Nice Games: Dominance and Rationalizability

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Abstract

A *compact-continuous* game is **nice** if, for each player, the action set is an interval and the payoff function is strictly quasi-concave in his own action given the actions of others. Many economic models satisfy these assumptions, e.g., oligopoly, network, and beauty-contest games. The noteworthy feature of nice games is that, in the analysis of strategic thinking, *we can restrict our attention to deterministic conjectures*. Therefore, if the outcome function is deterministic, results are independent of players' risk attitudes and the analysis is simplified.
[These slides summarize and complement material about nice games contained in Chapters 3 (3.3.2) and 4 (4.6) of "Game Theory: Analysis of Strategic Thinking".]

Example: Cournot oligopoly

Consider a Cournot (*quantity-setting*) *oligopoly* with n firms ($|I| = n$):

- inverse **market demand** $P(Q) = \max\left\{0, \bar{P} - \beta Q\right\}$, $Q = \sum_{i \in I} q_i$, $\bar{P}, \beta > 0$;
- **cost functions** $C_i(q_i) = cq_i$, $c < \bar{P}$ (*constant marginal cost*);
- and *large capacity* $\bar{q}$ for each firm: $(n-1)\bar{q} > \left(\bar{P} - c\right)/\beta$, i.e., $P((n-1)\bar{q}) < c$.
- Firms are expected profit maximizers (hence, *risk neutral*).
- All of the above is common knowledge (*complete information*).

Example: Cournot oligopoly (cont.)

- Let $Q_{-i} = \sum_{j \neq i} q_j$ denote i's competitors' total output. The payoff (profit) function is

$$u_i(q_i, q_{-i}) = \begin{cases} q_i\left(\bar{P} - \beta q_i - \beta Q_{-i}\right) - cq_i & \text{if } \beta(q_i + Q_{-i}) < \bar{P}, \\ -cq_i & \text{if } \beta(q_i + Q_{-i}) \geq \bar{P}. \end{cases}$$

  This function is *continuous*. It is *not concave* because of the kink due to the "0-floor" of price. But it is (strictly) "quasi-concave" (kind of "bell shaped"). See the picture in BBoard folder.

- We will show that (1) w.l.o.g., *we can focus on deterministic conjectures*; hence, (2) the *risk attitudes of the firms do not matter.*
- Each i has B.R. function $r(q_{-i}) = \max\left\{0, \frac{\bar{P}-c}{2\beta} - \frac{1}{2}\sum_{j\neq i} q_j\right\}$.
  Clearly, $r(0, ..., 0) = \frac{\bar{P}-c}{2\beta} = q^M$. Given the assumptions, $r(\bar{q}, ..., \bar{q}) = 0$. Iterating r we get different results according to $n = 2$, or $n > 2$ (see Rationalizability).

Quasi-Concavity

General definition

- Recall that the payoff function $u_i = v_i \circ g : A \to \mathbb{R}$ is i's vNM utility of action profiles; hence, it depends on i's *risk attitudes*.
- It is interesting to identify classes of games where risk attitudes do not matter for most purposes, i.e., where results are preserved under strictly increasing transformations of the payoff functions.

Definition

(Q-Concavity) Let X be a *convex* subset of a Euclidean space (or a real vector space). A function $f : X \to \mathbb{R}$ is **quasi-concave** if $\forall x', x'' \in X, \forall w \in [0,1]$,

$$f\left(wx' + (1-w)x''\right) \geq \min\left\{f\left(x'\right), f\left(x''\right)\right\};$$

it is **strictly quasi-concave** if $\forall x', x'' \in X$ with $x' \neq x''$, $\forall w \in (0,1)$,

$$f\left(wx' + (1-w)x''\right) > \min\left\{f\left(x'\right), f\left(x''\right)\right\}.$$

