivity of $\succsim_A$.

3. The proof is similar to the one of the previous point, hence it is left to the reader. □

Proof of Proposition 2 1. Fix $s \in S$. For each $c \in C$,

$$\rho_{\delta_a}(c \mid s) = \delta_a(\{b \in A : \rho(b, s) = c\}) = \begin{cases} 1 & c = \rho(a, s) \\ 0 & \text{otherwise} \end{cases} = \delta_{\rho(a,s)}(c)$$

that is $\rho_{\delta_a}(s) = \delta_{\rho(a,s)}$. Write x instead of δ_x if either $x \in A$ or $x \in C$, then

$$F(a) = F(\delta_a) = \begin{bmatrix} \rho_{\delta_a}(s_1) \\ \rho_{\delta_a}(s_2) \\ \vdots \\ \rho_{\delta_a}(s_n) \end{bmatrix} = \begin{bmatrix} \delta_{\rho(a,s_1)} \\ \delta_{\rho(a,s_2)} \\ \vdots \\ \delta_{\rho(a,s_n)} \end{bmatrix} = \begin{bmatrix} \rho(a, s_1) \\ \rho(a, s_2) \\ \vdots \\ \rho(a, s_n) \end{bmatrix} = \rho_a$$

as desired.

[48] Specifically, $\succsim_C = \{(c, d) \in C \times C : a_c \succsim a_d$ for some/all $a_c, a_d \in A$ such that $\rho_{a_c} \equiv c, \rho_{a_d} \equiv d\}$.

2. Fix $s \in S$. For each $c \in C$,

$$\begin{aligned}\rho_{q\alpha+(1-q)\beta}(c \mid s) &= (q\alpha + (1-q)\beta)(\{a \in A : \rho(a,s) = c\}) \\ &= q\alpha(\{a \in A : \rho(a,s) = c\}) + (1-q)\beta(\{a \in A : \rho(a,s) = c\}) \\ &= q\rho_\alpha(c \mid s) + (1-q)\rho_\beta(c \mid s)\end{aligned}$$

   that is, $\rho_{q\alpha+(1-q)\beta} = q\rho_\alpha + (1-q)\rho_\beta$.
3. Let D be an arbitrary nonempty subset of C. Then $\Delta(D)$ is convex in $\mathbb{R}^D$, and its extreme points are the point-masses $\{\delta_d\}_{d\in D}$. Therefore $\Delta(D)^S$ is convex too. Next we show that its extreme points are the vectors of point-masses. If $g \in \Delta(D)^S$ is not extreme there exist $g', g'' \in \Delta(D)^S$ with $g' \neq g''$, say $0 \leq g'(d \mid s) < g''(d \mid s) \leq 1$, and $q \in (0,1)$ such that $g = qg' + (1-q)g''$. Then $g(d \mid s) = qg'(d \mid s) + (1-q)g''(d \mid s) \in (0,1)$ and $g(s)$ is not a point-mass. Conversely, if $g \in \Delta(D)^S$ is not a vector of point-masses, then $g(s)$ is not a point-mass for some $s \in S$. Therefore, there exist $g'(s), g''(s) \in \Delta(D)$ with $g'(s) \neq g''(s)$ and $q \in (0,1)$ such that $g(s) = qg'(s) + (1-q)g''(s)$. But this implies that $g = qg' + (1-q)g''$ where g' (resp. g'') is obtained replacing the s-th element $g(s)$ of the vector g with $g'(s)$ (resp. $g''(s)$). Thus g is not extreme.
   In particular, the generic extreme point of $\Delta(D)^S$ is

$$\begin{bmatrix}\delta_{e(s_1)} \\ \delta_{e(s_2)} \\ \vdots \\ \delta_{e(s_n)}\end{bmatrix} \text{ with } \begin{bmatrix}e(s_1) \\ e(s_2) \\ \vdots \\ e(s_n)\end{bmatrix} = e \in D^S.$$

If part. Assume $\{\rho_a\}_{a\in A} = C^S$. For each $e \in C^S$, there exists $a_e \in A$ such that $\rho_{a_e} = e$. For each $f \in \Delta(C)^S$, set $D = \bigcup_{s\in S} \operatorname{supp} f(s)$. Then $\Delta(D)^S$ is compact in $\mathbb{R}^D$ since D is finite. By the Krein–Milman Theorem, there exists $\phi \in \Delta\left(D^S\right)$ such that

$$f = \sum_{e\in D^S} \phi(e) \begin{bmatrix}\delta_{e(s_1)} \\ \delta_{e(s_2)} \\ \vdots \\ \delta_{e(s_n)}\end{bmatrix} = \sum_{e\in D^S} \phi(e) \begin{bmatrix}\delta_{\rho(a_e,s_1)} \\ \delta_{\rho(a_e,s_2)} \\ \vdots \\ \delta_{\rho(a_e,s_n)}\end{bmatrix} = \sum_{e\in D^S} \phi(e) \begin{bmatrix}\rho_{\delta_{a_e}}(s_1) \\ \rho_{\delta_{a_e}}(s_2) \\ \vdots \\ \rho_{\delta_{a_e}}(s_n)\end{bmatrix} \tag{24}$$

where the last equality follows from point 1 of this proposition. By (24), and since F is affine (point 2 of this proposition),

$$f = \sum_{e\in D^S} \phi(e)\, \rho_{\delta_{a_e}} = \sum_{e\in D^S} \phi(e)\, F\left(\delta_{a_e}\right) = F\left(\sum_{e\in D^S} \phi(e)\, \delta_{a_e}\right)$$

where $\sum_{e \in D^S} \phi(e)\, \delta_{a_e} \in \Delta(A)$ since D^S is finite and $\Delta(A)$ is convex. Therefore, $\digamma$ is onto.

Only if part. Assume $\digamma(\Delta(A)) = \Delta(C)^S$. For each $e \in C^S$, there exists $\alpha \in \Delta(A)$ such that

$$\rho_\alpha(s) = \delta_{e(s)} \quad \forall s \in S. \tag{25}$$

Partition the finite support of α into realization equivalence classes so that

$$\operatorname{supp}\alpha = \bigsqcup_{i=1}^{k} A_i$$

with $\rho_{a_i} = \rho_{a_i'}$ for all $a_i, a_i' \in A_i$ and all $i = 1, 2, \ldots, k$ and $\rho_{a_i} \neq \rho_{a_j}$ if $a_i \in A_i$, $a_j \in A_j$ and $i \neq j$. Moreover, for each $i = 1, 2, \ldots, k$ arbitrarily select $b_i \in A_i$. By (25), for each $s \in S$,

$$\delta_{e(s)} = \sum_{a \in \operatorname{supp}\alpha} \alpha(a)\, \rho_{\delta_a}(s) = \sum_{i=1}^{k} \left(\sum_{a_i \in A_i} \alpha(a_i)\, \rho_{\delta_{a_i}}(s) \right)$$

but, for each $i = 1, 2, \ldots, k$, and all $a_i \in A_i$,

$$\rho_{\delta_{a_i}}(s) = \delta_{\rho(a_i, s)} = \delta_{\rho(b_i, s)} = \rho_{\delta_{b_i}}(s).$$

