Justifiability and Dominance

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Abstract

We analyze rational behavior for some given conjecture, without analyzing yet how conjectures are formed. An action of a player in a game is **justifiable** if it is a best reply to at least one probabilistic conjecture, it is **undominated** (or **admissible**) if there is no mixed action that yields a strictly higher expected payoff for every action profile of the co-players (i.e., for every deterministic conjecture). These slides go through a proof of the following duality result: *an action is justifiable if and only if it is undominated (admissible)*. The proof relies on the Maxmin Theorem applied to the incentive to deviate to a mixed action for a given conjecture: the smallest (with respect to conjectures) maximized incentive to deviate is non-positive [justifiability] if and only if the largest (with respect to mixed actions) minimized incentive to deviate is non-positive ["undominance"].

[These slides complement Ch. 3 of Game Theory: Analysis of Strategic Thinking (GT-AST). For the mathematical formalism see also the note *Mathematical Language and Game Theory*.]

Introduction

- We analyze rational behavior for some given conjecture, without analyzing yet how conjectures are formed.
- The *equivalence between justifiability and not being dominated* was "discovered for the last time" (in the same sense as Columbus was the last one to discover America) by the game theorist David Pearce (1984), who cited earlier versions in the statistical literature.
- Indeed, a related result called the *Complete Class Theorem* was discovered in the 40's by Abraham Wald in his seminal work relating statistical inference to game theory (e.g., Wald 1949). Thus, we refer to the equivalence result as *Wald-Pearce Lemma.*
- Both Wald and Pearce rely on the *Maxmin Theorem* (due to von Neumann) to prove their results. We do the same here, following the Appendix of Ch. 3 of GT-AST (see also the different approach in the main text of Ch. 3, which relies of Farkas Lemma).

Consider a *finite game with simultaneous moves* $\left\langle I, (A_i, u_i)_{i \in I} \right\rangle$, where

- I is the finite **player set**, with generic element i;
- A_i is the finite **set of feasible actions** of player i;
- $u_i : A_i \times A_{-i} \to \mathbb{R}$ is the "**payoff function**" of i, with $A_{-i} := \times_{j \in I \backslash \{i\}} A_j$.
- Each action a_i yields the payoff vector $\mathbf{u}_i(a_i) := (u_i(a_i, a_{-i}))_{a_{-i} \in A_{-i}} \in \mathbb{R}^{A_{-i}}$.

- Fix a *finite* set X. The set of probability mass functions on X is the simplex

$$\Delta(X) = \left\{ \mu \in \mathbb{R}_+^X : \sum_{x \in X} \mu(x) = 1 \right\},$$

a *convex* and *compact* subset of Euclidean space $\mathbb{R}^X$. (By finiteness of X, probability density functions are equivalent to probability measures $\mu \in \mathbb{R}_+^{2^X}$.)

- The Dirac mass function/measure supported by x is the function

$$\delta_x(x') = \begin{cases} 1 & \text{if } x' = x, \\ 0 & \text{if } x' \neq x. \end{cases}$$

Since $X \cong \{\delta_x\}_{x \in X} \subseteq \Delta(X)$, it makes sense to write $X \subseteq \Delta(X)$.

- The **support of** μ is the set $\mathrm{supp}\mu := \{x \in X : \mu(x) > 0\}$ of elements with strictly positive μ-probability.

- **Conjectures** are subjective probabilistic beliefs $\mu^i \in \Delta(A_{-i})$. Set $A_{-i} \subseteq \Delta(A_{-i})$ is also interpreted as the set of **deterministic conjectures.**
- **Mixed actions** are objectively randomized choices described by the induced vector of objective probabilities $\alpha_i \in \Delta(A_i)$.
- The payoff function u_i is extended to domain $\Delta(A_i) \times \Delta(A_{-i})$ by means of **subjective expected utility** (SEU) calculations:

$$u_i\left(\alpha_i, \mu^i\right) = \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} \alpha_i(a_i)\, u_i(a_i, a_{-i})\, \mu^i(a_{-i}).$$

- Each mixed action α_i yields (objective) EU $u_i(\alpha_i, a_{-i})$ for each a_{-i}, and EU vector $\mathbf{u}_i(\alpha_i) := (u_i(\alpha_i, a_{-i}))_{a_{-i} \in A_{-i}} \in \mathbb{R}^{A_{-i}}$; with this, we get the inner product SEU formula $u_i\left(\alpha_i, \mu^i\right) = \mathbf{u}_i(\alpha_i) \cdot \mu^i$.

