Rationalizability: The Calculus of Strategic Thinking, I

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Game Theory: Analysis of Strategic Thinking

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Abstract

The analysis of strategic thinking allows us to endogenize conjectures (beliefs about others' behavior). Recall: an action of a player in a game is consistent with rationality IFF (if and only if) it is justifiable. A rational player who believes in the rationality of others chooses actions justified as best replies to conjectures that assign probability 0 to the unjustifiable actions of others. If a player also believes that everybody else believes in rationality, a further restriction on conjectures is obtained. More generally, a chain of stronger and stronger restrictions on conjectures is obtained to represent the implications of sophisticated strategic thinking. An action is rationalizable if it is a best reply to some conjecture satisfying all these restrictions. Thus, rationalizability characterizes the actions consistent with rationality and sophisticated strategic thinking. The rationalization operator is an analytical tool for the "calculus" of such actions. Relying on the Wald-Pearce Lemma (an action is justifiable IFF it is not dominated) we show that rationalizability can be computed by iterated elimination of dominated actions even if the elimination procedure is not maximal at each stage.

[These slides summarize and complement Chapter 4.1-3 of GT-AST]

Heuristic example: duopoly

Consider a Cournot (*quantity-setting*) *duopoly* with

- inverse **market demand** $P(q_i + q_{-i}) = \max\left\{0, \bar{P} - \beta(q_i + q_{-i})\right\}$, $\bar{P}, \beta > 0$;
- **cost functions** $C_i(q_i) = cq_i$, $c < \bar{P}$ (*constant marginal cost*);
- and *large capacity* for each firm, $\bar{q} > \left(\bar{P} - c\right)/\beta$.
- Firms are expected profit maximizers (hence, *risk neutral*).
- All of the above is common knowledge (*complete information*).
- W.l.o.g., *we can focus on deterministic conjectures* (see Problem 10). Each i has B.R. function $r(q_{-i}) = \max\left\{0, \frac{\bar{P}-c}{2\beta} - \frac{1}{2}q_{-i}\right\}$.

Strategic thinking in Cournot duopoly

1. An action (output) q_i is *consistent with rationality* (justifiable) IFF it is a best reply to a conjecture, that is, $q_i \in r([0, \bar{q}]) = \left[0, \frac{\bar{P}-c}{2\beta}\right]$, where $r([a, b]) = [r(b), r(a)]$ because r is decreasing. Thus, $0 = r(\bar{q})$ and $\frac{\bar{P}-c}{2\beta} = r(0) = q^M$ is how much a monopolist would produce.
2. If firm i believes in the rationality of $-i$, it is certain that $q_{-i} \leq q^M$. Thus, q_i is *consistent with rationality and belief in rationality* IFF $q_i \in r(r([0, \bar{q}])) = \left[\frac{\bar{P}-c}{4\beta}, \frac{\bar{P}-c}{2\beta}\right]$, where $\frac{\bar{P}-c}{4\beta} = r\left(\frac{\bar{P}-c}{2\beta}\right) = r(r(0))$.
3. If, on top of 1 and 2, firm i is certain that $-i$ is certain that i is rational, then it is certain that $q_{-i} \geq \frac{\bar{P}-c}{4\beta}$. Thus, q_i is *consistent with rationality, belief in rationality, and belief of belief in rationality* IFF $q_i \in r(r(r([0, \bar{q}]))) = \left[\frac{\bar{P}-c}{4\beta}, \frac{3(\bar{P}-c)}{8\beta}\right]$, where $\frac{3(\bar{P}-c)}{8\beta} = r\left(\frac{\bar{P}-c}{4\beta}\right) = r(r(r(0)))$.

Strategic thinking in Cournot duopoly (cont.)