Therefore, setting $\beta(b_i) = \sum_{a_i \in A_i} \alpha(a_i)$ for all $i = 1, 2, \ldots, k$ and $\beta(a) = 0$ if $a \in A \setminus \{b_1, \ldots, b_k\}$, $\beta \in \Delta(A)$ is such that

$$\delta_{e(s)} = \sum_{i=1}^{k} \left(\sum_{a_i \in A_i} \alpha(a_i) \right) \rho_{\delta_{b_i}}(s) = \sum_{i=1}^{k} \beta(b_i)\, \rho_{\delta_{b_i}}(s) = \sum_{i=1}^{k} \beta(b_i)\, \delta_{\rho(b_i, s)}$$

and

$$\begin{bmatrix} \delta_{e(s_1)} \\ \delta_{e(s_2)} \\ \vdots \\ \delta_{e(s_n)} \end{bmatrix} = \sum_{i=1}^{k} \beta(b_i) \begin{bmatrix} \delta_{\rho(b_i, s_1)} \\ \delta_{\rho(b_i, s_2)} \\ \vdots \\ \delta_{\rho(b_i, s_n)} \end{bmatrix}. \tag{26}$$

The summands $\left[\delta_{\rho(b_i, s_1)} \ \ldots \ \delta_{\rho(b_i, s_n)}\right]^\intercal$ are distinct (extreme) points in $\Delta(C)^S$ and $\left[\delta_{e(s_1)} \ \ldots \ \delta_{e(s_n)}\right]^\intercal$ is an extreme point of $\Delta(C)^S$. Then $\beta = \delta_b$ for some $b \in \operatorname{supp}\beta \subseteq A$, and (26) implies $e = \rho_b$. The arbitrary choice of e allows us to conclude that $C^S \subseteq \{\rho_a\}_{a \in A}$ and the converse inclusion is trivial. □

Proof of Corollary 1 If $\beta \neq \delta_a$ there exists $b \neq a$ such that $\beta(b) \neq 0$. Since the framework is reduced $\rho_b \neq \rho_a$ and there exists $s \in S$ such that $\rho(a, s) \neq \rho(b, s)$. Setting $c = \rho(b, s)$, it follows

$$\rho_\beta(c \mid s) \geq \beta(b) > 0 = \delta_{\rho(a,s)}(c) = \rho_{\delta_a}(c \mid s)$$

and hence $\rho_\beta \neq \rho_{\delta_a}$ and $\delta_a \not\approx \beta$. By contrapositive $\delta_a \approx \beta \Longrightarrow \beta = \delta_a$, and the converse is trivial. □

Lemma 1 *Let (A, S, C, ρ) be a decision framework, $\alpha \in \Delta(A)$, $u \in \mathbb{R}^C$, and $\mu \in \Delta(S)$. Then:*

1. $\rho_\alpha(s) = \sum_{a \in A} \alpha(a)\, \delta_{\rho(a,s)}$ *for all $s \in S$;*
2. $u(\rho_\alpha(s)) = \mathbb{E}_{\rho_\alpha(s)}[u] = \sum_{a \in A} \alpha(a)\, u(\rho(a,s))$*;*
3. $\int_S u(\rho_\alpha(s))\, d\mu(s) = \mathbb{E}_{\alpha \times \mu}[u \circ \rho]$.

Moreover, if (A, S, C, ρ) is a Luce–Raiffa framework, then

$$\begin{array}{rccc} \epsilon\colon & \Delta(C) & \to & \Delta_\ell(A) \\ & \sum_{c \in C} \gamma(c)\, \delta_c & \mapsto & \sum_{c \in C} \gamma(c)\, \delta_{cS} \end{array}$$

is affine and bijective. In particular:

4. *for each $\alpha = \sum_{c \in C} \alpha(cS)\, \delta_{cS} \in \Delta_\ell(A)$ the only $\gamma \in \Delta(C)$ such that $\epsilon(\gamma) = \alpha$ is $\epsilon^{-1}(\alpha) = \sum_{c \in C} \alpha(cS)\, \delta_c$;*
5. *for each $\gamma \in \Delta(C)$ and each $s \in S$, $\rho_{\epsilon(\gamma)}(s) = \gamma$.*

Proof 1. Follows from points 1 and 2 of Proposition 2. Specifically,

$$\rho_\alpha = F\left(\sum_{a \in A} \alpha(a)\, \delta_a\right) = \sum_{a \in A} \alpha(a)\, F(\delta_a) = \sum_{a \in A} \alpha(a)\, \rho_{\delta_a}$$

and therefore for each $s \in S$

$$\rho_\alpha(s) = \sum_{a \in A} \alpha(a)\, \rho_{\delta_a}(s) = \sum_{a \in A} \alpha(a)\, \delta_{\rho(a,s)}.$$

2. By definition, for all $s \in S$,

$$u(\rho_\alpha(s)) = \mathbb{E}_{\rho_\alpha(s)}[u] = \mathbb{E}_{\sum_{a \in A} \alpha(a) \delta_{\rho(a,s)}}[u] = \sum_{a \in A} \alpha(a)\, \mathbb{E}_{\delta_{\rho(a,s)}}[u]$$
$$= \sum_{a \in A} \alpha(a)\, u(\rho(a,s)).$$

3. By definition,

$$\int_S u(\rho_\alpha(s))\, d\mu(s) = \sum_{s \in S} \mu(s) \sum_{a \in A} \alpha(a)\, u(\rho(a,s)) = \sum_{a \in A} \sum_{s \in S} \alpha(a)\, \mu(s)\, u(\rho(a,s))$$
$$= \mathbb{E}_{\alpha \times \mu}[u \circ \rho].$$

If (A, S, C, ρ) is a Luce–Raiffa framework, since all sure actions are conceivable for each $c \in C$ there exists a sure action $a_c \in A$ such that $\rho(a_c, \cdot) \equiv c$, reduction instead guarantees that such sure action is unique and we denote it by cS. In other words, the map $c \mapsto cS$ is well defined. It is also easy to check that its range is the set of all sure actions and that the map is injective. Therefore $c \hookrightarrow cS$ is an embedding of C onto the set of all sure actions. The fact that ϵ is affine and bijective immediately follows.