Fundamental Max-SEU Lemma & geometry

Lemma

Fix a mixed action α_i^* and a conjecture $\mu^i \in \Delta(A_{-i})$; α_i^* is optimal (SEU maximizing) under μ^i if and only if each pure action in the support of α_i^* is optimal under μ^i:

$$\alpha_i^* \in \arg\max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i) \iff \mathrm{supp}\alpha_i^* \subseteq A_i \cap \arg\max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i).$$

- **Geometrical intuition:** Set of "feasible vectors" $\mathbf{U}_i$
  - $\mathbf{U}_i := \left\{ \mathbf{u}_i \in \mathbb{R}^{A_{-i}} : \exists \alpha_i \in \Delta(A_i), \mathbf{u}_i = \left(u_i(\alpha_i, a_{-i})\right)_{a_{-i} \in A_{-i}} \right\}$, a *convex* **polytope** (multi-dim. version of poly-gone/hedron).
  - $\mathbf{H}(\mu^i, \bar{u}) := \left\{ \mathbf{u}_i \in \mathbb{R}^{A_{-i}} : \mathbf{u}_i \cdot \mu^i = \bar{u} \right\}$ is the "indifference hyperplane" with constant-SEU$=\bar{u}$ given conjecture μ^i.
  - *Equivalent problem*: $\max \bar{u}$, s.t. $\mathbf{H}(\mu^i, \bar{u}) \cap \mathbf{U}_i \neq \emptyset$.
  - Intersection at max contains a *vertex* $\left(u_i(a_i^*, a_{-i})\right)_{a_{-i} \in A_{-i}}$ of $\mathbf{U}_i$, and *vertices of* $\mathbf{U}_i$ are given by *pure actions*. ♡

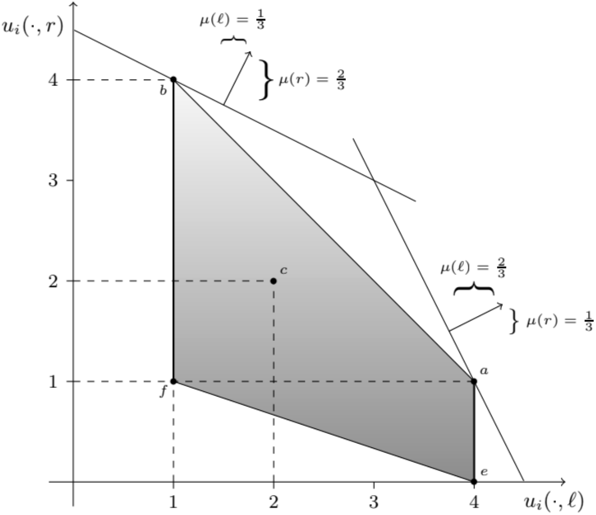
Example: payoff matrix and extreme points

- Two-person game, $A_i = \{a, b, c, e, f\}$, $A_{-i} = \{\ell, r\}$

u_i	ℓ	r
a	4	1
b	1	4
c	2	2
e	4	0
f	1	1

- The set of extreme points of the polytope $\mathbf{U}_i$ is $\{(4,1),(1,4),(4,0),(1,1)\}$, while $(2,2)$ is a convex combination of $(4,1)$, $(1,4)$, and $(1,1)$ (can you find the weights?).