- Generalizing to further reasoning steps, it can be shown that q_i is *consistent with rationality and* $k-1$ *levels of mutual belief in rationality* IFF $q_i \in r^k([0,\bar{q}])$ ($r^k = r \circ r^{k-1}$, r^0 by convention is the identity function).
- For $k = 1, 3, ...$ (odd), $r^k([0,\bar{q}]) = \left[r^{k-1}(0), r^k(0)\right]$, where $r^1(0) = q^M$ (the upper bound is decreased).
- For $k = 2, 4, ...$ (even), $r^k([0,\bar{q}]) = \left[r^k(0), r^{k-1}(0)\right]$ (the lower bound is increased).
- It turns out that the sequences of lower bounds and upper bounds (necessarily monotone) have the same limit q^*, which must satisfy $q^* = r(q^*)$: the Cournot-Nash equilibrium!

Strategic thinking in Cournot oligopoly

- Instead, with $n \geq 3$ firms, the process gets stuck after the first step:
  - On one hand, if i is "very pessimistic", it believes that each competitor produces q^M; since $n-1 \geq 2$,

$$q_i = r\left((n-1)\frac{\bar{P}-c}{2\beta}\right) = \max\left\{0, \frac{\bar{P}-c}{2\beta} - \frac{(n-1)}{2}\frac{\bar{P}-c}{2\beta}\right\} = 0.$$

  - On the other hand, if i is "very optimistic", it believes that everybody else is producing 0 and the best reply is $q_i = q^M$. Hence, we obtain the same bounds as in step 1 and nothing more can be excluded.
- **Take home message:** In some cases (here, $n = 2$), sophisticated strategic thinking leads to a unique prediction, which must be the unique equilibrium. In other cases (here $n \geq 3$) sophisticated strategic thinking allows for many possibilities. (Compare with the analysis of a perfectly competitive market in Ch. 1.)

Assumptions about players' beliefs

- In the example, we formally described *only* the *actions* consistent with higher and higher levels of strategic thinking, there was no mathematical representation of the events "i is rational", "i believes that $-i$ is rational", and so on.
- Such mathematical representation is studied in so called *epistemic game theory* (e.g., Dekel & Siniscalchi 2015), but it is too advanced to be considered in detail here. *We only give a hint and introduce some suggestive and helpful notation.*
- Ω is a set of **states of the world**, each state ω describes a possible configuration of behaviors and beliefs, including beliefs about the beliefs of others (a "conjecture" is a belief about others' behavior). **Events** are subsets $E \subseteq \Omega$ representing propositions such as "i is rational", denoted R_i, and "everybody is rational" (rationality), denoted $R = \cap_{i \in I} R_i$.
- $\mathrm{B}_i(E)$="i believes (is certain of) E". $\mathrm{B}(E) = \cap_{i \in I} \mathrm{B}_i(E)$.

The rationalization operator

Preliminaries

- We want to characterize the *behavioral implications* of R, $R \cap \mathrm{B}(R)$, $R \cap \mathrm{B}(R) \cap \mathrm{B}(\mathrm{B}(R))$ ... and so on in game

$$G = \left\langle I, (A_i, u_i)_{i \in I} \right\rangle$$

  under *complete information* (common knowledge of rules and preferences).
- We mostly focus on *finite games*, with some hints at compact-continuous games.
- A subset of action profiles $C \subseteq A := \times_{i \in I} A_i$ is **Cartesian** if $C = \times_{i \in I} C_i$ ($C_i \subseteq A_i$ for each i). The **collection of Cartesian subsets of** A is denoted $\mathcal{C}$.
- Next we define a self-map $\rho : \mathcal{C} \to \mathcal{C}$ called "rationalization operator": intuitively, $\rho(C)$ represents the set of action profiles consistent with rationality *if* each i believes others are choosing in C (that is, in "their part of C," C_{-i}).

The rationalization operator: example

- Consider the following 2-person 3x3 game:

	ℓ	c	r
T	3, 2	0, 1	0, 0
M	0, 2	3, 1	0, 0
B	1, 1	1, 2	3, 0

- $\mathcal{C}$ contains $\left(2^3 - 1\right) \times \left(2^3 - 1\right) = 7 \times 7 = 49$ nonempty sets.
- If we pick $C \in \mathcal{C}$ arbitrarily, there is no *necessary* inclusion relationship between C and $\rho(C)$.
- E.g.: $\rho(\{M, B\} \times \{\ell, c\}) = \{T, M\} \times \{\ell, c\}$, $\rho(\{T, M\} \times \{r\}) = \{B\} \times \{\ell\}$.
- But, if we start from A we get a decreasing sequence: $\rho(A) = \{T, M, B\} \times \{\ell, c\}$, $\rho(\{T, M, B\} \times \{\ell, c\}) = \{T, M\} \times \{\ell, c\}$, $\rho(\{T, M\} \times \{\ell, c\}) = \{T, M\} \times \{\ell\}$, $\rho(\{T, M\} \times \{\ell\}) = \{T\} \times \{\ell\}$.

