Nash Equilibrium

P. Battigalli
Bocconi University
Game Theory: Analysis of Strategic Thinking

September 29, 2023

Abstract
I introduce and criticize the Nash equilibrium concept, arguing that Nash's original motivation is valid in games with one rationalizable outcome, but dubious otherwise. Next I illustrate computation methods and explain existence of equilibria in the class of compact, convex, continuous and quasiconcave games, and of symmetric equilibria in symmetric nice games.
[These slides summarize and complement Chapter 5 of "Game Theory: Analysis of Strategic Thinking" and the note "Two-Person, Symmetric Nice Games".]

Introduction: the approach of Nash

- An action profile $(a_i^*)_{i \in I}$ is a **Nash equilibrium** (**NE**) if the action of each player is a best reply to the action(s) of the co-player(s), that is, $a_i^* \in r_i\left(a_{-i}^*\right)$ for each $i \in I$.
- In his PhD thesis, John Nash motivates his solution concept as follows:
  - "We proceed by investigating the question: what would be a 'rational' prediction of the behavior to be expected of rational playing the game in question? By using the principles that a rational prediction should be unique, that the players should be able to deduce and make use of it, and that such knowledge on the part of each player of what to expect the others to do should not lead him to act out of conformity with the prediction, one is led to the concept of a solution defined before." (Nash, 1950, p. 23.)
- *The weakness of this approach is the presumption of uniqueness*. **If** uniqueness obtains, then the prediction must be a Nash equilibrium, indeed the only one (some games have multiple NEs). But we have no reason to presume that this is always the case.

Introduction: the problem of Nash's approach

- With the benefit of our current understanding of the foundations of GT, we should interpret "rational prediction" and "rational playing the game" as the output of sophisticated strategic thinking.
- The best theory we have of the latter is the formal analysis of *rationality and common belief in rationality* (**RCBR**).
- The behavioral implications of RCBR are given by *rationalizability* (see Ch. 4 of GT-AST). The latter does not always give a unique prediction. When it does, it is the (necessarily) unique NE (see below).
- Even in games with a unique NE, it is not at all clear that this should be the only behavior consistent with sophisticated strategic thinking. For example, in many Cournot oligopolies with $n \geq 3$ firms, each output between 0 and q^M (the monopoly output) is rationalizable, i.e., consistent with RCBR.
- Yet, we must study NE in depth because it is what most of the profession uses to model strategic interaction.

Nash equilibrium and rationalizability

Definition

An action profile $a^* = (a_i^*)_{i \in I}$ is a **Nash equilibrium (NE)** if, for every $i \in I$, $a_i^* \in r_i(a_{-i}^*)$.

- In other words, a^* is an NE IFF, for every $i \in I$ and $a_i \in A_i$, $u_i\left(a_i^*, a_{-i}^*\right) \geq u_i\left(a_i, a_{-i}^*\right)$.
- **Observation.** By inspection of the definitions of NE and of the rationalization operator ρ, an action profile a^* is an NE IFF $\{a^*\} \subseteq \rho(\{a^*\})$.
- **Observation.** With this, by Theorem 3 of GT-AST, *every NE is rationalizable. If $\rho^\infty(A)$ is a singleton*, it contains the *unique NE*.
- **Observation.** Recall that an action of a player is rationalizable if it is part of a rationalizable action profile. With this, it follows from the previous observation that, *if an action of a player is part of some NE, then it is rationalizable.*

Computation of Nash equilibria

- **First**, by the previous observations, you may want to eliminate the non-rationalizable (iteratively dominated) actions. You may then focus on the restricted game with set of action profiles $A^* = \rho^\infty(A) = \mathrm{ND}^\infty(A)$. If it is a singleton, it is the unique Nash equilibrium.
- **Second**, if $A^* = \rho^\infty(A)$ is not a singleton, we may distinguish two main cases:
  - A^* is *convex*: if each u_i is smooth, *differential methods* can be used, e.g., for *interior NEs* solve the system of equations

    $$\forall i \in I, \frac{\partial}{\partial a_i} u_i(a_i, a_{-i}) = 0.$$

    Sufficient conditions for the existence of NE will be provided below.
  - A^* is *finite*: there are conceptually simple (although sometimes cumbersome) methods to find the NEs, *if any*. See below.

