Introduction to Game Theory: Simultaneous Moves
Lecture 9, *Experimental Econ. & Psychology*

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Abstract

Game theory is the *formal analysis* of interactive decision making, i.e., of situations with n individuals (called **players**), some or all of whom have to take actions, which affect the outcome (consequences) for everybody. In **static** games, the topic of this lecture, active players move simultaneously once and for all. In **dynamic** games, some or all moves are sequential. Game theory describes situations of interactive decision making with a *mathematical language*. It is crucial to distinguish the description of the "rules of the game" from the description of the exogenous personal features of the participating individuals, such as their tastes. The mathematical description of the rules of the game is called **game form**. A game form with a description of player's exogenous personal features is called **game.** Predictions about behavior in each game are obtained by means of **solution concepts**, such as Nash equilibrium, or iterated dominance. Sometimes solutions concepts have interesting foundations, often they don't. We focus more on how players' beliefs about others affect their behavior rather than explaining how such beliefs are shaped.

Introduction: games

- **Game theory** is the formal analysis of interactive decision making, i.e., of situations with n individuals (called **players**), some or all of whom have to take actions, which affect the outcome (consequences) for everybody. We will focus on *monetary outcomes.*
  - In **static** games, active players move simultaneously once and for all.
  - In **dynamic** games, moves are sequential, although some of them may be simultaneous.
- Game theory describes interactive situations using a *mathematical language*. It is crucial to distinguish:
  - the description of the "rules of the game" (in experiments, controlled by the experimenter), called **game form**,
  - from the description of the exogenous personal features of the participating individuals, such as their "tastes" (e.g., preferences over lotteries of outcomes).
  - Appending to a game form the description of players' exogenous features we obtain a **game**.

Example of static game form: Prisoners' Dilemma

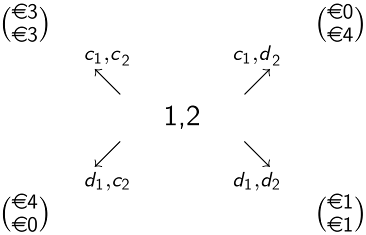
- Ann (pl. 1) and Bob (pl. 2) choose simultaneously between actions c (cooperate) and d (defect). Outcomes are *monetary* payoffs $\pi = (\pi_1, \pi_2)$ in, say, €=ECU (exper. currency units). *Static* game forms admit a *tabular description*, the **payoff matrix:**

PD
(not a "game"!)

π (in ECU)	c_2	d_2
c_1	€3, €3	€0, €4
d_1	€4, €0	€1, €1

- ... and also a *graphical description*, the **game tree** form:

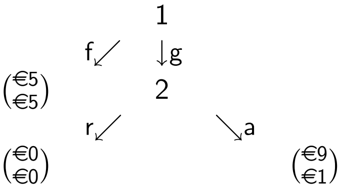
PD
(not a "game"!)



Example of dynamic game form: Ultimatum mini-Game

- There are €10 to split. Ann can implement the fair allocation (€5, €5) (action *f*) or make a greedy offer of only €1 to Bob (action *g*). *If* Ann makes the greedy offer, Bob can reject (*r*) or accept (*a*).
- The possible sequences of actions and the implied payoffs are described by the **game tree** form:

UmG
(not a "game"!)



Examples of "tastes" (utility functions)