(Strict) Quasi-Concavity: Independence of risk attitudes

- **Observation (Invariance).** (Strict) Quasi-concavity is preserved by strictly increasing transformations: if f is (strictly) quasi-concave and $\varphi : f(X) \to \mathbb{R}$ is strictly increasing, then $\varphi \circ f$ is (strictly) quasi-concave.
- **Proof:** We consider the case of *strict* quasi-concavity.
  - **Preliminaries:** For all $y', y'' \in f(X)$, by def. $y', y'' \geq \min\{y', y''\}$; since φ is *increasing*,
  $$\varphi(\min\{y', y''\}) = \min\{\varphi(y'), \varphi(y'')\}.$$
  - Fix $x', x'' \in X$ with $x' \neq x''$ and $w \in (0,1)$ arbitrarily. By *strict quasi-concavity* of f,
  $$f(wx' + (1-w)x'') > \min\{f(x'), f(x'')\}.$$
  By *strict* monotonicity of φ,
  $$\begin{aligned} \varphi(f(wx' + (1-w)x'')) &> \varphi(\min\{f(x'), f(x'')\}) \\ &= \min\{\varphi(f(x')), \varphi(f(x''))\}. \ \blacksquare \end{aligned}$$

Quasi-Concavity: Characterization

- **Observation.** A real-valued function f is *quasi-concave* on its convex domain X IFF, for every $y \in f(X) \subseteq \mathbb{R}$, the set $\{x \in X : f(x) \geq y\}$ is *convex*.
- **Proof:**
  - (**If**) Let $\{x \in X : f(x) \geq y\}$ be *convex* for every $y \in f(X)$. Fix $x', x'' \in X$, $w \in [0,1]$ arbitrarily. Let $y = \min\{f(x'), f(x'')\}$; with this, $f(x') \geq y$ and $f(x'') \geq y$; thus, $x', x'' \in \{x \in X : f(x) \geq y\}$. Since $\{x \in X : f(x) \geq y\}$ is convex, $(wx' + (1-w)x'') \in \{x \in X : f(x) \geq y\}$, that is, $f(wx' + (1-w)x'') \geq y = \min\{f(x'), f(x'')\}$.
  - (**Only if**) Let f be quasi-concave and fix $y \in f(X)$ arbitrarily. Fix $x', x'' \in \{x \in X : f(x) \geq y\}$ and $w \in [0,1]$ arbitrarily. Then, $f(x') \geq y$ and $f(x'') \geq y$, that is, $\min\{f(x'), f(x'')\} \geq y$. Quasi-concavity of f implies $f(wx' + (1-w)x'') \geq \min\{f(x'), f(x'')\} \geq y$. Thus, $(wx' + (1-w)x'') \in \{x \in X : f(x) \geq y\}$. This shows that $\{x \in X : f(x) \geq y\}$ is convex. ■

Strict Quasi-Concavity: 1-Variable Characterization

Strict quasi-concavity of a (continuous) function of one variable $f : [a, b] \to \mathbb{R}$ means that f is "bell-shaped", or strictly increasing, or strictly decreasing.

- **Observation.** Fix any *compact interval* $[a, b] \subseteq \mathbb{R}$ and let the function $f : [a, b] \to \mathbb{R}$ be *continuous*. Then, f is *strictly quasi-concave* IFF it has a *unique maximizer* x^*, it is *strictly increasing on* $[a, x^*]$ (if $a < x^*$), and it is *strictly decreasing on* $[x^*, b]$ (if $x^* < b$).
- Note, a (continuous) strictly quasi-concave function $f : [a, b] \to \mathbb{R}$ may attain its maximum at the boundary (endpoints), it is strictly *increasing* if $x^* = b$ and strictly *decreasing* if $x^* = a$.

Strict Quasi-Concavity: Proof of the 1-V Characterization

- **Proof:** By compactness and continuity there is at least one maximizer $x^* \in \arg\max_{x\in[a,b]} f(x)$ (Weierstrass).
  - Suppose that f is strictly quasi-concave and that $a < x^*$. Let $a \leq x' < x'' < x^*$. We prove that $f(x'') > f(x')$. This implies that f is strictly increasing on $[a, x^*]$. Indeed, $x'' = (wx' + (1-w)x^*)$, where $w = \frac{x^*-x''}{x^*-x'} \in (0,1)$. Thus, $f(x'') = f(wx' + (1-w)x^*) > \min\{f(x'), f(x^*)\} = f(x')$ where the inequality holds by strict quasi-concavity of f, and the second equlity holds because x^* is a maximizer. The proof for the case $x^* < x' < x'' \leq b$ is similar.
  - Suppose that f has a unique maximizer x^*, it is strictly increasing on $[a, x^*]$ (if $a < x^*$), and it is strictly decreasing on $[x^*, b]$ (if $x^* < b$). Fix $x', x'' \in [a,b]$ with $x' < x''$, and $w \in (0,1)$. Let $x(w) := wx' + (1-w)x''$, then, $x' < x(w) < x''$. If $x(w) \leq x^*$, then $f(x(w)) > f(x') \geq \min\{f(x'), f(x'')\}$ because f is strictly increasing on $[a, x^*]$. If $x(w) \geq x^*$, then $f(x(w)) > f(x'') \geq \min\{f(x'), f(x'')\}$ because f is strictly decreasing on $[x^*, b]$. Thus, f is strictly quasi-concave. ■