Differential approach: a beauty contest example

- Consider the following beauty contest game: $|I| = n \geq 2$, for every $i \in I$,
  - $A_i = [0,1]$, where $[0,1]$ represents a set of candidates, and a_i is the one for whom i votes;
  - let $\widehat{a_{-i}} = (n-1)^{-1} \sum_{j \neq i} a_j$, then $u_i(a_i, a_{-i}) = -\left(a_i - (1-\omega)\theta_i - \omega\widehat{a_{-i}}\right)^2$ $(0 < \omega < 1)$, $\theta_i \in [0,1]$ is i's preferred candidate, ω is a measure of conformism (desire to vote like others).
- This is a *nice game* where u_i is smooth. With this,
  - $\frac{\partial u_i}{\partial a_i}(a) = -2\left(a_i - (1-\omega)\theta_i - \omega\widehat{a_{-i}}\right)$.
  - The FOC for an interior equilibrium yields $a_i = (1-\omega)\theta_i + \omega\widehat{a_{-i}} \in [0,1]$ (extremes attained if $\theta_i = 0 = \widehat{a_{-i}}$ or $\theta_i = 1 = \widehat{a_{-i}}$). The constraints $0 \leq a_i \leq 1$ do not bind.
  - Thus, $r_i(a_{-i}) = (1-\omega)\theta_i + \omega\widehat{a_{-i}}$. With $|I| = 2$, the NE is

$$\forall i \in I, a_i^* = \frac{1}{1+\omega}\theta_i + \frac{\omega}{1+\omega}\theta_{-i}.$$

Finite (restricted) games: the "children algorithm"

- Represent the game in matrix form: a double-entry matrix in 2-person games, a set of triple-entry matrices in 3-person g.s, etc.
- **2-person games:**
  - For each *column*, mark all the cells where the payoff of the *row* player is maximal in the column (e.g., using $*$).
  - For each *row*, mark all the cells where the payoff of the *column* player is maximal in the row (e.g., using $\circ$).
  - Each cell with a double mark is an NE. Note: there may be no such cell, *existence is not guaranteed.*
- **3-person games**: similar method. Now the third player chooses the matrix. For the row player fix each matrix and column. For the column player fix each matrix and row. For the matrix player, fix each row and column, find each matrix where his payoff is maximal in the cell of that row and column, mark the cell.

Payoffs of the **row pl.** (*column pl.*) in **bold** (*Italics); $*$* : max for row pl.; $\circ$: max for column pl.

	ℓ	c	r
t	**4**, *2*	**1**, *1*	**4**, *1*
m	**1**, *1*	**2**, *4*	**1**, *2*
b	**2**, *0*	**1**, *0*	**0**, *1*

$\left(\frac{1}{2}\delta_t + \frac{1}{2}\delta_m \text{ dom } b\right) \longrightarrow$

	ℓ	c	r
t	**4**, *2*	**1**, *1*	**4**, *1*
m	**1**, *1*	**2**, *4*	**1**, *2*

$\left(\frac{1}{2}\delta_\ell + \frac{1}{2}\delta_c \text{ dom } r\right) \longrightarrow$

	ℓ	c
t	$_*$**4**, *2*$_\circ$	**1**, *1*
m	**1**, *1*	$_*$**2**, *4*$_\circ$

$\longrightarrow$

NEs:	$\{(t,\ell),(m,c)\}$

Example of the "children algorithm": 2 players, no NE

Payoffs of the **row pl.** (*column pl.*) in **bold** (*Italics*); $*$: max for row pl.; $\circ$: max for column pl.

