Static Games and Expected Utility Maximization

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Game Theory: Analysis of Strategic Thinking

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ABSTRACT: A **static game** is a one-stage game with simultaneous moves. Static games are summarily represented specifying the set of feasible actions and the payoff function of each player. We (initially) focus on two cases concerning players' knowledge of the game: either there is complete information, or—at least—each player knows her payoff function. Mathematically, we consider either finite games, or games with compact action spaces and continuous utility functions. We assume that each player holds subjective probabilistic beliefs, called **conjectures**, about the co-players' actions. We also consider the possibility that they choose objective randomizations over actions, called **mixed actions**. Players evaluate (mixed) action-conjecture pairs by means of expected utility (EU) calculations. A rational player chooses an action that maximizes her EU *given* her conjecture.
[These slides summarize and complement Chapter 2 and Sections 3.1-3.2 of "Game Theory: Analysis of Strategic Thinking" (GT-AST).]

Introduction: static games

- A static game (form) has only one stage in which players move simultaneously.
- The realization of the simultaneous choices is an **action profile,** that is, a *function* $(a_i)_{i \in I}$ (or $i \mapsto a_i$) that associates each player i with her chosen action a_i. To ease notation, we often write action profiles as letters without subscripts: $a = (a_i)_{i \in I}$. (Similarly, we write $A := \times_{i \in I} A_i$ for the set of action profiles; thus, $a \in A$).
- We let $-i$ denote the **co-players** of i (the other players). With this, $a_{-i} = (a_j)_{j \in I \setminus \{i\}} = (a_j)_{j \neq i}$.
- An action profile a yields an outcome $y = g(a)$.
- See the examples of the introductory Lecture 1.

Games with simultaneous moves

We *summarily* represent a **game with simultaneous moves** (static game) as a mathematical structure $G = \left\langle I, (A_i, u_i)_{i \in I} \right\rangle$, where

- I is the *finite* **player set**, with generic element i;
- A_i is the **set of feasible actions** of player i;
- $u_i : \times_{j \in I} A_j \to \mathbb{R}$ is the **payoff function** of i (note, here "payoff" = "utility of actions")
  - *Recall* that u_i is derived from i's vNM **utility function** $v_i : Y \to \mathbb{R}$ (representing i's preferences over lotteries) and the **outcome function** $g : \times_{j \in I} A_j \to Y$, that is, $u_i = v_i \circ g$, which means $u_i\left((a_j)_{j \in I}\right) = v_i\left(g\left((a_j)_{j \in I}\right)\right)$ for all $(a_j)_{j \in I} \in \times_{j \in I} A_j$.
- To emphasize the perspective of player i, write action profiles as (a_i, a_{-i}), with $a_{-i} \in A_{-i} := \times_{j \in I \setminus \{i\}} A_j$, and write the payoff function of i as $u_i : A_i \times A_{-i} \to \mathbb{R}$.

Assumptions on knowledge about the game

We consider a strong and a weak assumption concerning players' knowledge of the game.

- The *strong* assumption is **complete information:** there is common knowledge (CK) of the interactive situation represented by G, that is, CK of the rules of the game and players' preferences.
- The *weak* assumption is that each i knows the action sets $(A_j)_{j\in I}$ and her payoff function $u_i : \times_{j\in I} A_j \to \mathbb{R}$.
- *Even the weak assumption is substantive*: we take for granted that i knows her preferences over (lotteries of) outcomes, that is, the vNM utility $v_i : Y \to \mathbb{R}$; but—in general— i may not know the outcome function $g : \times_{j\in I} A_j \to Y$; hence, she may not know $u_i = v_i \circ g$.

