Iterated Dominance and Rationalizability: The Calculus of Strategic Thinking, II

P. Battigalli
Bocconi University
Game Theory: Analysis of Strategic Thinking

September 22, 2023

Abstract

Rationalizability characterizes the behavioral implications of rationality and sophisticated strategic thinking (that is, common belief in rationality). The rationalization operator provides a first tool for a "calculus of strategic thinking" via a kind of iterated deletion of unjustifiable actions. To facilitate such calculus, we rely on the coincidence between justifiable and undominated actions: the iterated deletion of dominated actions provides an algorithm to progressively simplify the game and compute the set of rationalizable actions of each player. It turns out that (in all "non-pathological" games) the order of elimination does not matter. Iterated dominance (in any order) is thus a second tool for the "calculus of strategic thinking".

[These slides summarize and complement Chapter 4.5 of GT-AST]

Rationalizability and iterated dominance: example

Discretized Cournot duopoly

- We illustrate how rationalizable actions can be obtained by iteratively deleting dominated actions.
- Consider a standard, symmetric, quantity-setting duopoly with the following features:
  - Market (inverse) demand function:
    $P(q_i + q_{-i}) = \max\{0, 7 - q_i - q_{-i}\}$.
  - Capacity $\bar{q} = 4$, that is, $0 \leq q_i \leq 4$.
  - Cost function: $C_i(q_i) = q_i$ (marg. cost=average cost$= 1$).
- Also assume that firms have a discrete feasible set: $q_i \in \{0, 1^-, 2, 3, 4\}$, where $1^- = 1 - \varepsilon$, with $\varepsilon > 0$ *small* (to break annoying ties due to the discretization).

Rationalizability and iterated dominance: example

Payoff matrix of the discretized Cournot duopoly

Taking into account that (1) market price hits the 0-bound for $q_i + q_{-i} \geq 7$, and (2) $u_i(q_i, q_{-i}) = q_i(6 - q_{-i}) - q_i^2$ for $q_i + q_{-i} \leq 7$, we obtain:

$u_i(q_i, q_{-i})$	$q_{-i} = 0$	$q_{-i} = 1^-$	$q_{-i} = 2$	$q_{-i} = 3$	$q_{-i} = 4$
$q_i = 0$	0	0	0	0	0
$q_i = 1^-$	5^-	4^-	3^-	2^-	$\mathbf{1}^-$
$q_i = 2$	8	6^+	**4**	**2**	0
$q_i = 3$	**9**	$\mathbf{6}^{++}$	3	0	-3
$q_i = 4$	8	4^+	0	-4	-4

($n^+ - n$, $n^{++} - n^+$, and $n - n^-$ are positive small numbers for any integer n; to check it, do the algebra with $1^- = 1 - \varepsilon$. The **maximum** in each column is in **bold.**)

Rationalizability and iterated dominance: example

Rationalizability in the discretized Cournot duopoly

It can be verified that

- $q^M = 3$ is the monopolistic output, and $q^* = 2$ is the Cournot-Nash equilibrium output.
- $\rho^1(A) = \rho\left(\{0, 1^-, 2, 3, 4\}^2\right) = \{1^-, 2, 3\}^2$ (it is unjustifiable to produce more than q^M; since the competitor's capacity is not large, it is also unjustifiable to produce 0).
- $\rho^2(A) = \rho\left(\{1^-, 2, 3\}^2\right) = \{2, 3\}^2$ (assuming that $q_{-i} \leq q^M$, also the low production $q_i = 1^-$ cannot be a best response).
- $\rho^3(A) = \rho\left(\{2, 3\}^2\right) = \{2\} \times \{2\}$ (assuming that $q_{-i} \geq 2$, q^M cannot be a best response).

- We can compute each $\rho^k(A)$ as follows: delete all dominated actions $\rightarrow$ obtain a smaller game and delete all the dominated actions in the smaller game $\rightarrow$ obtain an even smaller game ...
- Note: $q_i = 1^-$ dominates $q_i = 0$, and $q_i = 3$ dominates $q_i = 4$, while $1^- = r(4)$, $2 = r(2) = r(3)$, $3 = r(0) = r(1^-)$. Obtain the smaller and smaller games:

$q_i \backslash q_{-i}$	1^-	2	3
1^-	4^-	3^-	2^-
2	6^+	**4**	**2**
3	$\mathbf{6}^{++}$	3	0

$\overset{(2 \text{ dom } 1^-)}{\longrightarrow}$

$q_i \backslash q_{-i}$	2	3
2	**4**	**2**
3	3	0

$\overset{(2 \text{ dom } 3)}{\longrightarrow}$

$q_i \backslash q_{-i}$	2
2	**4**

Rationalizability and iterated dominance: example

"Slow" iterated dominance and order independence

- Both $q_i = 0$ and $q_i = 4$ are dominated $\Rightarrow$ 2 eliminations in the first step. This is the "maximal speed" in iterated elimination.
- But we can go at a "slower pace" and delete, for example, just one action at each step: the order does not matter!