The rationalization operator

Definition

- We let $\Delta(C_{-i})$ denote the set of conjectures μ^i such that $\mu^i(C_{-i}) = 1$ ($C_{-i} = \times_{j \neq i} C_j$). Thus, the set of best replies to such conjectures is

$$r_i(\Delta(C_{-i})) = \bigcup_{\mu^i : \mu^i(C_{-i})=1} r_i(\mu^i).$$

Definition

The **rationalization operator** is the self-map

$$\begin{array}{cccc} \rho : & \mathcal{C} & \rightarrow & \mathcal{C}, \\ & C & \mapsto & \times_{i \in I} r_i(\Delta(C_{-i})). \end{array}$$

- Thus, $\rho(C)$ is the set of behaviors "rationalized" by the belief that (other) players are choosing in C.

The rationalization operator

Monotonicity

- We endow the collection of subsets $\mathcal{C}$ with the partial order $\subseteq$ of (weak) set inclusion: E is "weakly smaller" than F IFF $E \subseteq F$.
- With this, a map $\gamma : \mathcal{C} \to \mathcal{C}$ is **monotone** (in the weak sense) if $\gamma(E) \subseteq \gamma(F)$ whenever $E \subseteq F$.
- **Observation.** *The rationalization operator is monotone.*
- **Proof.** Fix $E, F \in \mathcal{C}$ so that $E \subseteq F$ (i.e., $\forall j \in I$, $E_j \subseteq F_j$).
  - For each $i \in I$, $E_{-i} \subseteq F_{-i}$;
  - since $\mu^i(E_{-i}) = 1 \Rightarrow \mu^i(F_{-i}) = 1$, we have $\Delta(E_{-i}) \subseteq \Delta(F_{-i})$,
  - and $r_i(\Delta(E_{-i})) \subseteq r_i(\Delta(F_{-i}))$ (a smaller set of conjectures justifies a smaller set of actions).
  - Hence

$$\rho(E) = \times_{i \in I} r_i(\Delta(E_{-i})) \subseteq \times_{i \in I} r_i(\Delta(F_{-i})) = \rho(F). \blacksquare$$

The rationalization operator

Iterations

- For every self-map $f : X \to X$ define the iterations f^k of f as follows: $f^0 = \mathrm{Id}_X$ (the identity map on X), and $f^k = f \circ f^{k-1}$ for every $k \in \mathbb{N}$. Thus, for all $x \in X$, $k \in \mathbb{N}$,

$$f^0(x) = x,\ f^k(x) = f\left(f^{k-1}(x)\right).$$

- Let $X = \mathcal{C}$ and $f = \rho$ to obtain, for all $C \in \mathcal{C}$, $k \in \mathbb{N}$,

$$\rho^0(C) = C,\ \rho^k(C) = \rho\left(\rho^{k-1}(C)\right).$$

For example,

- $\rho^1(C) = \rho(C)$,
- $\rho^2(C) = \rho(\rho(C))$,
- $\rho^3(C) = \rho(\rho(\rho(C)))$.

The rationalization operator

Proofs by (mathematical) induction

- Proofs by induction are *essential* to understand GT! To prove by induction that a property $P(k)$ (indexed by natural numbers) holds for every $k \in \mathbb{N}_0$, one has to prove two things:
  - *Basis.* $P(0)$ is true.
  - *Inductive step.* For any given $k \in \mathbb{N}_0$, if $P(k)$ holds (**inductive hypothesis, IH**), then $P(k+1)$ holds as well.
- Next, we rely on the monotonicity of ρ to prove that *for every* $k \in \mathbb{N}_0$, $\rho^{k+1}(A) \subseteq \rho^k(A)$.