Example: polytope and geometry of Max-EU



Fundamental Max-SEU Lemma & algebra

- Let $r_i\left(\mu^i\right) := A_i \cap \arg\max_{\alpha_i \in \Delta(A_i)} u_i\left(\alpha_i, \mu^i\right)$ denote the **set of pure best replies** to μ^i.
- The **Fundamental Max-SEU Lemma** says that *there is no need to randomize because*

$$\alpha_i^* \in \arg\max_{\alpha_i \in \Delta(A_i)} u_i\left(\alpha_i, \mu^i\right) \Longleftrightarrow \mathrm{supp}\alpha_i^* \subseteq r_i\left(\mu^i\right).$$

- **Algebraic intuition** [of the proof in GT-AST that you must learn]**:**
  - ($\Rightarrow$) By *contraposition*, suppose there is $a_i^* \in \mathrm{supp}\alpha_i^* \backslash r_i\left(\mu^i\right)$ (show $\alpha_i^* \notin \arg\max...$). Then, $\exists \alpha_i$, $u_i\left(\alpha_i, \mu^i\right) > u_i\left(a_i^*, \mu^i\right)$; hence, $\exists a_i' \in \mathrm{supp}\alpha_i$, $u_i\left(a_i', \mu^i\right) > u_i\left(a_i^*, \mu^i\right)$ ($u_i\left(\alpha_i, \mu^i\right)$ is a weighted average). Given α_i^*, shift all prob. mass from a_i^* to a_i' to obtain α_i' s.t.
$\left[u_i\left(\alpha_i', \mu^i\right) - u_i\left(\alpha_i^*, \mu^i\right)\right] = \alpha_i^*\left(a_i'\right)\left[u_i\left(a_i', \mu^i\right) - u_i\left(a_i^*, \mu^i\right)\right] > 0.$
  - ($\Leftarrow$) Let $\hat{u}_i\left(\mu^i\right) = \max_{\alpha_i \in \Delta(A_i)} u_i\left(\alpha_i, \mu^i\right)$. If $\mathrm{supp}\alpha_i^* \subseteq r_i\left(\mu^i\right)$ then $u_i\left(\alpha_i^*, \mu^i\right) = \sum_{a_i \in r_i(\mu^i)} u_i\left(a_i, \mu^i\right) = \hat{u}_i\left(\mu^i\right)$. ♡

Justifiability

- Intuitively, an action is justifiable as the choice of a rational player with preferences represented by u_i if there exists a conjecture that justifies it as a best reply.

Definition

An action $\bar{a}_i \in A_i$ is **justifiable** if $\bar{a}_i \in \bigcup_{\mu^i \in \Delta(A_{-i})} r_i\left(\mu^i\right)$, that is, if

$$\exists \mu^i \in \Delta\left(A_{-i}\right), \forall \alpha_i \in \Delta\left(A_i\right), u_i\left(\alpha_i, \mu^i\right) \leq u_i\left(\bar{a}_i, \mu^i\right).$$

- **Observation** *(incentive to deviate).* The difference $\left[u_i\left(\alpha_i, \mu^i\right) - u_i\left(\bar{a}_i, \mu^i\right)\right]$ measures the incentive to deviate from the given action $\bar{a}_i$ to mixed action α_i. Action $\bar{a}_i$ is justifiable IFF

$$\exists \mu^i \in \Delta\left(A_{-i}\right), \forall \alpha_i \in \Delta\left(A_i\right), \left[u_i\left(\alpha_i, \mu^i\right) - u_i\left(\bar{a}_i, \mu^i\right)\right] \leq 0.$$