4. Clearly $\gamma = \sum_{c \in C} \alpha(cS)\,\delta_c \in \Delta(C)$ and $\gamma(c) = \alpha(cS)$ for all $c \in C$, by definition of ϵ it follows $\epsilon(\gamma) = \sum_{c \in C} \gamma(c)\,\delta_{cS} = \alpha$.
5. By definition $\epsilon(\gamma) = \sum_{c \in \operatorname{supp}\gamma} \gamma(c)\,\delta_{cS}$, therefore, for each $s \in S$,

$$\begin{aligned}\rho_{\epsilon(\gamma)}(s) &= \rho_{\sum_{c \in \operatorname{supp}\gamma} \gamma(c)\delta_{cS}}(s) = \sum_{c \in \operatorname{supp}\gamma} \gamma(c)\,\rho_{\delta_{cS}}(s) \\ &= \sum_{c \in \operatorname{supp}\gamma} \gamma(c)\,\delta_{\rho(cS,s)} = \sum_{c \in \operatorname{supp}\gamma} \gamma(c)\,\delta_c = \gamma.\end{aligned} \tag{27}$$

□

Proof of Theorem 1 (i) $\Longrightarrow$ (ii). Since $\succsim$ satisfies Axiom A.1, then

$$\Gamma_{\succsim}(\mathcal{A}) = \left\{\alpha \in \mathcal{A} : \alpha \succsim \beta \ \forall \beta \in \mathcal{A}\right\} \quad \forall \mathcal{A} \in \wp(\Delta(A))$$

is a rational choice correspondence and generates $\succsim$. Since $\succsim$ also satisfies Axiom A.2, then $\succsim_{\Delta(C)}$ satisfies Axioms 1, 2, and 3 of Herstein and Milnor (1953) and there exists $u \in \mathbb{R}^C$ such that, if $\gamma, \zeta \in \Delta(C)$, then

$$\gamma \succsim_{\Delta(C)} \zeta \iff \sum_{c \in C} \gamma(c)\,u(c) \geq \sum_{c \in C} \zeta(c)\,u(c)$$

and, by definition of $\succsim_{\Delta(C)}$ this means

$$\epsilon(\gamma) \succsim \epsilon(\zeta) \iff \mathbb{E}_\gamma[u] \geq \mathbb{E}_\zeta[u].$$

Therefore, if $\alpha, \beta \in \Delta(A)$, then

$$\begin{aligned}\alpha \succcurlyeq_S \beta &\iff \rho_\alpha(s) \succsim_{\Delta(C)} \rho_\beta(s) \quad \forall s \in S \\ &\iff u(\rho_\alpha(s)) \geq u\left(\rho_\beta(s)\right) \quad \forall s \in S \\ &\iff \sum_{a \in A} \alpha(a)\,u(\rho(a,s)) \geq \sum_{a \in A} \beta(a)\,u(\rho(a,s)) \quad \forall s \in S \\ &\iff \alpha \geqslant_u \beta.\end{aligned}$$

Now Axiom A.3 implies

$$\alpha \geqslant_u \beta \implies \alpha \succsim \beta \tag{28}$$

moreover, if $\alpha, \beta \in \Delta_\ell(A)$, then $\alpha = \epsilon(\gamma)$ and $\beta = \epsilon(\zeta)$ for some $\gamma, \zeta \in \Delta(C)$, and

$$\alpha \succsim \beta \iff \gamma \succsim_{\Delta(C)} \zeta \iff \rho_{\epsilon(\gamma)}(s) \succsim_{\Delta(C)} \rho_{\epsilon(\zeta)}(s) \quad \forall s \in S \iff \alpha \geqslant_u \beta$$

that is $\succsim$, $\succcurlyeq_S$, and $\geqslant_u$ coincide on $\Delta_\ell(A)$ where

$$\begin{aligned} \alpha \succsim \beta &\iff \gamma \succsim_{\Delta(C)} \zeta \iff \mathbb{E}_\gamma[u] \geq \mathbb{E}_\zeta[u] \\ &\iff \sum_{c\in C} \alpha(cS)\, u(c) \geq \sum_{c\in C} \beta(cS)\, u(c). \end{aligned}$$

This immediately implies, for each $\mathcal{A} \in \wp(\Delta(A))$ such that $\mathcal{A} \subseteq \Delta_\ell(A)$,

$$\begin{aligned} \Gamma_{\succsim}(\mathcal{A}) &= \left\{\alpha \in \mathcal{A} : \alpha \succsim \beta\ \forall \beta \in \mathcal{A}\right\} \\ &= \{\alpha \in \mathcal{A} : \alpha \geqslant_u \beta\ \forall \beta \in \mathcal{A}\} = \Upsilon_u(\mathcal{A}) \end{aligned}$$

For a generic $\mathcal{A} \in \wp(\Delta(A))$, by (28),

$$\begin{aligned} \Upsilon_u(\mathcal{A}) &= \{\alpha \in \mathcal{A} : \alpha \geqslant_u \beta\ \forall \beta \in \mathcal{A}\} \\ &\subseteq \left\{\alpha \in \mathcal{A} : \alpha \succsim \beta\ \forall \beta \in \mathcal{A}\right\} = \Gamma_{\succsim}(\mathcal{A}). \end{aligned}$$

(ii) $\Longrightarrow$ (i). If $\succsim$ is generated by a rational choice correspondence $\Gamma : \wp(\Delta(A)) \to \wp(\Delta(A))$, then $\succsim$ satisfies Axiom A.1. By (ii) there exists also $u \in \mathbb{R}^C$ such that:

$$\Upsilon_u(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) \text{ for all } \mathcal{A} \in \wp(\Delta(A)) \text{ and equality holds if } \mathcal{A} \subseteq \Delta_\ell(A).$$

Therefore, if $\alpha, \beta \in \Delta(A)$,

$$\begin{aligned} \alpha \succcurlyeq_S \beta &\iff \rho_\alpha(s) \succsim_{\Delta(C)} \rho_\beta(s) \quad \forall s \in S \\ &\iff \epsilon(\rho_\alpha(s)) \succsim \epsilon\left(\rho_\beta(s)\right) \quad \forall s \in S \\ &\iff \epsilon(\rho_\alpha(s)) \in \Gamma\left(\left\{\epsilon(\rho_\alpha(s)), \epsilon\left(\rho_\beta(s)\right)\right\}\right) \quad \forall s \in S \end{aligned}$$

but $\left\{\epsilon(\rho_\alpha(s)), \epsilon\left(\rho_\beta(s)\right)\right\} \subseteq \Delta_\ell(A)$ for all $s \in S$, that is

$$\begin{aligned} \alpha \succcurlyeq_S \beta &\iff \epsilon(\rho_\alpha(s)) \in \Upsilon_u\left(\left\{\epsilon(\rho_\alpha(s)), \epsilon\left(\rho_\beta(s)\right)\right\}\right) \quad \forall s \in S \\ &\iff \epsilon(\rho_\alpha(s)) \geqslant_u \epsilon\left(\rho_\beta(s)\right) \quad \forall s \in S. \end{aligned} \tag{29}$$