The rationalization operator

Example of a proof by induction

- **Observation.** *For every* $k \in \mathbb{N}_0$, $\rho^{k+1}(A) \subseteq \rho^k(A)$.
- **Proof**
  - *Basis.* $\rho^1(A) \overset{\text{(def)}}{=} \rho(A) \overset{\text{(def)}}{\subseteq} A \overset{\text{(def)}}{=} \rho^0(A)$ (everything holds by definition).
  - *Inductive step.* Suppose that $\rho^{k+1}(A) \subseteq \rho^k(A)$ (IH), we have to prove $\rho^{k+2}(A) \subseteq \rho^{k+1}(A)$, which follows from the chain of inclusions:

$$\rho^{k+2}(A) \overset{\text{(def)}}{=} \rho\left(\rho^{k+1}(A)\right) \overset{\text{(IH \& mon)}}{\subseteq} \rho\left(\rho^k(A)\right) \overset{\text{(def)}}{=} \rho^{k+1}(A). \blacksquare$$

- Thus, the sequence of subsets $\left(\rho^k(A)\right)_{k\in\mathbb{N}} = \left(\rho^1(A), \rho^2(A), \rho^3(A), ...\right)$ is (weakly) decreasing.

- Although we do not provide here a mathematical analysis of the events/assumptions R, $R \cap \mathrm{B}(R)$, $R \cap \mathrm{B}(R) \cap \mathrm{B}(\mathrm{B}(R))$ ..., we can use ρ to obtain intuitively a characterization of their behavioral implications (recall: our intuitive analysis is mathematized by epistemic game theory).
- Behavior (action profile) $(a_i)_{i \in I}$ is *consistent with*

1. *Rationality* (R) IFF it is justifiable, that is, $(a_i)_{i \in I} \in \rho(A) = \times_{i \in I} r_i(\Delta(A_{-i}))$.
2. *Rationality and belief in rationality* $(R \cap \mathrm{B}(R))$ IFF it is justified by conjectures that assign probability 0 to unjustifiable actions of others, that is, IFF $(a_i)_{i \in I} \in \rho(\rho(A))$.
3. *... and belief of belief in rationality* $(R \cap \mathrm{B}(R) \cap \mathrm{B}(\mathrm{B}(R)))$ IFF $(a_i)_{i \in I} \in \rho(\rho(\rho(A)))$.

Since $E \mapsto \mathrm{B}(E)$ is a self-map, write $\mathrm{B}^k(R) = \mathrm{B}\left(\mathrm{B}^{k-1}(R)\right)$. Since $\left(\rho^k(A)\right)_{k=1}^{\infty}$ is decreasing, define $\rho^{\infty}(A) = \bigcap_{k\geq 1} \rho^k(A)$. We **claim** that $\rho^{\infty}(A)$ *characterizes the behavioral implications of rationality and common belief in rationality:*

Assumptions	Behavioral implications
R	$\rho(A)$
$R \cap \mathrm{B}(R)$	$\rho(\rho(A)) = \rho^2(A)$
...	...
$R \cap \bigcap_{k=1}^{n} \mathrm{B}^k(R)$	$\rho^{n+1}(A)$
...	...
$R \cap \bigcap_{k=1}^{\infty} \mathrm{B}^k(R)$	$\rho^{\infty}(A)$

Rationalizability

Existence, fixed point property

Definition

Set $\rho^{\infty}(A)$ is the set of **rationalizable** action profiles. An action is rationalizable if it is part of a rationalizable profile.