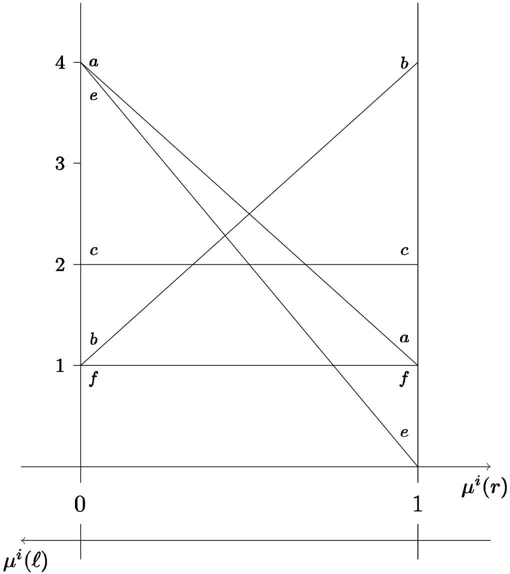
Justifiability: example

- Two-person game, $A_i = \{a, b, c, e, f\}$, $A_{-i} = \{\ell, r\}$

u_i	ℓ	r
a	4	1
b	1	4
c	2	2
e	4	0
f	1	1

- Intuitively, c dominates f, which is obviously un-justifiable (never a best reply).
- Is c justifiable? *No*: a (respectively, b) is strictly better if $\mu^i(r) \leq \frac{1}{2}$ (respectively, $\mu^i(r) \geq \frac{1}{2}$).
- Actually, c is dominated by $\frac{1}{2} : \frac{1}{2}$ mixture of a and b which yields 2.5 on average under both ℓ and r (see Fig.s 3.2, 3.3 GT-AST).

Expected utility lines



Justifiability and minmax

- Let X and Y be *compact* subsets of Euclidean spaces.
  - **Observation** If $\bar{f} : X \to \mathbb{R}$ is *continuous*, then

$$\left(\forall x \in X,\ \bar{f}(x) \leq 0\right) \iff \max_{x \in X} \bar{f}(x) \leq 0.$$

    If $h : Y \to \mathbb{R}$ is *continuous*, then

$$\left(\exists y \in Y,\ h(y) \leq 0\right) \iff \min_{y \in Y} h(y) \leq 0.$$

  - **Observation** If $f : X \times Y \to \mathbb{R}$ is (jointly) *continuous*, then function

$$\begin{array}{cccc} h: & Y & \to & \mathbb{R} \\ & y & \mapsto & \max_{x \in X} f(x, y) \end{array}$$

    is well defined and continuous (Maximum Theor.). Therefore,

$$\exists y \in Y, \forall x \in X, f(x, y) \leq 0 \iff \min_{y \in Y} \max_{x \in X} f(x, y) \leq 0.$$

- In particular, *an action $\bar{a}_i$ is justifiable IFF*

$$\min_{\mu^i \in \Delta(A_{-i})} \max_{\alpha_i \in \Delta(A_i)} \left[u_i\left(\alpha_i, \mu^i\right) - u_i\left(\bar{a}_i, \mu^i\right)\right] \leq 0.$$

Dominance

- Recall, each $\alpha_i \in \Delta(A_i)$ is associated with a vector of objective expected utilities

$$\mathbf{u}_i(\alpha_i) = (u_i(\alpha_i, a_{-i}))_{a_{-i} \in A_{-i}} \in \mathbb{R}^{A_{-i}}.$$

- The dominance relation between (pure and mixed) actions is based on the incomplete coordinate-wise order over vectors in $\mathbb{R}^{A_{-i}}$: α_i dominates β_i if $\mathbf{u}_i(\alpha_i) > \mathbf{u}_i(\beta_i)$ (a so called "lattice" order). Thus:

Definition

An action $\bar{a}_i \in A_i$ is **dominated** if

$$\exists \alpha_i \in \Delta(A_i), \forall a_{-i} \in A_{-i}, u_i(\alpha_i, a_{-i}) > u_i(\bar{a}_i, a_{-i}),$$

and it is **undominated** if

$$\forall \alpha_i \in \Delta(A_i), \exists a_{-i} \in A_{-i}, u_i(\alpha_i, a_{-i}) \leq u_i(\bar{a}_i, a_{-i}).$$

Dominance and maxmin

Let X and Y be *compact* subsets of Euclidean spaces. Arguments similar to those given for justifiability and minmax yield the following:

- **Observation** If $f : X \times Y \to \mathbb{R}$ is (jointly) *continuous*, then function

$$\begin{array}{rccc} \ell : & X & \to & \mathbb{R} \\ & x & \mapsto & \min_{y \in Y} f(x,y) \end{array}$$

  is well defined and continuous. Therefore,

$$(\forall x \in X, \exists y \in Y, f(x,y) \leq 0) \Longleftrightarrow \max_{x \in X} \min_{y \in Y} f(x,y) \leq 0.$$

- **Observation.** In particular, *an action $\bar{a}_i$ is undominated IFF*

$$\max_{\alpha_i \in \Delta(A_i)} \min_{a_{-i} \in A_{-i}} \left[u_i(\alpha_i, a_{-i}) - u_i(\bar{a}_i, a_{-i}) \right] \leq 0.$$

The equivalence result of Wald and Pearce

Lemma

For each player $i \in I$ and action $\bar{a}_i \in A_i$ in a finite game, $\bar{a}_i$ is justifiable if and only if it is undominated.