Assumptions on knowledge about the game: oligopoly

- Each firm i in an oligopolistic market with *quantity setting* just cares about (expected) profit, which depends on its output (q_i) and competitors' output (q_{-i}), *via* the market inverse demand function $P\left(\sum_{j\in I} q_j\right)$ and its cost function $C_i(q_i)$, which is known to i. Thus, $u_i(q_i, q_{-i}) = v_i\left(q_i P\left(\sum_{j\in I} q_j\right) - C_i(q_i)\right)$ $(v_i' > 0)$.
- Assume *CK of selfishness and of risk-neutrality* $(v_i(\pi_i) = \pi_i)$.
  - If CK of demand and cost functions $P(\cdot)$ and $(C_i(\cdot))_{i\in I}$, then the *strong assumption holds*: there is *complete information*.
  - If each i knows $P(\cdot)$ [on top of $C_i(\cdot)$], then the *weak assumption holds*: in particular, each i knows $u_i(q_i, q_{-i}) = q_i P\left(\sum_{j\in I} q_j\right) - C_i(q_i)$.
  - If some i does not know $P(\cdot)$, *not even the weak assumption holds.*

Finite and compact-continuous games

We focus on either finite or compact-continuous games:

- Game G is **finite** if, for each player $i \in I$, the action set A_i is finite.
- Game G is **compact-continuous** if, for each $i \in I$,
  - A_i is a *compact* subset of a Euclidean space (characterization: *a subset of a Euclidean space is compact if and only if it is closed and bounded*; not true for more general metric spaces),
  - the payoff function $u_i : \times_{j \in I} A_j \to \mathbb{R}$ is *continuous*.
- When we analyze *probabilistic concepts*, we mostly focus on *finite* games (such concepts are mathematically advanced for infinite games).

Probabilities

- Fix a *finite* set X. The set of probability mass functions (or distributions) on X is the **simplex** in $\mathbb{R}^X$

$$\Delta(X) = \left\{ \mu \in \mathbb{R}_+^X : \sum_{x \in X} \mu(x) = 1 \right\},$$

a *convex* and *compact* subset of Euclidean space $\mathbb{R}^X$.

  - [By finiteness of X, probability mass functions are equivalent to probability measures $\mu \in \mathbb{R}_+^{2^X}$ *via* formula $\mu(E) = \sum_{x \in E} \mu(x)$.]

- The **Dirac** measure/mass function supported by x is the function

$$\delta_x(x') = \begin{cases} 1 & \text{if } x' = x, \\ 0 & \text{if } x' \neq x. \end{cases}$$

Since the set $\{\delta_x\}_{x \in X} \subseteq \Delta(X)$ is isomorphic to X ($\{\delta_x\}_{x \in X} \cong X$, read "*essentially the same as* X"), it *makes sense to write* $X \subseteq \Delta(X)$.

Geometry of the simplex

Consider the *3-point case*:

- $X = \{x^1, x^2, x^3\}$. Write $\mu(x^i) = \mu_i$.
- The simplex $\Delta(X)$ is the triangle (a 2-dimensional object) in $\mathbb{R}^X \cong \mathbb{R}^3$ with vertices

$$\begin{aligned} \delta_{x^1} &= (1, 0, 0), \\ \delta_{x^2} &= (0, 1, 0), \\ \delta_{x^1} &= (0, 0, 1), \end{aligned}$$

  the classical base of the 3-dimensional Euclidean space.
- This triangle is obtained from intersection of the plane $\left\{\mu \in \mathbb{R}^3 : \sum_{i=1}^3 \mu_i = 1\right\}$ with the positive orthant $\mathbb{R}^X_+$.

Conjectures

- Player i cannot know what the co-players $-i$ are going to do, she can only make a (perhaps educated) guess.
- We call "**conjectures**" of player i the subjective probabilistic beliefs $\mu^i \in \Delta(A_{-i})$. The set $A_{-i} \subseteq \Delta(A_{-i})$ of co-players' action profiles is also interpreted as the set of **deterministic conjectures** (Dirac beliefs) of i about $-i$. See above.
- **Note**, the probabilities $\mu^i(a_{-i})$ are *subjective*, because a_{-i} is the result of the free choices of the co-players, not the realization of a random variable with known distribution.