- One can show that $\rho(\rho^{\infty}(A)) \subseteq \rho^{\infty}(A)$, but in some "pathological" games the inclusion is strict: i.e., once we reach the limit $\rho^{\infty}(A)$, we have to start all over again.
- The claim of the previous page is true IFF $\rho(\rho^{\infty}(A)) = \rho^{\infty}(A)$ (fixed point). It would also be nice to have existence: $\rho^{\infty}(A) \neq \emptyset$. Both hold in every sufficiently regular game. In particular:

Theorem

(Limit) *In every finite or compact-continuous game, $\rho(\rho^{\infty}(A)) = \rho^{\infty}(A) \neq \emptyset$. If the game is finite $\emptyset \neq \rho^{k}(A) = \rho^{\infty}(A)$ for $k \geq \left(\sum_{i \in I} |A_i|\right) - |I|$.*

- If $\rho^{k+1}(A) = \rho^k(A)$, then $\rho^\ell(A) = \rho^k(A)$ for all $\ell > k$. By finiteness, $r_i(\mu^i) \neq \emptyset$ for every μ^i, which implies that $\rho(C) \neq \emptyset$ for every $C \neq \emptyset$. An easy inductive argument then shows that $\rho^k(A) \neq \emptyset$ for every k (*check that you can prove it*).
- Since $\left(\rho^k(A)\right)_{k=0}^{\infty}$ is (weakly) decreasing, either $\rho^k(A) = A$ for all k, or in each step $k = 1, 2, ...$ one action is deleted for at least one player until—by finiteness—no more actions can be deleted and the sequence becomes constant from some k: $\rho^\infty(A) = \rho^k(A)$. The *slowest possible pace* is to delete just 1 action for just 1 player at each step k until (by the previous existence result) only one action is left. Thus, $\rho^k(A) = \rho^\infty(A)$ for every $k \geq \left(\sum_{i\in I}|A_i|\right) - |I|$. ■

Definition

Set $C \in \mathcal{C}$ has the **best reply property (BRP)** if $C \subseteq \rho(C)$.

- **Observations.** (1) The BRP may hold as an equality $[C = \rho(C)]$. (2) Consider singletons, $C = \left\{(a_i^*)_{i\in I}\right\} = \times_{i\in I}\left\{a_i^*\right\}$ for some $a^* = (a_i^*)_{i\in I} \in A$: *singleton* $\left\{(a_i^*)_{i\in I}\right\}$ *has the BRP IFF* a^* *is a Nash equilibrium* (that is, $a_i^* \in r_i\left(a_{-i}^*\right)$ for every $i \in I$).

Theorem

(BRP) In every finite or compact-continuous game, for every $\bar{a} \in A$, $\bar{a}$ *is rationalizable IFF* $\bar{a} \in C$ *for some set* C *with the BRP.*

- Since $\rho^\infty(A) = \rho(\rho^\infty(A))$ (limit theorem), the set of rationalizable profiles $\rho^\infty(A)$ is the largest set with the BRP.

- **(Only IF)** Let $a \in \rho^{\infty}(A)$ (rationalizability). By the limit theorem, $\rho^{\infty}(A) = \rho(\rho^{\infty}(A))$; hence, a belongs to a set with the BRP (satisfied with $=$, a special case of $\subseteq$).
- **(IF)** Let $a \in C \subseteq \rho(C)$ (hypothesis, Hp). We prove below by induction (using the monotonicity of ρ) that $C \subseteq \rho^k(C) \subseteq \rho^k(A)$ for every $k \in \mathbb{N}_0$. Thus, $a \in C \subseteq \rho^{\infty}(A)$.
  - *Basis.* $C \overset{\text{(def)}}{\subseteq} \rho^0(C) \overset{\text{(def)}}{\subseteq} \rho^0(A)$ because $\rho^0(E) = E$ for every E (the first $\subseteq$ holds as $=$).
  - *Inductive step.* Suppose that $C \subseteq \rho^k(C) \subseteq \rho^k(A)$ (IH), then

$$C \overset{\text{(Hp)}}{\subseteq} \rho(C) \overset{\text{(IH \& mon)}}{\subseteq} \rho(\rho^k(C)) \overset{\text{(IH \& mon)}}{\subseteq} \rho(\rho^k(A)).$$

By definition, $\rho \circ \rho^k = \rho^{k+1}$; thus,

$$C \subseteq \rho^{k+1}(C) \subseteq \rho^{k+1}(A). \blacksquare$$

Example: Back to Cournot Oligopoly, I

- Consider the symmetric Cournot oligopoly model presented at the beginning. It is a compact-continuous game with $A = [0, \bar{q}]^I$. It also has additional "nice" properties, which allow to prove the following:
- *Suppose that, for each $i \in I$, C_i is a closed interval, then*

$$\rho(C) = \times_{i \in I} r_i(C_{-i})$$

[where $r_i(C_{-i}) = \bigcup_{q_{-i} \in C_{-i}} r_i(q_{-i})$], that is, *we can restrict our attention to deterministic conjectures*.