Moreover, given $\eta, \kappa \in \Delta(A)$

$$\begin{aligned} \eta \geqslant_u \kappa &\iff \sum_{a\in A} \eta(a)\, u(\rho(a,w)) \geq \sum_{a\in A} \kappa(a)\, u(\rho(a,w)) \quad \forall w \in S \\ &\iff \mathbb{E}_{\rho_\eta(w)}[u] \geq \mathbb{E}_{\rho_\kappa(w)}[u] \quad \forall w \in S \end{aligned}$$

and (29) becomes

$$\alpha \succcurlyeq_S \beta \iff \mathbb{E}_{\rho_{\epsilon(\rho_\alpha(s))}(w)}[u] \geq \mathbb{E}_{\rho_{\epsilon\left(\rho_\beta(s)\right)}(w)}[u] \quad \forall w, s \in S$$

but, $\rho_{\epsilon(\gamma)} \equiv \gamma$ for all $\gamma \in \Delta(C)$, therefore

$$\begin{aligned}\alpha \succcurlyeq_S \beta &\iff \mathbb{E}_{\rho_\alpha(s)}[u] \geq \mathbb{E}_{\rho_\beta(s)}[u] \quad \forall w, s \in S \\ &\iff \sum_{a\in A} \alpha(a)\, u(\rho(a,s)) \geq \sum_{a\in A} \beta(a)\, u(\rho(a,s)) \quad \forall s \in S \\ &\iff \alpha \geqslant_u \beta \implies \alpha \in \Upsilon_u(\{\alpha,\beta\}) \subseteq \Gamma(\{\alpha,\beta\}) \implies \alpha \succsim \beta \end{aligned} \tag{30}$$

and $\succsim$ satisfies Axiom A.3.

Finally, if $\alpha, \beta \in \Delta_\ell(A)$, then $\alpha = \epsilon(\gamma)$ and $\beta = \epsilon(\zeta)$ for some $\gamma, \zeta \in \Delta(C)$, and

$$\begin{aligned}\alpha \succsim \beta &\iff \epsilon(\gamma) \succsim \epsilon(\zeta) \iff \gamma \succsim_{\Delta(C)} \zeta \\ &\iff \rho_{\epsilon(\gamma)}(s) \succsim_{\Delta(C)} \rho_{\epsilon(\zeta)}(s)\, \forall s \in S \iff \alpha \succcurlyeq_S \beta\end{aligned}$$

but suppα, supp$\beta \subseteq \{cS\}_{c\in C}$ and by (30)

$$\begin{aligned}\alpha \succsim \beta &\iff \sum_{a\in A} \alpha(a)\, u(\rho(a,s)) \geq \sum_{a\in A} \beta(a)\, u(\rho(a,s)) \quad \forall s \in S \\ &\iff \sum_{c\in C} \alpha(cS)\, u(\rho(cS,s)) \geq \sum_{c\in C} \beta(cS)\, u(\rho(cS,s)) \quad \forall s \in S \\ &\iff \sum_{c\in C} \alpha(cS)\, u(c) \geq \sum_{c\in C} \beta(cS)\, u(c)\end{aligned}$$

and so $\succsim$ satisfies Axioms 2 and 3 of Herstein and Milnor (1953) on $\Delta_\ell(A)$, that is, Axiom A.2. □

Recall that if $\gamma \in \Delta(C)$ and $u \in \mathbb{R}^C$ we indifferently write $\mathbb{E}_\gamma[u]$ or $u(\gamma)$; with a similar abuse of notation, if $f \in \Delta(C)^S$ we denote by $u(f)$ the element of $\mathbb{R}^S$ defined by

$$u(f) = \begin{bmatrix} u(f(s_1)) \\ u(f(s_2)) \\ \vdots \\ u(f(s_n)) \end{bmatrix} = \begin{bmatrix} \mathbb{E}_{f(s_1)}[u] \\ \mathbb{E}_{f(s_2)}[u] \\ \vdots \\ \mathbb{E}_{f(s_n)}[u] \end{bmatrix}.$$

Lemma 2 *Let (A, S, C, ρ) be a Luce–Raiffa framework, $\mathcal{F} = F(\Delta(A))$, $u \in \mathbb{R}^C$, and $\succsim$ be a preorder on $\Delta(A)$. Then:*

1. *$\mathcal{F}$ is a convex subset of $\Delta(C)^S$ containing all constant Anscombe–Aumann acts;*
2. *if $\succsim$ satisfies Axiom A.3, then $\succsim$ satisfies mixed consequentialism;*

3. *if all bets are conceivable,* $\{u(f) : f \in \mathcal{F}\} = (\mathrm{co}(u(C)))^S = u(\Delta(C))^S$.

Moreover, if $\succsim$ *satisfies mixed consequentialism, then* $\succsim_F$ *is a well-defined preorder on* $\mathcal{F}$ *and:*

4. *for* $N = 1, 3, 4, 6, 7, 9$, $\succsim_F$ *satisfies Axiom AA.N if and only if* $\succsim$ *satisfies Axiom A.N;*
5. *Axiom AA.2 for* $\succsim_F$ *is implied by Axiom A.2 for* $\succsim$ *(and they are equivalent under Axiom A.4);*
6. *Axiom AA.5 for* $\succsim_F$ *is equivalent to Axiom A.5 for* $\succsim$ *when* $\succsim_F^{\#}$ *is defined by*

$$f \succsim_F^{\#} g \iff f(s) \succsim_F g(s) \quad \forall s \in S;$$

7. *Axiom AA.8 for* $\succsim_F$ *implies Axiom A.8 for* $\succsim$ (*and they are equivalent under Axioms* A.1 *and* A.4).

Proof 1. Follows from the fact that F is affine and the fact that $\rho_{\epsilon(\gamma)} \equiv \gamma$ for all $\gamma \in \Delta(C)$.
2. If $\alpha, \beta \in \Delta(A)$ and $\alpha \approx \beta$, then $\rho_\alpha(s) = \rho_\beta(s)$ for all $s \in S$; since $\succsim$ is a preorder $\epsilon(\rho_\alpha(s)) \succsim \epsilon\left(\rho_\beta(s)\right)$, that is, $\rho_\alpha(s) \succsim_{\Delta(C)} \rho_\beta(s)$ for all $s \in S$, and Axiom A.3 implies $\alpha \succsim \beta$. By a symmetric argument $\beta \succsim \alpha$.
3. Obviously, $\{u(f) : f \in \mathcal{F}\} \subseteq (\mathrm{co}(u(C)))^S = u(\Delta(C))^S$. Set $U = \mathrm{co}(u(C))$ and assume $u(C)$ is not a singleton, otherwise the result is trivial. For every vector $\varkappa = [x_1 \ \ldots \ x_n]^\mathsf{T} \in U^S$ there exist $c \neq d$ in C and $q_1, q_2, \ldots, q_n \in [0, 1]$ such that

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_1 u(c) + (1-q_1) u(d) \\ q_2 u(c) + (1-q_2) u(d) \\ \vdots \\ q_n u(c) + (1-q_n) u(d) \end{bmatrix} = \begin{bmatrix} u(q_1\delta_c + (1-q_1)\delta_d) \\ u(q_2\delta_c + (1-q_2)\delta_d) \\ \vdots \\ u(q_n\delta_c + (1-q_n)\delta_d) \end{bmatrix}.$$