- **Standard economics** describes tastes by means of utilities of outcomes (e.g., monetary outcomes), $v_i : Y \to \mathbb{R}$, where typically $Y \subseteq \mathbb{R}^I$ with I=set of individuals, for example (*non exhaustive* list, y_j=monetary payoff of $j \in I$):
  - $v_i\left((y_j)_{j\in I}\right) = y_i$ (selfish and risk neutral),
  - $v_i\left((y_j)_{j\in I}\right) = V_i(y_i)$ with $V_i' > 0$, $V_i'' < 0$ (selfish, risk averse),
  - $v_i\left((y_j)_{j\in I}\right) = y_i + \sum_{j\neq i} V_{ij}(y_j)$, $0 < V_{ij}' \leq 1$ (partially altruist, own-risk neutral).
- **Non-standard economics** allows utility to depend on more than the material outcomes, e.g.:
  - chosen actions matter per se (as in the "warm glow of giving"); this is still consistent with **traditional game theory**: $u_i\left((a_j)_{j\in I}\right)$,
  - in **psychological game theory (PGT)** also beliefs matter, including those of others: $u_i\left((a_j, \mathrm{belief}_j)_{j\in I}\right)$.

Static Games

- **Game form:** mathematical structure $\langle I, (A_i, \pi_i)_{i \in I} \rangle$ describing the "rules", where
  - I, *finite* set of **players** ($i \in I$ is a role)
  - A_i, *finite* set of feasible **actions**, or alternatives, of player (role) i
  - $\pi_i : \times_{j \in J} A_j \to Y_i$, material (e.g., monetary $Y_i \subseteq€\mathbb{R}$) **payoff function** of i
- **Game:** mathematical structure $\langle I, (A_i, \pi_i, v_i)_{i \in I} \rangle$ describing the *specific situation of interaction*, given the individual players' exogenous personal features (e.g., "tastes"), where
  - in traditional GT, $v_i : \times_{j \in I} Y_j \to \mathbb{R}$.
  - in PGT, $v_i : \times_{j \in I} (Y_j \times B_j) \to \mathbb{R}$ (B_j=belief set of j, to be defined).
- **Game in "action form":** $\langle I, (A_i, u_i)_{i \in I} \rangle$, where
  - in traditional GT, $u_i : \times_{j \in I} A_j \to \mathbb{R}$, $u_i\left((a_j)_{j \in I}\right) = v_i\left(\pi\left((a_j)_{j \in I}\right)\right)$, $\pi = (\pi_k)_{k \in I}$; *WARNING:* u_i is often called "payoff function", but it is the *utility* of actions!
  - in PGT, $u_i : \times_{j \in I} A_j \times B_j \to \mathbb{R}$.

Static game forms: examples

Very simple static game forms are used to model/simulate *social dilemmas* and are often implemented in the lab (all numbers are *material* payoffs, e.g., monetary payoffs):

PD	c_2	d_2
c_1	3, 3	0, 4
d_1	4, 0	1, 1

Co	ℓ_2	r_2
ℓ_1	1, 1	0, 0
r_1	0, 0	1, 1

SH	b_2	s_2
b_1	5, 5	0, 3
s_1	3, 0	3, 3

- **Prisoners Dilemma**: selfish behavior (*d*efection) yields inefficiency (unlike perfectly competitive markets!).
- **Coordination**: find a convention to coordinate [e.g., drive on the right (EU), or on the left (UK)].
- **Stag Hunt** (from a story of Rousseau): coordination on a risky action (hunt for a **b**ig prey like a stag, together) achieves first **b**est, the alternative is a *s*afe action (hunt for a **s**mall prey, alone).

Beliefs

- In order to decide what to do, players form subjective beliefs (subjective probability measures) about the relevant unknowns, such as the actions of others.
- *Preliminary*: fix a *finite* uncertainty space X, the set of probability measures on X is $\Delta(X) = \left\{\mu \in \mathbb{R}_{+}^{X} : \sum_{x \in X} \mu(x) = 1\right\}$, where $\mathbb{R}_{+}^{X}$ is the set of nonnegative real-valued functions (vectors) on X.
- The definition of $\Delta(X)$ is generalized for the case of *infinite* uncertainty spaces.
- *NOTE*: some—possibly all but one—x's may be assigned probability 0; $\mu(x) = 1$ means **certainty** of x.