Definition

(Nice Games) A *compact-continuous* game $\langle I, (A_i, u_i)_{i \in I} \rangle$ is **nice** if, for every player $i \in I$, $A_i = [\underline{a}_i, \bar{a}_i] \subseteq \mathbb{R}$ is an *interval* and each section $u_i(\cdot, a_{-i}) : A_i \to \mathbb{R}$ of the payoff function $(a_{-i} \in A_{-i})$ is *strictly quasi-concave.*

- **Examples.** Many economic models, such as oligopolies (e.g., the aforementioned Cournot model), beauty contests, and network interactions are nice games.
- Many models satisfying all the nice-game properties except compactness can be "compactified". For example, in the above-mentioned oligopoly model, the set of justifiable outputs is $[0, q^M]$ (with q^M=monopoly output) with any capacity $\bar{q} > q^M$ such that $P((|I| - 1)\bar{q}) < c$, and *also without capacity constraints*.

- **Terminology.** A correspondence $\varphi : X \rightrightarrows Y$ is called **function** if $\varphi(x)$ is a *singleton* for every $x \in X$.

Lemma

(Continuous BR) In every nice game, for each player $i \in I$, the best reply correspondence r_i restricted to A_{-i} is a continuous function.

- **Proof.** Each section $u_i(\cdot, a_{-i})$ has a unique maximizer $r_i(a_{-i})$ by strict quasi-concavity.
  - Suppose that $a^k_{-i} \to a^*_{-i}$. By compactness, we may assume w.l.o.g. that $\lim_{k\to\infty} r_i(a^k_{-i}) = a^*_i$ for some $a^*_i \in A_i$.
  - We prove $a^*_i = r_i(a^*_{-i})$. Since $r_i(a^k_{-i})$ is the maximizer for each k and by continuity of u_i (as $k \to \infty$),

$$\begin{aligned} \forall a_i &\in A_i, \forall k \in \mathbb{N}, u_i\left(r_i\left(a^k_{-i}\right), a^k_{-i}\right) \geq u_i\left(a_i, a^k_{-i}\right) \\ \forall a_i &\in A_i, u_i\left(a^*_i, a^*_{-i}\right) \geq u_i\left(a_i, a^*_{-i}\right), \text{ thus } a^*_i = r_i\left(a^*_{-i}\right). \blacksquare \end{aligned}$$

Nice Games

Dominance by pure actions and best replies

- Let
  $ND_{i,p} := \{\bar{a}_i \in A_i : \forall a_i \in A_i, \exists a_{-i} \in A_{-i}, u_i(\bar{a}_i, a_{-i}) \geq u_i(a_i, a_{-i})\}$
  denote the set of pure actions not dominated by *pure* actions.

Lemma

(Equivalence to Certainty) *Fix any player $i \in I$ in a nice game. The set of best replies to deterministic conjectures is a compact interval and it coincides with the set of pure actions undominated by pure actions; hence, it also coincides with the set of justifiable actions and undominated actions: $r_i(A_{-i}) = r_i(\Delta(A_{-i})) = ND_i = ND_{i,p}$.*

- This result greatly simplifies the computation of justifiable and rationalizable actions in nice games, allowing us to neglect probabilistic conjectures.