Set $D = \{c, d\}$, set $B = \{a \in A : \rho_a(S) \subseteq \{c, d\}\}$, and consider the Luce–Raiffa framework (B, S, D, ρ). Since all the bets are conceivable in (A, S, C, ρ), then $\{\rho_b\}_{b \in B} = D^S$. In particular, by point 3 of Proposition 2 for each $f \in \Delta(D)^S$, there exists $\beta_f \in \Delta(B)$ such that

$$\beta_f(\{b \in B : \rho(b, s) = c\}) = f(c|s) \quad \forall s \in S.$$

Choose $f \in \Delta(D)^S$ such that $f(c \mid s_i) = q_i = 1 - f(d \mid s_i)$ for all $i = 1, \ldots, n$, and set $\alpha(a) = \beta_f(a)$ if $a \in B$ and $\alpha(a) = 0$ if $a \in A \backslash B$. Then, $\alpha \in \Delta(A)$ and, for all $i = 1, \ldots, n$,

$$\rho_\alpha(c \mid s_i) = \beta_f(\{b \in B : \rho(b, s_i) = c\}) = f(c \mid s_i) = q_i$$
$$\rho_\alpha(d \mid s_i) = f(d \mid s_i) = 1 - q_i$$

so that $\rho_\alpha(s_i) = q_i\delta_c + (1-q_i)\delta_d$ and $u(\rho_\alpha(s_i)) = q_i u(c) + (1-q_i) u(d)$, that is, $u(\rho_\alpha) = \varkappa$.

Assume $\succsim$ satisfies mixed consequentialism. Let $f, g \in \mathcal{F}$ and $\alpha_f, \beta_f, \alpha_g, \beta_g \in \Delta(A)$, not necessarily distinct, be such that

$$\rho_{\alpha_f} = \rho_{\beta_f} = f \quad \text{and} \quad \rho_{\alpha_g} = \rho_{\beta_g} = g. \tag{31}$$

These mixed actions exist since $\mathcal{F} = F(\Delta(A))$. By mixed consequentialism, (31) implies $\alpha_f \sim \beta_f$ and $\alpha_g \sim \beta_g$ Therefore, by transitivity of $\succsim$,

$$\alpha_f \succsim \alpha_g \iff \beta_f \succsim \beta_g$$

and $\succsim_F$ is well defined.[49] Reflexivity and transitivity of $\succsim_F$ follow from reflexivity and transitivity of $\succsim$.

The verification of points 4, 5, and 6 is routine.

7. Assume $\succsim_F$ satisfies Axiom AA.8. If $\alpha, \beta \in \Delta(A)$ and $\alpha \sim \beta$, then $\rho_\alpha \sim_F \rho_\beta$. By Axiom AA.8, $2^{-1}\rho_\alpha + 2^{-1}\rho_\beta \succsim_F \rho_\alpha$, therefore $\rho_{2^{-1}\alpha+2^{-1}\beta} \succsim_F \rho_\alpha$ and $2^{-1}\alpha + 2^{-1}\beta \succsim \alpha$, so that $\succsim$ satisfies Axiom A.8.

Conversely, assume $\succsim$ satisfies Axioms A.1 and A.4. Next we show that Axiom A.8 implies that for all $\alpha, \beta \in \Delta(A)$ such that $\alpha \sim \beta$ and all $q \in (0,1)$ it holds $q\alpha + (1-q)\beta \succsim \alpha$. Per contra, assume there exist $\alpha, \beta \in \Delta(A)$ and $p \in (0,1)$ such that $\alpha \sim \beta$ and $p\alpha + (1-p)\beta \prec \alpha$. Set

$$T = \{t \in [0,1] : t\alpha + (1-t)\beta \prec \alpha\}.$$

Clearly $p \in T$. Moreover, by Axioms A.1 and A.4, T is open in $[0,1]$ and hence there exists O open in $\mathbb{R}$ such that $T = O \cap [0,1]$, but $T \subseteq (0,1)$, therefore $T = O \cap (0,1)$ is open in $\mathbb{R}$. Therefore there exists an open interval in T that contains p. The set

$$I = \bigcup_{p \ni (q,r) \subseteq T} (q, r)$$

is a union of pairwise overlapping open intervals, and so it is an open interval itself: $p \in I = (\bar{q}, \bar{r}) \subseteq T \subseteq (0,1)$. If $\bar{q} \in T$, there would exist $\varepsilon > 0$ such that $(\bar{q}-\varepsilon, \bar{q}+\varepsilon) \subseteq T$, and then $p \in (\bar{q}-\varepsilon, \bar{r}) \subseteq T$, whence

$$(\bar{q}-\varepsilon, \bar{r}) \subseteq I = (\bar{q}, \bar{r})$$

a contradiction. Therefore $\bar{q} \notin T$ and (analogously) $\bar{r} \notin T$, that is,

$$\bar{q}\alpha + (1-\bar{q})\beta \succsim \alpha \quad \text{and} \quad \bar{r}\alpha + (1-\bar{r})\beta \succsim \alpha.$$

[49] In fact, $\{(f,g) \mid \alpha \succsim \beta, \forall \alpha, \beta \in \Delta(A) : \rho_\alpha = f \text{ and } \rho_\beta = g\} = \{(f,g) \mid \exists \alpha, \beta \in \Delta(A) : \rho_\alpha = f, \rho_\beta = g, \text{and } \alpha \succsim \beta\} \subseteq \mathcal{F} \times \mathcal{F}$ and the relation $\succsim_F$ they define on $\mathcal{F}$ is such that, if $\alpha, \beta \in \Delta(A)$ then $\alpha \succsim \beta \iff \rho_\alpha \succsim_F \rho_\beta$.

Since $(\bar{q},\bar{r}) \subseteq T$ is nonempty, eventually, $\bar{q}+n^{-1}$ and $\bar{r}-n^{-1}$ belong to T, and by Axiom A.4

$$\bar{q}\alpha + (1-\bar{q})\,\beta \precsim \alpha \quad \text{and} \quad \bar{r}\alpha + (1-\bar{r})\,\beta \precsim \alpha.$$

That is, $\bar{q}\alpha + (1-\bar{q})\,\beta \sim \bar{r}\alpha + (1-\bar{r})\,\beta \sim \alpha$, and Axiom A.8 implies

$$\left(\frac{\bar{q}}{2}+\frac{\bar{r}}{2}\right)\alpha + \left(1-\frac{\bar{q}}{2}-\frac{\bar{r}}{2}\right)\beta = \frac{1}{2}\left(\bar{q}\alpha + (1-\bar{q})\,\beta\right)$$
$$+\frac{1}{2}\left(\bar{r}\alpha + (1-\bar{r})\,\beta\right) \succsim \bar{r}\alpha + (1-\bar{r})\,\beta \sim \alpha$$

but this is a contradiction since $2^{-1}\bar{q}+2^{-1}\bar{r} \in (\bar{q},\bar{r}) \subseteq T$.