Rationalizability and iterated dominance: example

"Slow" iterated dominance and order independence: delete 0 first, then 4

- Deleting $q_i = 0$ first, then $q_i = 4$, we obtain:

$q_i \backslash q_{-i}$	1^-	2	3	4
1^-	4^-	3^-	2^-	$\mathbf{1}^-$
2	6^+	**4**	**2**	0
3	$\mathbf{6}^{++}$	3	0	-3
4	4^+	0	-4	-4

$\overset{(3 \text{ dom } 4)}{\longrightarrow}$

$q_i \backslash q_{-i}$	1^-	2	3
1^-	4^-	3^-	2^-
2	6^+	**4**	**2**
3	$\mathbf{6}^{++}$	3	0

$\overset{(2 \text{ dom } 1^-)}{\longrightarrow}$

$q_i \backslash q_{-i}$	2	3
2	**4**	**2**
3	3	0

$\overset{(2 \text{ dom } 3)}{\longrightarrow}$

$q_i \backslash q_{-i}$	2
2	**4**

- But we can follow other orders...

Rationalizability and iterated dominance: example

"Slow" iterated dominance and order independence: delete 4 first, then 0

- Deleting $q_i = 4$ first, then 0, we obtain:

$q_i \backslash q_{-i}$	0	1^-	2	3
0	0	0	0	0
1^-	5^-	4^-	3^-	2^-
2	8	6^+	**4**	**2**
3	**9**	$\mathbf{6}^{++}$	3	0

$\overset{(1^- \text{ dom } 0)}{\longrightarrow}$

$q_i \backslash q_{-i}$	1^-	2	3
1^-	4^-	3^-	2^-
2	6^+	**4**	**2**
3	$\mathbf{6}^{++}$	3	0

$\overset{(2 \text{ dom } 1^-)}{\longrightarrow}$

$q_i \backslash q_{-i}$	2	3
2	**4**	**2**
3	3	0

$\overset{(2 \text{ dom } 3)}{\longrightarrow}$

$q_i \backslash q_{-i}$	2
2	**4**

- Yet other orders of elimination are possible, and *all reach the same result*: $q_i = 2$ for each firm i.
- **Warning:** Here we used only dominance by pure actions, but—in general—we must *check for dominance by mixed actions* as well.

Restricted dominance

In the heuristic example we looked at dominance within restrictions of the original game with smaller and smaller sets of actions for each player. This motivates the following (recall that $\mathcal{C}$ is the collection of Cartesian subsets of $A = \times_{i \in I} A_i$):

Definition

Fix any $C \in \mathcal{C}$ with $C \neq \emptyset$, $i \in I$, and $a_i \in A_i$. Say that a_i is **dominated in** C if $a_i \in C_i$ and there is some $\alpha_i \in \Delta(C_i)$ such that

$$\forall a_{-i} \in C_{-i}, u_i(a_i, a_{-i}) < u_i(\alpha_i, a_{-i}).$$

- For each $C \in \mathcal{C}$, let $\mathrm{ND}_i(C)$ denote the set of actions in C_i that are not dominated in C, and let $\mathrm{ND}(C) = \times_{i \in I} \mathrm{ND}_i(C)$.
- After the rationalization operator $\rho : \mathcal{C} \to \mathcal{C}$, we obtain another operator, $\mathrm{ND} : \mathcal{C} \to \mathcal{C}$, the "undominance operator".

Restricted dominance: comments

- The rationalization operator ρ and the "undominance operator" ND have *different properties*:
  - On the one hand, $\mathrm{ND}(C) \subseteq C$ by definition (ND is a "restriction operator"), instead we can have $\rho(C) \nsubseteq C$ and even $\rho(C) \cap C = \emptyset$ (e.g., when—for some i—C_i is a set of unjustifiable actions).
  - On the other hand, ρ is *monotone*, but ND is *not* monotone, because—when dominance is restricted to C—it is like being in a smaller game: for any $i \in I$, mixed actions outside $\Delta(C_i)$ cannot be used to dominate actions in C_i (e.g., let $\bar{a}_i$ be dominated in the *unrestricted* game, fix $\bar{a}_{-i}$ arbitrarily, then—by definition—$\bar{a} = (\bar{a}_i, \bar{a}_{-i}) \notin \mathrm{ND}(A)$, but $\bar{a} \in \mathrm{ND}(\{\bar{a}\}) = \{\bar{a}\}$).
- Operators ρ and ND have *different roles*: ρ shows how players may best respond, with no constraint, when the support of their conjectures is restricted; ND instead shows how dominance works in restricted games obtained by constraining players' choices. Operator ND is merely an *auxiliary tool* to facilitate calculations: eliminated actions can be "forgotten."