First-order beliefs

- **First-order beliefs** are (probabilistic) beliefs about "primitive uncertainty", such as: "How will the game be played?" We denote by $-i$ the co-player(s) of i.
- At the planning stage, i does not know what actions are going to be played (although she may be sure about her own).
  - $\mathbf{1}^{st}$**-order belief about others**: $\alpha_{i,-i} \in \Delta\left(A_{-i}\right)$, with $A_{-i} = \times_{j \neq i} A_j$.
  - $\mathbf{1}^{st}$**-order belief about oneself** (*plan* of i): $\alpha_{i,i} \in \Delta\left(A_i\right)$, typically (not always), $\alpha_{i,i}\left(a_i^*\right) = 1$, that is, i is certain of her (planned) action.
  - $\mathbf{1}^{st}$**-order belief**: $\alpha_i = \alpha_{i,i} \times \alpha_{i,-i} \in \Delta\left(A_i \times A_{-i}\right)$, that is, $\alpha_i\left(a_i, a_{-i}\right) = \alpha_{i,i}\left(a_i\right) \times \alpha_{i,-i}\left(a_{-i}\right)$ for all $\left(a_i, a_{-i}\right) \in A_i \times A_{-i}$ (self-*vs*-others *independence*).
- We let Δ_i^1 denote the space of $\mathbf{1}^{st}$-order beliefs of i (that satisfy the foregoing independence condition).

Best replies: standard GT

- Given utility function $v_i : Y \to \mathbb{R}$, we obtain the utility of actions $u_i : \times_{j \in I} A_j \to \mathbb{R}$, and we can compute the expected utility (EU) of taking any given action a_i given i's 1^{st}-order belief $\alpha_{i,-i}$:

$$\bar{u}_i\left(a_i, \alpha_{i,-i}\right) := \mathbb{E}_{a_i, \alpha_{i,-i}}\left(u_i\right) = \sum_{a_{-i} \in A_{-i}} u_i\left(a_i, a_{-i}\right) \alpha_{i,-i}\left(a_{-i}\right).$$

- **Best-reply correspondence:** it associates each 1^{st}-order belief about others, $\alpha_{i,-i}$, with the set $BR_i\left(\alpha_{i,-i}\right) \subseteq A_i$ of actions that maximize EU given $\alpha_{i,-i}$ (*note*: it may be multi-valued):

$$\begin{array}{rccc} BR_i : & \Delta\left(A_{-i}\right) & \rightrightarrows & A_i \\ & \alpha_{i,-i} & \mapsto & \arg\max_{a_i \in A_i} \bar{u}_i\left(a_i, \alpha_{i,-i}\right) \end{array}$$

- **Exercise:** Find the best reply correspondences of PD, Co, and SH, assuming that
  - either $v_i\left(y_i, y_{-i}\right) = y_i$,
  - or $v_i\left(y_i, y_{-i}\right) = y_i + \frac{1}{2} y_{-i}$.

Best replies: PGT, part I

- To model psychological factors, PGT allows the utility of outcomes/actions of everybody to depend on the beliefs of everybody. We just consider dependence on first-order beliefs:

$$u_i : \times_{j \in I} \left(A_j \times \Delta_j^1 \right) \to \mathbb{R}.$$

- Two cases:
  - *Own-plan-independence:* $u_i \left((a_j, \alpha_j)_{j \in I} \right)$ does not depend on i's plan $\alpha_{i,i}$ (possibly, it does not depend on α_i at all). *Example*: other things equal (OTE), i dislikes to make others earn less material payoff than they expect (not to live up to others' expectations).
  - *Own-plan-dependence:* $u_i \left((a_j, \alpha_j)_{j \in I} \right)$ depends also on $\alpha_{i,i}$. *Example*: OTE, i dislikes to be disappointed (to get less material payoff than she expected). *Note*: How much i expects to get depends also on her plan, because i's payoff depends also on what she does. *Warning:* this may involve difficulties!
- *To get BR's we need* beliefs about actions *and* (1^{st}-order) beliefs.