- For every number of firms, the singleton $\{(q^*, ..., q^*)\} \subseteq [0, \bar{q}]^I$ containing just the Cournot-Nash equilibrium profile is a BRP set.

Example: Back to Cournot Oligopoly, II

- With $n \geq 3$ firms $\left[0, q^M\right]^I$ (where q^M is the monopolistic output) is a BRP set.
- Indeed,

$$r_i(0) = q^M,\ r_i\left(q^M, ..., q^M\right) = \max\left\{0, \frac{\bar{P}-c}{2\beta} - \frac{(n-1)}{2}\frac{\bar{P}-c}{2\beta}\right\} = 0$$

thus, $\rho\left(\left[0, q^M\right]^I\right) = \left[0, q^M\right]^I$. This implies that $\left[0, q^M\right]^I \subseteq \rho^\infty(A)$, by the previous theorem.

- Also, $\rho(A) = \rho\left([0, \bar{q}]^I\right) = \left[0, q^M\right]^I$ (note: $r_i(\bar{q}, ..., \bar{q}) = 0$).
  - Thus, $\left[0, q^M\right]^I = \rho^\infty(A)$ is precisely the rationalizable set,
  - because $\rho^1(A) = \rho(A) = \rho(\rho(A)) = \rho^2(A)$, thus $\forall k \in \mathbb{N}$, $\rho^k(A) = \left[0, q^M\right]^I$.

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COURNOT DUOPOLY (SYMMETRIC)

RATIONALIZABILTY & NASH EQUILIBRIUM 1: BEST REPLIES

Market (inverse) demand: $P(Q) = \max\{0, \bar{P} - \beta Q\}$ $(\bar{P}, \beta > 0)$