Now let $f, g \in \mathcal{F}$ be such that $f \sim_F g$ and arbitrarily choose $q \in (0,1)$. Let $\alpha, \beta \in \Delta(A)$ be such that $f = \rho_\alpha$ and $g = \rho_\beta$, then $f \sim_F g$ implies $\alpha \sim \beta$, Axioms A.1, A.4, and A.8 imply $q\alpha + (1-q)\,\beta \succsim \alpha$, whence $\rho_{q\alpha+(1-q)\beta} \succsim_F \rho_\alpha$ and $qf + (1-q)\,g = q\rho_\alpha + (1-q)\,\rho_\beta \succsim_F \rho_\alpha = f$. As wanted. □

Proof of Proposition 3 It immediately follows from Lemma 2. □

In order to obtain representation (16), due to Cerreia-Vioglio et al. (2011b), we need two final pieces of notation. First, given $u \in \mathbb{R}^C$, we denote by $U = \mathrm{co}\,(u\,(C))$ the smallest interval containing $u\,(C)$ and by $\mathcal{G}\,(U, \Delta\,(S))$ the set of functions $G : U \times \Delta\,(S) \to (-\infty, \infty]$ such that:

1. G is quasiconvex,
2. $\inf_{\sigma\in\Delta(S)} G\,(x,\sigma) = x$ for all $x \in U$,
3. G is increasing in the first component,
4. the function $\varkappa \mapsto \inf_{\sigma\in\Delta(S)} G\,(\varkappa\cdot\sigma,\sigma)$ is continuous on U^S.

Second, for each $l \in -U$ and each $\sigma \in \Delta\,(S)$, we denote by $\mathcal{B}\,(l,\sigma)$ the set of all mixed actions that have Bayes risk level l under σ, that is,

$$\mathcal{B}\,(l,\sigma) = \{\beta \in \Delta\,(A) : r\,(\beta,\sigma) = l\}$$

and by $v\,(l,\sigma)$ the indirect payoff $v\,(\mathcal{B}\,(l,\sigma))$ of decision problem $(\mathcal{B}\,(l,\sigma)\,, S, C, \rho)$, as defined in (8).

Theorem 5 *Let (A, S, C, ρ) be a Marschak–Radner framework and $\succsim$ a binary relation on $\Delta\,(A)$. The following conditions are equivalent:*

(i) *$\succsim$ is a non-trivial and continuous rational preference that satisfies uncertainty aversion;*
(ii) *there exist a nonconstant $u \in \mathbb{R}^C$ and $G \in \mathcal{G}\,(U, \Delta\,(S)) \to (-\infty, \infty]$ such that, if $\alpha, \beta \in \Delta\,(A)$,*

$$\alpha \succsim \beta \iff \inf_{\sigma\in\Delta(S)} G\,(\mathbb{E}_{\alpha\times\sigma}\,[u\circ\rho]\,,\sigma) \geq \inf_{\sigma\in\Delta(S)} G\left(\mathbb{E}_{\beta\times\sigma}\,[u\circ\rho]\,,\sigma\right). \quad (32)$$

In this case, u is cardinally unique and, for each u, the minimal element of $\mathcal{G}(U, \Delta(S))$ satisfying (32) is the indirect payoff

$$G_u(x, \sigma) = v(-x, \sigma) \quad \forall (x, \sigma) \in U \times \Delta(S). \tag{33}$$

Continuous and uncertainty averse rational preferences are thus represented by

$$V(\alpha) = \inf_{\sigma \in \Delta(S)} G(\mathbb{E}_{\alpha \times \sigma}[u \circ \rho], \sigma) \quad \forall \alpha \in \Delta(A) \tag{34}$$

and (16) is obtained by setting

$$R(\alpha, \sigma) = G(\mathbb{E}_{\alpha \times \sigma}[u \circ \rho], \sigma) \quad \forall (\alpha, \sigma) \in \Delta(A) \times \Delta(S)$$

and by choosing Σ as the projection on $\Delta(S)$ of the domain of R. In particular, when the minimal G_u described by (33) is considered,

$$R_u(\alpha, \sigma) = v(r(\alpha, \sigma), \sigma)$$

is the indirect payoff of the decision problem in which the only available mixed actions are those with the same Bayes risk as α under σ. This is worth mentioning since the indirect payoff $v(l, \sigma)$ can be seen as a comparative index of uncertainty aversion (see Cerreia-Vioglio et al. 2011b).

Proofs of Theorems 2, 3, 4, and 5 Let $\succsim$ be a rational preference. As shown in the first part of the proof of Theorem 1, there exists $u \in \mathbb{R}^C$ such that:

- if $\gamma, \zeta \in \Delta(C)$, then $\gamma \succsim_{\Delta(C)} \zeta \iff \sum_{c \in C} \gamma(c) u(c) \geq \sum_{c \in C} \zeta(c) u(c)$;
- if $\alpha, \beta \in \Delta(A)$, then $\alpha \succcurlyeq_S \beta \iff \alpha \geqslant_u \beta \iff \sum_{a \in A} \alpha(a) u(\rho(a, s)) \geq \sum_{a \in A} \beta(a) u(\rho(a, s))$ for all $s \in S$;
- if $\alpha, \beta \in \Delta_\ell(A)$, then $\alpha \succsim \beta \iff \alpha \succcurlyeq_S \beta \iff \sum_{c \in C} \alpha(cS) u(c) \geq \sum_{c \in C} \beta(cS) u(c)$.

Moreover, if $\succsim$ is continuous, for every $\alpha \in \Delta(A)$ there exists $\alpha_\ell \in \Delta(C)$ such that $\alpha \sim \alpha_\ell$, more precisely, $\alpha \sim \epsilon(\alpha_\ell)$. In fact, choosing w and m in S so that

$$\rho_\alpha(m) \succsim_{\Delta(C)} \rho_\alpha(s) \succsim_{\Delta(C)} \rho_\alpha(w) \quad \forall s \in S \tag{35}$$

it follows $\epsilon(\rho_\alpha(m)) \succcurlyeq_S \alpha \succcurlyeq_S \epsilon(\rho_\alpha(w))$. Whence $\beta = \epsilon(\rho_\alpha(m))$ and $\eta = \epsilon(\rho_\alpha(w))$ belong to $\Delta_\ell(A)$ and, by Axiom A.3, $\beta \succsim \alpha \succsim \eta$. Therefore, the non-empty and closed (by Axiom A.4) sets

$$\{q \in [0,1] : q\eta + (1-q)\beta \succsim \alpha\} \quad \text{and} \quad \{q \in [0,1] : \alpha \succsim q\eta + (1-q)\beta\}$$

cover (by Axiom A.1) the connected set $[0,1]$. In particular, they cannot be disjoint, and there is $q_\ell \in [0,1]$ such that $q_\ell \eta + (1-q_\ell)\beta \sim \alpha$; convexity of $\Delta_\ell(A) = \epsilon(\Delta(C))$ implies that $q_\ell \eta + (1-q_\ell)\beta = \epsilon(\alpha_\ell)$ for some $\alpha_\ell \in \Delta(C)$.