Rationalizability and (maximal) iterated dominance

Equivalence theorem

- Consider the iterations of the self-map $\mathrm{ND} : \mathcal{C} \to \mathcal{C}$, starting from the set A of *all* action profiles. This gives the maximal iterated elimination of dominated actions, or **maximal iterated dominance** algorithm $\mathrm{ND}^1(A)$, $\mathrm{ND}^2(A)$, $\mathrm{ND}^3(A)$, ..., where "maximal" means that, at each step, *all* the dominated actions in the restricted game are eliminated.
- Despite the aforementioned differences between the two operators ρ and ND, their iterations starting from A yield the *same sequence of subsets*:

Theorem

(Equivalence) *Fix a finite or compact-continuous game. For every* $k \in \mathbb{N}_0$, $\rho^k(A) = \mathrm{ND}^k(A)$; *therefore, rationalizability and (maximal) iterated dominance coincide.*

- For each $i \in I$ and nonempty $C_i \subseteq A_i$, define the C_i-constrained B.R. correspondence $\mu^i \mapsto r_i\left(\mu^i | C_i\right) = \arg\max_{a_i \in C_i} u_i\left(a_i, \mu^i\right)$ $[\mu^i \in \Delta\left(A_{-i}\right)]$.
- With this, define the "restriction operator" $C \mapsto \bar{\rho}\left(C\right) = \times_{i \in I} r_i\left(\Delta\left(C_{-i}\right) | C_i\right)$ $[C \in \mathcal{C}]$.
- Use "independence of irrelevant alternatives" to show by induction that $\rho^k\left(A\right) = \bar{\rho}^k\left(A\right)$ for every $k \in \mathbb{N}$ (by def., $\rho^1\left(A\right) = \bar{\rho}^1\left(A\right)$; in step 2 it does not matter whether actions outside $\bar{\rho}^1\left(A\right)$ are available or not, because they cannot be best replies; and so on).
- Use the *Wald-Pearce Lemma* to prove by induction that $\bar{\rho}^k\left(A\right) = \mathrm{ND}^k\left(A\right)$ for every $k \in \mathbb{N}$. Thus, $\rho^k\left(A\right) = \bar{\rho}^k\left(A\right) = \mathrm{ND}^k\left(A\right)$ for every $k \in \mathbb{N}$. ♡

Rationalizability and (maximal) iterated dominance

Numerical example

To solve the following game, we must use dominance by mixed actions (to ease reading, payoffs of i are in *Italics*, payoff of $-i$ in **bold**):

$i\backslash -i$	a_{-i}	b_{-i}	c_{-i}
a_i	*4*, **3**	*2*, **1**	*0*, **0**
b_i	*1*, **0**	*1*, **5**	*1*, **0**
c_i	*0*, **0**	*2*, **1**	*3*, **4**

$\overset{(\frac{1}{2}\delta_{a_i}+\frac{1}{2}\delta_{c_i} \text{ dom } b_i)}{\longrightarrow}$

$i\backslash -i$	a_{-i}	b_{-i}	c_{-i}
a_i	*4*, **3**	*2*, **1**	*0*, **0**
c_i	*0*, **0**	*2*, **1**	*3*, **4**

$\overset{(\frac{1}{2}\delta_{a_{-i}}+\frac{1}{2}\delta_{c_{-i}} \text{ dom } b_{-i})}{\longrightarrow}$

$i\backslash -i$	a_{-i}	c_{-i}
a_i	*4*, **3**	*0*, **0**
c_i	*0*, **0**	*3*, **4**

END

Note that $C = \{a_i, c_i\} \times \{a_{-i}, c_{-i}\}$ is a set with the Best Reply Property in the original game, hence every element of C is rationalizable.