Expected payoff/utility maximization

- The payoff function u_i is extended to domain $A_i \times \Delta(A_{-i})$ by means of **subjective expected utility** (SEU) calculations:

$$u_i\left(a_i, \mu^i\right) = \sum_{a_{-i} \in A_{-i}} u_i\left(a_i, a_{-i}\right) \mu^i\left(a_{-i}\right).$$

- This function is *affine* (convex *and* concave) in μ^i over the *convex set* $\Delta(A_{-i})$: for each $a_i \in A_i$ and all $\mu^i, \bar{\mu}^i \in \Delta(A_{-i})$, $\omega \in [0,1]$,
  - $\left(\omega\mu^i + (1-\omega)\bar{\mu}^i\right) \in \Delta(A_{-i})$ and
  - $u_i\left(a_i, \omega\mu^i + (1-\omega)\bar{\mu}^i\right) = \omega u_i\left(a_i, \mu^i\right) + (1-\omega) u_i\left(a_i, \bar{\mu}^i\right)$.
- A **rational** player i with conjecture μ^i chooses an action in

$$\arg\max_{a_i \in A_i} u_i\left(a_i, \mu^i\right) := \left\{a_i^* \in A_i : \forall a_i \in A_i, u_i\left(a_i^*, \mu^i\right) \geq u_i\left(a_i, \mu^i\right)\right\}.$$

Extension of choice to mixed actions

- For analytical convenience, we consider the following *contrived assumption*: players can delegate their choice of actions to objective randomization devices, such as a roulette spin, or a fair coin toss.
- What players choose is the randomization, e.g., if $a_i \in \{t, b\}$ (top or bottom) i can delegate the choice to a fair coin toss: t if Tail, b if Head; thus, i is effectively choosing to play t and b with objective probability $\frac{1}{2}$. This is an example of "mixed action."
- **Mixed actions** are objectively randomized choices described by the induced objective probabilities $\alpha_i \in \Delta(A_i)$.
- **Note:** *pure actions are a special case of mixed actions*, as choosing a_i is like choosing δ_{a_i} (a device that selects a_i with probability 1); Thus, formally, we can write $A_i \subseteq \Delta(A_i)$.

Extended optimization problem

- The payoff function u_i is extended to domain $\Delta(A_i) \times \Delta(A_{-i})$ by means of **expected utility** calculations, under the assumption that a_i is *statistically independent* of a_{-i}:
$$u_i(\alpha_i, \mu^i) = \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \alpha_i(a_i) u_i(a_i, a_{-i}) \mu^i(a_{-i}),$$
which is jointly *continuous* and separately *affine* (convex *and* concave) in both arguments.
- With this "trick", the problem of a rational player i with conjecture μ^i is $\max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i)$ (the max problem has a solution, because $\Delta(A_i)$ is compact and u_i is continuous in α_i).
- It turns out that the set of solutions has a simple structure, indeed, *an optimal mixed action is a randomization over optimal pure actions:*
$$\arg\max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i) = \Delta\left(\arg\max_{a_i \in A_i} u_i(a_i, \mu^i)\right).$$

Example

Payoff matrix representing u_i:

a_{-i}

a_i

u_i	ℓ	r
t	0	2
m	v	v
b	2	0

- Three cases:
  - $v < 1$: $\arg\max_{\alpha_i \in \Delta(A_i)} u_i\left(\alpha_i, \mu^i\right) = \{t\}$ if $\mu^i(r) > 1/2$, ... $= \Delta(\{t, b\})$ if $\mu^i(r) = 1/2$, ...$= \{b\}$ if $\mu^i(r) < 1/2$.
  - $v > 1$: *solve in detail as exercise*; $\arg\max_{\alpha_i \in \Delta(A_i)} u_i\left(\alpha_i, \mu^i\right) = \{t\}$ for $\mu^i(r)$ close to 1, ...$= \{m\}$ for $\mu^i(r)$ close to $1/2$, ...$= \{b\}$ for $\mu^i(r)$ close to 0, plus two cases of indifference: ...$= \Delta(\{t, m\})$ if $\mu^i(r) =?$, ...$= \Delta(\{m, b\})$ if $\mu^i(r) =?$
  - $v = 1$: $\arg\max_{\alpha_i \in \Delta(A_i)} u_i\left(\alpha_i, \mu^i\right) = \{t\}$ if $\mu^i(r) > 1/2$, ... $= \Delta(\{t, m, b\})$ if $\mu^i(r) = 1/2$, ...$= \{b\}$ if $\mu^i(r) < 1/2$.

References

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