If $\alpha, \beta \in \Delta(A)$, then

$$\alpha \succsim \beta \iff \epsilon(\alpha_\ell) \succsim \epsilon(\beta_\ell) \iff \alpha_\ell \succsim_{\Delta(C)} \beta_\ell \iff \sum_{c\in C} \alpha_\ell(c)\,u(c) \geq \sum_{c\in C} \beta_\ell(c)\,u(c)$$

that is, the functional

$$\begin{aligned} V\colon \Delta(A) &\to \mathbb{R} \\ \alpha &\mapsto \sum_{c\in C} \alpha_\ell(c)\,u(c) \end{aligned}$$

represents $\succsim$ on $\Delta(A)$.[50]

For each $\alpha \in \Delta(A)$, by Eq. (35) and Axiom A.3, $\epsilon(\rho_\alpha(m)) \succsim \alpha \succsim \epsilon(\rho_\alpha(w))$ so that

$$\begin{aligned} \max_{s\in S} \sum_{a\in A} \alpha(a)\,u(\rho(a,s)) &= \max_{s\in S} \mathbb{E}_{\rho_\alpha(s)}[u] = \mathbb{E}_{\rho_\alpha(m)}[u] = V(\epsilon(\rho_\alpha(m))) \\ &\geq V(\alpha) \\ &\geq V(\epsilon(\rho_\alpha(w))) = \mathbb{E}_{\rho_\alpha(w)}[u] = \min_{s\in S} \mathbb{E}_{\rho_\alpha(s)}[u] \\ &= \min_{s\in S} \sum_{a\in A} \alpha(a)\,u(\rho(a,s)). \end{aligned}$$

Up until this point, we only assumed that $\succsim$ satisfies Axioms A.1–A.4.
Theorem 2. If, in addition, Axiom A.5 holds, then $\epsilon(\alpha_\ell) \precsim \alpha$ implies $\alpha \succcurlyeq_S \epsilon(\alpha_\ell)$, that is $\rho_\alpha(w) \succsim_{\Delta(C)} \alpha_\ell$, and

$$\min_{s\in S} \sum_{a\in A} \alpha(a)\,u(\rho(a,s)) = \min_{s\in S} \mathbb{E}_{\rho_\alpha(s)}[u] = \mathbb{E}_{\rho_\alpha(w)}[u] \geq \mathbb{E}_{\alpha_\ell}[u] = V(\alpha).$$

To sum up, $V(\alpha) = \min_{s\in S} \sum_{a\in A} \alpha(a)\,u(\rho(a,s))$ for all $\alpha \in \Delta(A)$, which proves (10). The converse is routine, hence omitted. □

If $\succsim$ satisfies Axioms A.1–A.4, by Lemma 2, $\succsim_{F}$ is well defined and satisfies Axioms AA.1–AA.4 on $\mathcal{F} = F(\Delta(A))$. For each $f = \rho_\alpha \in \mathcal{F}$, $\alpha \sim \epsilon(\alpha_\ell)$ implies $f \sim_F \rho_{\epsilon(\alpha_\ell)} = (\alpha_\ell)_S$. In the Anscombe–Aumann jargon, α_ℓ is a *certainty equivalent* γ_f of $f = \rho_\alpha$, more precisely,

[50] Notice that V is well defined since whenever $\alpha'_\ell, \alpha''_\ell \in \Delta_\ell(A)$ are such that $\epsilon(\alpha'_\ell) \sim \alpha$ and $\epsilon(\alpha''_\ell) \sim \alpha$, then $\alpha'_\ell \sim_{\Delta(C)} \alpha''_\ell$ and $\sum_{c\in C} \alpha'_\ell(c)\,u(c) = \sum_{c\in C} \alpha''_\ell(c)\,u(c)$. Also observe that if $\alpha = \epsilon(\gamma) \in \Delta_\ell(A)$, one can choose $\alpha_\ell = \gamma$ and obtain $V(\epsilon(\gamma)) = \sum_{c\in C} \gamma(c)\,u(c) = \mathbb{E}_\gamma[u]$.

$$\{\xi \in \Delta(C) : \alpha \sim \epsilon(\xi)\} = \{\gamma \in \Delta(C) : \rho_\alpha \sim \gamma_S\}.$$

Moreover, if $g = \rho_\beta \in \mathcal{F}$, then

$$\begin{aligned} f \succsim_F g &\iff \alpha \succsim \beta \iff \alpha_\ell \succsim_{\Delta(C)} \beta_\ell \iff \mathbb{E}_{\alpha_\ell}[u] \geq \mathbb{E}_{\beta_\ell}[u] \\ &\iff u\left(\gamma_f\right) \geq u\left(\gamma_g\right). \end{aligned}$$

In particular, for every $\gamma, \zeta \in \Delta(C)$,

$$\begin{aligned} \gamma_S \succsim_F \zeta_S &\iff \rho_{\epsilon(\gamma)} \succsim_F \rho_{\epsilon(\zeta)} \iff \epsilon(\gamma) \succsim \epsilon(\zeta) \iff \gamma \succsim_{\Delta(C)} \zeta \\ &\iff \mathbb{E}_\gamma[u] \geq \mathbb{E}_\zeta[u] \iff u(\gamma) \geq u(\zeta). \end{aligned}$$

Set $U = \mathrm{co}(u(C)) = u(\Delta(C))$ and $u(\mathcal{F}) = \{u(f) : f \in \mathcal{F}\} \subseteq U^S \subseteq \mathbb{R}^S$, and for each $\varkappa \in u(\mathcal{F})$ define

$$I(\varkappa) = u\left(\gamma_f\right) \quad \text{if } \varkappa = u(f).$$

Using the techniques of Cerreia-Vioglio et al. (2011a, b) it can be shown that $I : u(\mathcal{F}) \to \mathbb{R}$ is well defined, monotone and normalized (that is $I(x1_S) = x$ for all $x \in U$). Moreover, for each $\alpha \in \Delta(A)$, $V(\alpha) = \mathbb{E}_{\alpha_\ell}[u] = u(\alpha_\ell) = I(u(\rho_\alpha))$ and if $f = \rho_\alpha, g = \rho_\beta \in \mathcal{F}$

$$\begin{aligned} f \succsim_F g &\iff V(\alpha) \geq V(\beta) \iff I(u(\rho_\alpha)) \geq I\left(u\left(\rho_\beta\right)\right) \iff I(u(f)) \\ &\geq I(u(g)). \end{aligned} \tag{36}$$

Theorems 3 and 4. If $\succsim$ satisfies Axiom A.6, then $\succsim_F$ satisfies Axiom AA.6 by Lemma 2, which, together with (36), implies that I is affine. Theorem 3 follows by the finite dimensional versions of the Krein-Rutman Extension Theorem (see, e.g. Ok 2007, p. 496) and the Riesz Representation Theorem. The additional assumptions of Theorem 4, non-triviality of $\succsim$ and conceivability of all bets, guarantee that u is nonconstant and μ is unique. □

Theorem 5. Follows from Theorem 3 of Cerreia-Vioglio et al. (2011a, b) and Lemma 2 again, which guarantees that $\{u(f) : f \in \mathcal{F}\} = U^S$ when all bets are conceivable and that $\succsim_F$ satisfies Axiom AA.8 when $\succsim$ satisfies Axioms A.1–A.4 and A.8. □

References

Anscombe, F.J., Aumann, R.J.: A definition of subjective probability. Ann. Math. Stat. **34**, 199–205 (1963)