Failure of equivalence for lack of compactness or continuity

Consider the following *infinite* game with *discontinuous payoffs* (payoff of pl. **1** in **bold**): $A_1 = \{0, 1\}$, $A_2 = \left\{0, \frac{1}{2}, \frac{2}{3}, ..., 1 - \frac{1}{k}, ..., 1\right\}$,

$\mathbf{1} \backslash 2$	0	...	$1 - \frac{1}{k}$	...	1
0	**1**, *0*	...	**1**, $1 - \frac{1}{k}$	...	**1**, *0*
1	**0**, *0*		**0**, *0*	...	**0**, *1*

- Note: $\lim_{k \to \infty} u_2\left(0, 1 - \frac{1}{k}\right) = 1 \neq 0 = u_2(0, 1)$ (discontinuity).
- $\mu^2(0) > \frac{1}{2} \Rightarrow r_2(\mu^2) = \emptyset$, $\mu^2(0) \leq \frac{1}{2} \Rightarrow r_2(\mu^2) = \{1\}$.
- Thus, $\rho(A) = \{0\} \times \{1\}$,
  $\rho^2(A) = \rho(\{0\} \times \{1\}) = \emptyset \neq \mathrm{ND}(\{0\} \times \{1\}) = \{0\} \times \{1\}$.

Iterated dominance procedures

Consider *finite* games. We define sequences of eliminations of dominated actions that may be "slower" than maximal iterated dominance:

Definition

Fix a *finite* game G. An **iterated dominance procedure** is a finite sequence $\left(C^k\right)_{k=0}^{K} \in \mathcal{C}^{K+1}$ of nonempty Cartesian subsets of action profiles such that (i) $C^0 = A$, (ii) if $K > 0$, for each $k \in \{1, ..., K\}$, $\mathrm{ND}\left(C^{k-1}\right) \subseteq C^k \subset C^{k-1}$ ($\subset$ means *strict* inclusion), and (iii) $\mathrm{ND}\left(C^K\right) = C^K$.

- Condition
    - (i) requires to start from A (*initial condition*),
    - (ii) says that at least one action has to be eliminated at each step $k > 0$,
    - (iii) says that nothing can be eliminated from C^K (*terminal condition*).

Iterated dominance: order independence

The following result says that the *order of elimination does not matter*: the end result is always the rationalizable set (see the heuristic example).

Theorem

(Order Independence) Fix a finite game. For every iterated dominance procedure $\left(C^k\right)_{k=0}^{K} \in \mathcal{C}^{K+1}$, C^K *is the set of rationalizable action profiles.*

Note: The result can be extended to compact-continuous games (see Dufwenberg & Stegeman 2002). It fails in some "pathological" games that are either non-compact, or discontinuous (or both). See below.

Take home message: *Don't be anxious*: even if you fail to delete some actions along the way and go "slow", it is *enough that you stop only when no action is dominated in the restricted game*.

Failure of order independence (optional)

Consider the following *infinite* game with *discontinuous payoffs* (payoff of pl. **1** in **bold**): $A_i = \{0, \frac{1}{2}, \frac{2}{3}, ..., 1 - \frac{1}{k}, ..., 1\}$ ($i = \mathbf{1}, 2$)

$\mathbf{1}\backslash 2$	0	...	$1-\frac{1}{k}$	...	1
0	$\mathbf{0}$, 0	...	$\mathbf{0}$, $1-\frac{1}{k}$	...	$\mathbf{0}$, 0
...					
$1-\frac{1}{k}$	$\mathbf{1-\frac{1}{k}}$, 0		$\mathbf{1-\frac{1}{k}}$, $1-\frac{1}{k}$		$\mathbf{1-\frac{1}{k}}$, 0
...					
1	$\mathbf{0}$, 0		$\mathbf{0}$, $1-\frac{1}{k}$	...	$\mathbf{1}$, 1

- If $a_i < 1$, $u_i(a_i, a_{-i}) = a_i$; if $a_i = 1$ and $a_{-i} < 1$, $u_i(a_i, a_{-i}) = 0$; $u_i(1, 1) = 1$.
- Each $a_i < 1$ is dominated; indeed, for each $k \in \mathbb{N}$, $a_i' = 1 - \frac{1}{k+1}$ dominates $a_i = 1 - \frac{1}{k}$. Thus, $\mathrm{ND}(A) = \{1\} \times \{1\}$.
- **Partial elimination:** For each i, delete all dominated actions except $a_i = 1/2$. Obtain: $\mathrm{ND}(\{1/2, 1\} \times \{1/2, 1\}) = \{1/2, 1\} \times \{1/2, 1\}$.

References

Battigalli, P., E. Catonini, and N. De Vito (2023): *Game Theory: Analysis of Strategic Thinking*. Typescript, Bocconi University.

Battigalli, P. (2023): *Mathematical Language and Game Theory*. Typescript, Bocconi University.

Dufwenberg, M., and M. Stegeman (2002): “Existence and Uniqueness of Maximal Reductions Under Iterated Strict Dominance”, *Econometrica*, 70, 2007-2023.