Cost function $C_i(q_i) = c \cdot q_i$, $(c < \bar{P})$

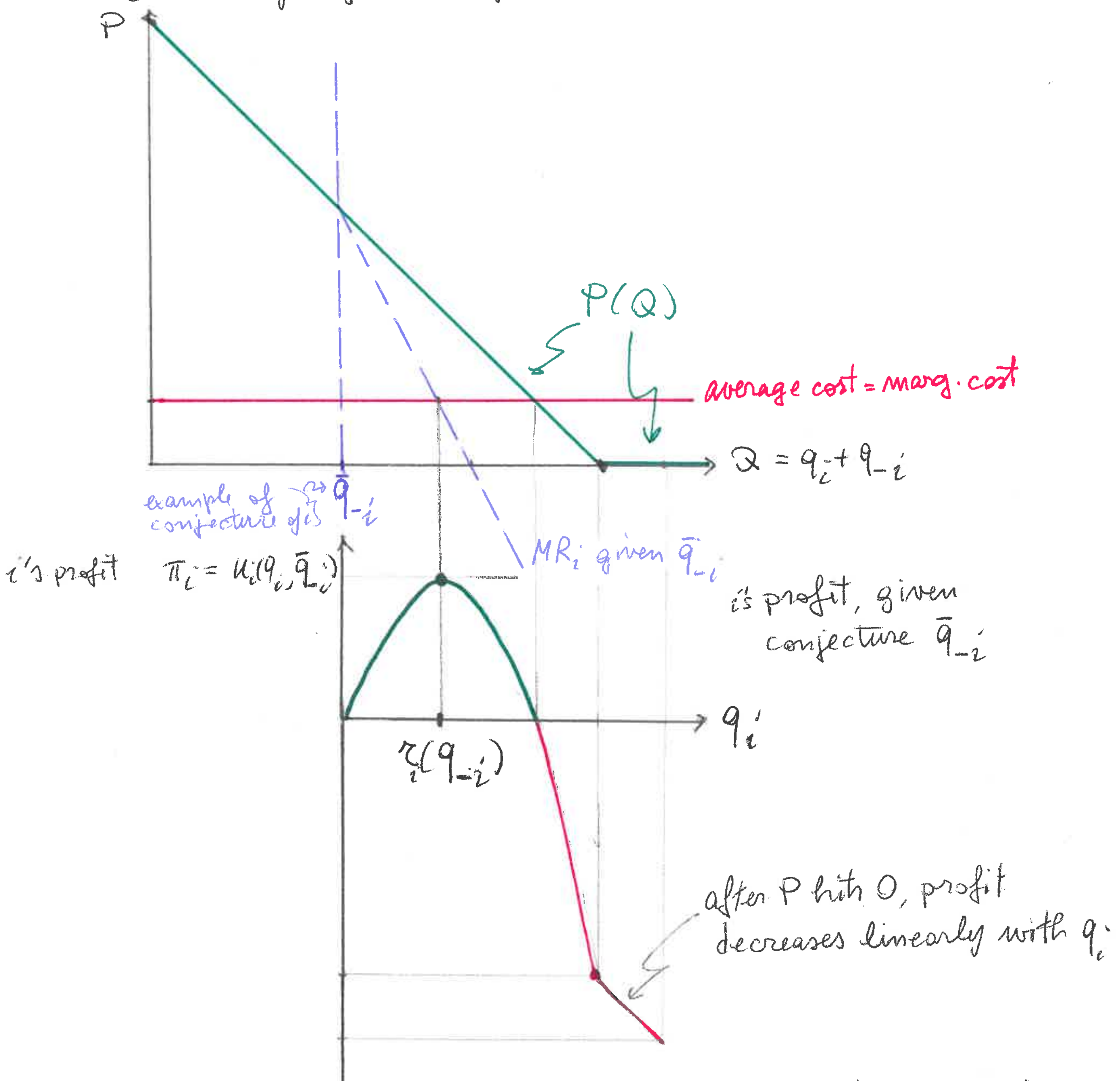
Capacity: $0 \leq q_i \leq \bar{q}$, with $\bar{q} > (\bar{P} - c)/\beta$

From unconstrained FOC, obtain the B.R. function

$$r_i(q_{-i}) = \max\left\{0, \frac{\bar{P} - c}{2\beta} - \frac{1}{2} q_{-i}\right\}$$

where $\frac{\bar{P} - c}{2\beta} = q^M$ is the monopolistic output

$(r_i = r$ by symmetry$)$



The profit function is not concave in q_i, but it is strictly quasi-concave in q_i