- **Proof.** It is trivially true that $r_i(A_{-i}) \subseteq r_i(\Delta(A_{-i})) \subseteq ND_i \subseteq ND_{i,p}$ (actually, $r_i(\Delta(A_{-i})) = ND_i$ by the WP Lemma). We prove $ND_{i,p} \subseteq r_i(A_{-i})$, or—equivalently—that every $a_i \in A_i \backslash r_i(A_i)$ is dominated by some $a_i' \in A_i$.
  - The image of a compact and connected set through a real-valued continuous function is a *compact interval*.
  - Thus, continuity of r_i (see previous lemma) yields $r_i(A_{-i}) = [\underline{a}_i^*, \bar{a}_i^*]$, where $\underline{a}_i^* := \min r_i(A_{-i})$, $\bar{a}_i^* := \max r_i(A_{-i})$.
  - If $\underline{a}_i \leq a_i < \underline{a}_i^*$, for every $a_{-i} \in A_{-i}$, $a_i < \underline{a}_i^* \leq r_i(a_{-i})$ and strict q-concavity yields $u_i(a_i, a_{-i}) < u_i(\underline{a}_i^*, a_{-i})$ because $u_i(\cdot, a_{-i})$ is strictly increasing on $[\underline{a}_i, r_i(a_{-i})]$; hence, a_i is dominated by $\underline{a}_i^*$.
  - A similar argument shows that if $\bar{a}_i^* < a_i \leq \bar{a}_i$ then a_i is dominated by $\bar{a}_i^*$. ■

- Looking at the iterated elimination of actions that are not justified by *deterministic* conjectures, we obtain the "**point rationalizability**" concept (conjectures are concentrated on one point). This is *just an auxiliary analytical tool* with no conceptual justification. It is typically easy to calculate and it yields a subset of the rationalizable set.
- Recall that $\mathcal{C}$ is the collection of **c**losed (hence, **c**ompact) **C**artesian subsets of $A := \times_{i \in I} A_i$. For each $C \in \mathcal{C}$, let

$$\mathrm{r}(C) := \times_{i \in I} r_i(C_{-i}).$$

  Since r_i is continuous and C_{-i} is closed, $r_i(C_{-i})$ is closed, for each i; hence, $\mathrm{r}(C)$ is closed and $\mathrm{r}(C) \in \mathcal{C}$. Thus, $C \mapsto \mathrm{r}(C)$ is a *self-map* on $\mathcal{C}$ and we can define r^k in the usual way (r^0 is the identity on $\mathcal{C}$, $\mathrm{r}^k = \mathrm{r} \circ \mathrm{r}^{k-1}$ for $k \in \mathbb{N}$).
- **Observation.** $\mathrm{r} : \mathcal{C} \to \mathcal{C}$ *is monotone.*

- Similarly, we can look at the iterated elimination of pure actions *dominated by pure actions* by iterating the operator ND_p defined (for each $C \in \mathcal{C}$) by $\mathrm{ND}_{i,p}(C) :=$

$$:= \{\bar{a}_i \in C_i : \forall a_i \in C_i, \exists a_{-i} \in C_{-i}, u_i(\bar{a}_i, a_{-i}) \geq u_i(a_i, a_{-i})\},$$

$$\mathrm{ND}_p(C) := \times_{i \in I} \mathrm{ND}_{i,p}(C).$$

- It can be shown that each $\mathrm{ND}_{i,p}(C)$ is a closed set. Hence, $\mathrm{ND}_p(C) \in \mathcal{C}$, and $C \mapsto \mathrm{ND}_p(C)$ is a self-map on $\mathcal{C}$. Thus, we can define $(\mathrm{ND}_p)^k = \mathrm{ND}_p \circ (\mathrm{ND}_p)^{k-1}$ for every $k \in \mathbb{N}$ (where $(\mathrm{ND}_p)^0$ is the identity on $\mathcal{C}$).
- Like ND also ND_p is a "restriction operator": $\mathrm{ND}_p(C) \subseteq C$ (by definition), but it is *not* a *monotone* operator (show it by example).

- The following noteworthy result simplifies the computation of the rationalizable set, showing that it is independent of risk attitudes (if the outcome function g is deterministic). In particular, *rationalizability coincides with point rationalizability and* with the *iterated elimination of pure actions dominated by pure actions*.