Arrow, K.J.: Rational choice functions and orderings. Economica **26**, 121–127 (1959)

Battigalli, P., Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M.: Mixed Extensions of Decision Problems under Uncertainty, IGIER Working Paper 485. Università Bocconi (2013)

Berger, A.: Statistical Decision Theory and Bayesian Analysis, 2nd edn. Springer, New York (1985)

Cerreia-Vioglio, S.: Maxmin Expected Utility on a Subjective State Space: Convex Preferences under Risk, manuscript. Columbia University (2009)

Cerreia-Vioglio, S., Ghirardato, P., Maccheroni, F., Marinacci, M., Siniscalchi, M.: Rational preferences under ambiguity. Econ. Theory **48**, 341–375 (2011a)

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., Montrucchio, L.: Uncertainty averse preferences. J. Econ. Theory **146**, 1275–1330 (2011b)
Cerny, A.: Mathematical Techniques in Finance: Tools for Incomplete Markets, 2nd edn. Princeton University Press, Princeton (2009)
Debreu, G.: Theory of Value. Wiley, New York (1959)
Dekel, E., Lipman, B.L., Rustichini, A.: Representing preferences with a unique subjective state space. Econometrica **69**, 891–934 (2001)
Dominiak, A., Schnedler, W.: Attitudes toward uncertainty and randomization: an experimental study. Econ. Theory. **48**, 289–312 (2011)
Eichberger, J., Grant, S., Kelsey, D.: Randomization and Dynamic Consistency. manuscript, University of Exeter (2013)
Ellsberg, D.: Risk, ambiguity, and the savage axioms. Q. J. Econ. **75**, 643–669 (1961)
Epstein, L.G., Halevy, Y.: No Two Experiments are Identical. manuscript, University of British Columbia (2014)
Fishburn, P.C.: Utility Theory for Decision Making. Wiley, New York (1970)
Ghirardato, P., Maccheroni, F., Marinacci, M., Siniscalchi, M.: A subjective spin on roulette wheels. Econometrica **71**, 1897–1908 (2003)
Ghirardato, P., Maccheroni, F., Marinacci, M.: Differentiating ambiguity and ambiguity attitude. J. Econ. Theory **118**, 133–173 (2004)
Ghirardato, P., Maccheroni, F., Marinacci, M.: Certainty independence and the separation of utility and beliefs. J. Econ. Theory **120**, 129–136 (2005)
Gilboa, I., Maccheroni, F., Marinacci, M., Schmeidler, D.: Objective and subjective rationality in a multiple prior model. Econometrica **78**, 755–770 (2010)
Gilboa, I., Marinacci, M.: Ambiguity and the Bayesian Paradigm. In: Acemoglu, D., Arellano, M., Dekel, E. (eds.) Advances in Economics and Econometrics: Theory and Applications. Cambridge University Press, Cambridge (2013)
Gilboa, I., Schmeidler, D.: Maxmin expected utility with non-unique prior. J. Math. Econ. **18**, 141–153 (1989)
Glazer, J., Rubinstein, A.: An extensive game as a guide for solving a normal game. J. Econ. Theory **70**, 32–42 (1996)
Hansen, L.P., Sargent, T.J.: Robust control and model uncertainty. Am. Econ. Rev. **91**, 60–66 (2001)
Hansen, L.P., Sargent, T.J.: Robustness. Princeton University Press, Princeton (2008)
Herstein, I.N., Milnor, J.: An axiomatic approach to measurable utility. Econometrica **21**, 291–297 (1953)
Karni, E., Maccheroni, F., Marinacci, M.: Ambiguity and nonexpected utility. In: Young, P., Zamir, S. (eds.) Handbook of Game Theory IV, pp. 901–948. Elsevier, Amsterdam (2014)
Karni, E., Viero, M.L.: Awareness of Unawareness: A Theory of Decision Making in the Face of Ignorance, QED Working Paper 1322, Queen's University (2014)
Klibanoff, P.: Uncertainty, Decision, and Normal Form Games, manuscript. Northwestern University (1992)
Kreps, D.M.: A representation theorem for preference for flexibility. Econometrica **47**, 565–577 (1979)
Kreps, D.M.: Notes on the Theory of Choice. Westview, Boulder (1988)
Kuhn, H.W.: Extensive games and the problem of information. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games II, pp. 193–216. Princeton, Princeton University Press (1953)
Kuzmics, C.: An Alternative Subjective Expected Utility Representation Theorem, manuscript. Bielefeld University (2012)
Lehrer, E., Teper, R.: Extension rules or what would the sage do? Am. Econ. J. Microecon. **6**, 5–22 (2014)
LeRoy, S.F., Werner, J.: Principles of Financial Economics. Cambridge University Press, Cambridge (2000)
Luce, R.D., Raiffa, H.: Games and Decisions: Introduction and Critical Survey. Wiley, New York (1957)
Maccheroni, F.: Maxmin under Risk. Econ. Theory **19**, 823–831 (2002)
Maccheroni, F., Marinacci, M., Rustichini, A.: Ambiguity aversion, robustness, and the variational representation of preferences. Econometrica **74**, 1447–1498 (2006)
Mariotti, M.: Is bayesian rationality compatible with strategic rationality? Econ. J. **105**, 1099–1109 (1995)
Marschak, J., Radner, R.: Economic Theory of Teams. Yale University Press, New Haven (1972)
Milnor, J.: Games against nature. In: Thrall, R.M., Coombs, C.H., Davis, D.L. (eds.) Decision Processes, pp. 49–59. Wiley, New York (1954)
Office of the Comptroller of the Currency: Bank Derivatives Report First Quarter, report. U.S, Department of Treasury (1999)
Ok, E.A.: Real Analysis with Economic Applications. Princeton University Press, Princeton (2007)

Raiffa, H.: Risk, ambiguity, and the savage axioms: comment. Q. J. Econ. **75**, 690–694 (1961)
Rubinstein, A.: Comments on the interpretation of game theory. Econometrica **59**, 909–924 (1991)
Saito, K.: Preference for Flexibility and Preference for Randomization under Ambiguity, manuscript. Caltech (2013)
Sarin, R., Wakker, P.P.: A single-stage approach to Anscombe and Aumann's expected utility. Rev. Econ. Stud. **64**, 399–409 (1997)
Savage, L.J.: The Foundations of Statistics. Wiley, New York (1954)
Schmeidler, D.: Subjective probability and expected utility without additivity. Econometrica **57**, 571–587 (1989)
Seo, K.: Ambiguity and second-order belief. Econometrica **77**, 1575–1605 (2009)
Strzalecki, T.: Axiomatic foundations of multiplier preferences. Econometrica **79**, 47–73 (2011)
Swinkels, J.: Subgames and the Reduced Normal Form, manuscript. Princeton University (1989)
Wakker, P.P.: Prospect Theory: For Risk and Ambiguity. Cambridge University Press, Cambridge (2010)
Wald, A.: Statistical Decision Functions. Wiley, New York (1950)