Best replies: PGT, part II

- **Second-order beliefs:** i's subjective probability measures about primitive uncertainty and others' beliefs (i knows her own beliefs by introspection) $\beta_i \in \Delta\left(A_i \times \left(\times_{j\neq i}\left(A_j \times \Delta_j^1\right)\right)\right)$.
- Thus, we can recover 1^{st}-order from 2^{nd}-order beliefs: for each profile of actions $(a_j)_{j\in I}$,

$$\alpha_i\left((a_j)_{j\in I}\right) = \beta_i\left(\underbrace{\{a_i\}}_{\text{self}} \times \underbrace{\left(\times_{j\neq i}\left(\{a_j\} \times \Delta_j^1\right)\right)}_{\text{others}}\right),$$

where $\{a_j\}$ is the singleton containing only action a_j, and $\alpha_i\left((a_j)_{j\in I}\right) = \alpha_{i,i}(a_i) \times \alpha_{i,-i}(a_{-i})$. The really important part of β_i is $\beta_{i,-i} \in \Delta\left(\left(\times_{j\neq i}A_j \times \Delta_j^1\right)\right)$.
- *NOTE*: Actions and beliefs of others *are not independent*. Hence, marg. beliefs about actions (1^{st}-order b.) and about beliefs (a feature of the 2^{nd}-order b.) cannot determine the joint belief $\beta_{i,-i}$.

Best replies: PGT, part III

We are now ready to define (a special case of) the BR correspondence in PGT.

- **Best reply correspondence** (given *own-plan independence*): Let $\bar{u}_i\left(a_i, \beta_{i,-i}\right) = \mathbb{E}_{a_i,\beta_{i,-i}}\left(u_i\right)$ denote the "psychological" EU of a_i given $\beta_{i,-i}$ (if $\beta_{i,-i}$ is discrete, we have the usual weighted summation formula). Then,

$$BR_i\left(\beta_{i,-i}\right) = \arg\max_{a_i \in A_i} \bar{u}_i\left(a_i, \beta_{i,-i}\right).$$

- **Exercise:** Let $[x]^+ = \max\{0, x\}$, and let δ_a denote the **deterministic belief** that $a = (a_i)_{i\in I}$ is played with certainty (=with prob. 1). Suppose that, in the *PD*,

$$u_1\left(a_1, a_2, \alpha_2\right) = \pi_1\left(a_1, a_2\right) - \frac{1}{2}\left[\mathbb{E}_{\alpha_2}\left(\pi_2\right) - \pi_2\left(a_1, a_2\right)\right]^+$$

Compute the best reply of 1 to any 2^{nd}-order belief $\beta_{1,2}$ such that $\beta_{1,2}\left(c_2, \delta_{(c_1,c_2)}\right) + \beta_{1,2}\left(d_2, \delta_{(c_1,d_2)}\right) = 1$.

Predictions

- GT derives predictions from **solution concepts** that associate each game with a corresponding set of possible action (and belief) profiles $(a_i)_{i\in I}$ (or $(a_i,\alpha_i)_{i\in I}$ or $(a_i,\beta_i)_{i\in I}$). Examples:
- **Nash equilibrium (pure):** in traditional GT, any profile $(a_i^*)_{i\in I}$ such $a_i^* \in BR_i\left(a_{-i}^*\right) = \arg\max_{a_i\in A_i} u_i\left(a_i, a_{-i}^*\right)$.
- **Iterated elimination of never best replies (rationalizability):**
  - 1. Eliminate all actions that are not best replies to any belief.
  - $n > 1$. Eliminate all the (remaining) actions that are not best replies to beliefs consistent with steps $1, ..., n-1$ (hence, which assign probability 0 to eliminated actions).
- We are *interested in experiments*, where there is *little reason to use NE to make predictions*. Sometimes predictions are derived from 2-3 rounds of iterated elimination of never best replies. Often we *"elicit" (=measure) players' beliefs and make predictions based on BR correspondences.*

References

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