COURNOT DUOPOLY (SYMMETRIC)
RATIONALIZABILITY AND NASH EQUILIBRIUM 2:
BEST REPLY FUNCTIONS AND STRATEGIC REASONING

$P(Q) = \max\{0, \bar{P} - \beta Q\}$ $(\bar{P}, \beta > 0,\ Q = q_i + q_{-i})$

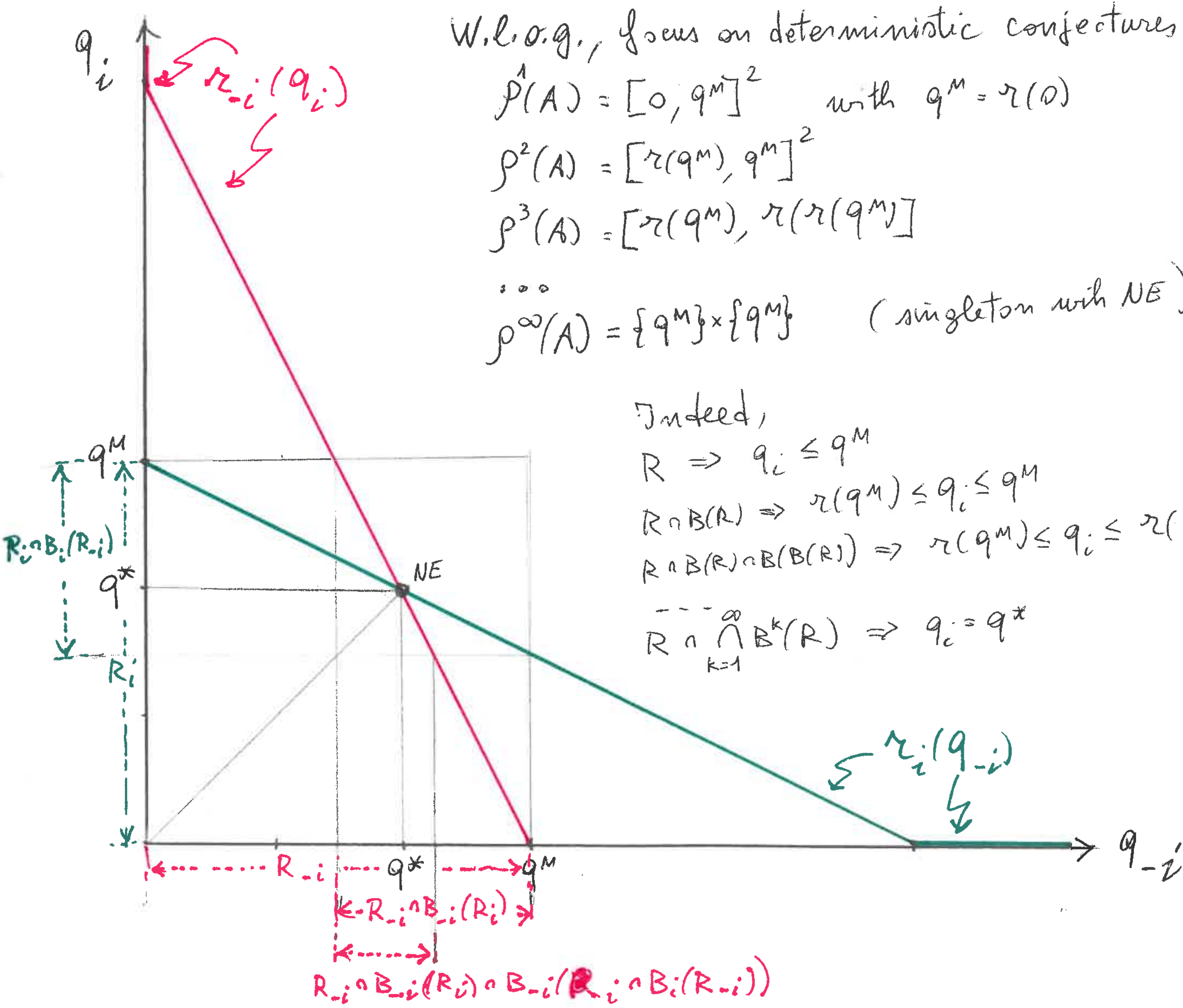
$C_i(q_i) = c q_i$ $(0 \le c < \bar{P})$ (constant average and marg. cost = c)

$0 \le q_i \le \bar{q}$, $\bar{q} > \frac{\bar{P} - c}{\beta}$

B.R. function $r_i(q_{-i}) = \max\left\{0, \frac{\bar{P}-c}{2\beta} - \frac{1}{2} q_{-i}\right\}$

$q^M = \frac{\bar{P}-c}{2\beta}$ monopolistic quantity

$q^* = \frac{\bar{P}-c}{3\beta}$ Cournot-Nash equilibrium



W.l.o.g., focus on deterministic conjectures

$\rho^1(A) = [0, q^M]^2$ with $q^M = r(0)$

$\rho^2(A) = [r(q^M), q^M]^2$

$\rho^3(A) = [r(q^M), r(r(q^M))]$

...

$\rho^\infty(A) = \{q^M\} \times \{q^M\}$ (singleton with NE)

Indeed,

$R \Rightarrow q_i \le q^M$

$R \cap B(R) \Rightarrow r(q^M) \le q_i \le q^M$

$R \cap B(R) \cap B(B(R)) \Rightarrow r(q^M) \le q_i \le r(r(q^M))$

...

$R \cap \bigcap_{k=1}^{\infty} B^k(R) \Rightarrow q_i = q^*$