Theorem

(Nice Rationalizability) In every nice game, for every $k \in \mathbb{N}_0 \cup \{\infty\}$, $\mathrm{r}^k(A)$ is a product of compact intervals and

$$\mathrm{r}^k(A) = \rho^k(A) = \mathrm{ND}^k(A) = (\mathrm{ND}_p)^k(A).$$

Proof of Nice Rationalizability Theorem

- **Proof.** For each $C \in \mathcal{C}$, define the "deterministic-conjecture analog" of $\bar{\rho}(C)$: $\bar{\mathrm{r}}(C) := \times_{i \in I} \bigcup_{a_{-i} \in C_{-i}} \arg\max_{a_i \in C_i} u_i(a_i, a_{-i})$. As we did for ρ and $\bar{\rho}$, one can show that $\bar{\mathrm{r}}^k(A) = \mathrm{r}^k(A)$ for every k. Next, we prove the theorem by induction.
  - *Basis.* Since $\gamma^0(A) = A$ for each $\gamma \in \{\mathrm{r}, \rho, \mathrm{ND}, \mathrm{ND}_p\}$, the result holds trivially for $k = 0$.
  - *Inductive step.* Suppose that $\mathrm{r}^k(A)$ is a product of compact intervals and $\mathrm{r}^k(A) = \rho^k(A) = \mathrm{ND}^k(A) = (\mathrm{ND}_p)^k(A)$ (IH). Then, the restricted game with constrained set of action profiles $C = \bar{\mathrm{r}}^k(A) = \mathrm{r}^k(A)$ is a *nice* game; thus, the previous Lemma and (IH) yield
    $\bar{\mathrm{r}}\left(\mathrm{r}^k(A)\right) = \bar{\rho}\left(\rho^k(A)\right) = \mathrm{ND}\left(\mathrm{ND}^k(A)\right) = \mathrm{ND}_p\left((\mathrm{ND}_p)^k(A)\right)$,
    where $\bar{\mathrm{r}}\left(\mathrm{r}^k(A)\right)$ is a product of compact intervals. Since $\bar{\mathrm{r}}^{k+1}(A) = \mathrm{r}^{k+1}(A)$, $\bar{\rho}^{k+1}(A) = \rho^{k+1}(A)$, and $\gamma^{k+1} = \gamma \circ \gamma^k$ for each $\gamma \in \{\mathrm{r}, \bar{\mathrm{r}}, \rho, \mathrm{ND}, \mathrm{ND}_p\}$, we obtain
    $\mathrm{r}^{k+1}(A) = \rho^{k+1}(A) = \mathrm{ND}^{k+1}(A) = (\mathrm{ND}_p)^{k+1}(A)$. ■

Rationalizability in the Cournot oligopoly

- The game is *nice* and we can *apply the previous results*. By symmetry, each firm has the same BR function, $r:[0,\bar{q}]^{n-1}\to[0,\bar{q}]$ ($n=|I|$ is the number of firms). Also, $r(0,...,0)=q^M$, and $r(q,...,q)=0$ for each q s.t. $P((n-1)q)\leq c$. With this, $\mathrm{r}(C)=\times_{i\in I}r(C_{-i})$ for every $C\in\mathcal{C}$.
- For $n=2$, $r:[0,\bar{q}]\to[0,\bar{q}]$ is a *self-map*, and it can be iterated, with $r^0(0)=0$, $r^1(0)=r(0)=q^M$, $r^2(0)=r\left(r^1(0)\right)=r\left(q^M\right)$, etc. With this

$$\begin{aligned}\rho^{2\ell+1}(A) &= \left[r^{2\ell}(0),r^{2\ell+1}(0)\right]^2,\ \ell=0,1,\ldots\text{ (odd iterations)}\\ \rho^{2\ell}(A) &= \left[r^{2\ell}(0),r^{2\ell-1}(0)\right]^2,\ \ell=1,2,\ldots\text{ (even iterations)}\end{aligned}$$

$$\lim_{\ell\to\infty}r^{2\ell}(0)\nearrow q^*\swarrow\lim_{\ell\to\infty}r^{2\ell+1}(0),\text{ with }q^*=r(q^*)\text{ symm. NE}$$

- For $n\geq 3$, $\forall k\in\mathbb{N}\cup\{\infty\}$, $\rho^k(A)=\left[0,q^M\right]^n$.

References

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