- By the previous observations, the equivalence can be restated as a minmax=maxmin equality.
- To abbreviate formulas, write the incentive to deviate from $\bar{a}_i$ to α_i given μ^i as

$$D\left(\alpha_i, \mu^i\right) = \left[u_i\left(\alpha_i, \mu^i\right) - u_i\left(\bar{a}_i, \mu^i\right)\right].$$

- With this, the lemma holds if and only if

$$\min_{\mu^i \in \Delta(A_{-i})} \max_{\alpha_i \in \Delta(A_i)} D\left(\alpha_i, \mu^i\right) = \max_{\alpha_i \in \Delta(A_i)} \min_{a_{-i} \in A_{-i}} D\left(\alpha_i, a_{-i}\right) \qquad (\bigstar)$$

(note: beside the minmax/maxmin inversion, we have probabilistic conjectures in the LHS, and deterministic ones in the RHS).

Proof of the Wald-Pearce Lemma

Preliminary result: the MaxMin Theorem of von Neumann

- Fix a real-valued function $d : A \times B \to \mathbb{R}$, with A and B *finite* sets.
- Note, by definition,
  $\forall b \in B, \max_{a \in A} \min_{b' \in B} d(a, b') \leq \max_{a \in A} d(a, b)$.
- Hence, $\max_{a \in A} \min_{b \in B} d(a, b) \leq \min_{b \in B} \max_{a \in A} d(a, b)$ ...
- ... and inequality $\leq$ can be strict ($<$). Yet, ...
- Let $D : \Delta(A) \times \Delta(B) \to \mathbb{R}$ denote the bi-affine extension of d from $A \times B$ to $\Delta(A) \times \Delta(B)$: for all $\alpha \in \Delta(A)$, $\beta \in \Delta(B)$,

$$D(\alpha, \beta) = \sum_{a \in A} \sum_{b \in B} \alpha(a) \, d(a, b) \beta(b) .$$

- **MaxMin Theorem (von Neumann)**

$$\max_{\alpha \in \Delta(A)} \min_{\beta \in \Delta(B)} D(\alpha, \beta) = \min_{\beta \in \Delta(B)} \max_{\alpha \in \Delta(A)} D(\alpha, \beta) .$$

Proof of the Wald-Pearce Lemma

- To prove equality (★),

$$\min_{\mu^i\in\Delta(A_{-i})}\max_{\alpha_i\in\Delta(A_i)} D\left(\alpha_i,\mu^i\right)=\max_{\alpha_i\in\Delta(A_i)}\min_{a_{-i}\in A_{-i}} D\left(\alpha_i,a_{-i}\right)$$

look at RHS.

- Since, both $u_i\left(\alpha_i,\mu^i\right)$ and $u_i\left(\bar{a}_i,\mu^i\right)$ are affine in μ^i, so is $D\left(\alpha_i,\mu^i\right)$. The min of an affine function on a compact and convex polyhedron $[\Delta\left(A_{-i}\right)]$ is attained at some extreme point [in A_{-i}]; thus,

$$\forall\alpha_i\in\Delta\left(A_i\right),\ \min_{a_{-i}\in A_{-i}} D\left(\alpha_i,a_{-i}\right)=\min_{\mu^i\in\Delta(A_{-i})} D\left(\alpha_i,\mu^i\right).$$

- Thus, by substitution in the RHS of (★), we must prove that

$$\min_{\mu^i\in\Delta(A_{-i})}\max_{\alpha_i\in\Delta(A_i)} D\left(\alpha_i,\mu^i\right)=\max_{\alpha_i\in\Delta(A_i)}\min_{\mu^i\in\Delta(A_{-i})} D\left(\alpha_i,\mu^i\right),$$

which follows from the Maxmin Theorem of von Neumann. ■

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