	ℓ	c	r
t	**4**, *1*	**1**, *2*	**4**, *1*
m	**1**, *4*	**2**, *1*	**1**, *2*
b	**2**, *0*	**1**, *0*	**0**, *1*

$\left(\frac{1}{2}\delta_t+\frac{1}{2}\delta_m \text{ dom } b\right) \longrightarrow$

	ℓ	c	r
t	**4**, *1*	**1**, *2*	**4**, *1*
m	**1**, *4*	**2**, *1*	**1**, *2*

$\left(\frac{1}{2}\delta_\ell+\frac{1}{2}\delta_c \text{ dom } r\right) \longrightarrow$

	ℓ	c
t	$_*$**4**, *1*	**1**, *2*$_\circ$
m	**1**, *4*$_\circ$	$_*$**2**, *1*

$\longrightarrow$ no NE

Example of the "children algorithm": 3 players, 2 NEs

Payoffs of the **row pl.** (*column pl.*) in **bold** (*Italics);* actions of matrix pl.: $\{L, C, R\}$.

L	ℓ	r
t	$\mathbf{2}$, *1*, 3	$\mathbf{1}$, *0*, 0
m	$\mathbf{1}$, *0*, 0	$\mathbf{0}$, *1*, 3

C	ℓ	r
t	$\mathbf{0}$, *0*, 1	$\mathbf{0}$, *2*, 1
m	$\mathbf{2}$, *0*, 1	$\mathbf{2}$, *2*, 1

R	ℓ	r
t	$\mathbf{1}$, *0*, 0	$\mathbf{2}$, *1*, 3
m	$\mathbf{0}$, *1*, 3	$\mathbf{1}$, *0*, 0

Delete first C (why?), then m: obtain game with col. and mtx players; $\circ$: max for col. pl.; $\#$: max for mtx pl.;

L	ℓ	r
t	$1_\circ, 3_\#$	*0*, 0

R	ℓ	r
t	*0*, 0	$1_\circ, 3_\#$

$\rightarrow$

NEs:	$\{(t, \ell, L), (t, r, R)\}$

Existence of Nash Equilibrium

- The following result provides sufficient conditions for the existence of NEs in compact-continuous games.

Theorem

(Existence) *Consider a compact-continuous game* $G = \langle I, (A_i, u_i)_{i \in I} \rangle$. *If, for every player* $i \in I$, A_i *is convex and the function* $u_i(\cdot, a_{-i}) : A_i \to \mathbb{R}$ *is quasi-concave for every* $a_{-i} \in A_{-i}$, *then* G *has a Nash equilibrium.*

- The proof of this **Existence Theorem** is preceded by two preliminary results, of which we prove only the first:
  - Recall that nice games have continuous best reply functions. The **Closed Graph Lemma** generalizes this result to compact-continuous games, which have best reply correspondences with a closed graph.
  - The **Kakutani Fixed Point Theorem** provides sufficient conditions for a correspondence $\varphi : X \rightrightarrows X$ to have a **fixed point** $x^* \in \varphi(x^*)$.

The Closed Graph Lemma

Lemma

If A is compact and $u_i : A \to \mathbb{R}$ is continuous, then $r_i(a_{-i}) \neq \emptyset$ for all $a_{-i} \in A_{-i}$, and $r_i|_{A_{-i}} : A_{-i} \rightrightarrows A_i$ [the restriction of $r_i : \Delta(A_{-i}) \rightrightarrows A_i$ to $A_{-i} \subseteq \Delta(A_{-i})$] has a closed graph, that is, set $\mathrm{Gr}\left(r_i|_{A_{-i}}\right) := \{(a_{-i}, a_i) \in A_{-i} \times A_i : a_i \in r_i(a_{-i})\}$ is closed.

- **Proof:**
  - For every $a_{-i} \in A_{-i}$, $u_i(\cdot, a_{-i})$ is *continuous* with *compact domain*; thus, $r_i(a_{-i}) \neq \emptyset$ by *Weierstrass* theorem.
  - We must prove that, for every $\left(a_{-i}^n, a_i^n\right)_{n=1}^{\infty} \in \mathrm{Gr}\left(r_i|_{A_{-i}}\right)^{\mathbb{N}}$ such that $\lim_{n\to\infty}\left(a_{-i}^n, a_i^n\right) = (\bar{a}_{-i}, \bar{a}_i)$, we must have $\bar{a}_i \in r_i(\bar{a}_{-i})$. Since $a_i^n \in r_i\left(a_{-i}^n\right)$ for every n [cf., definition of $\mathrm{Gr}\left(r_i|_{A_{-i}}\right)$] and u_i is *continuous*

$$\begin{aligned} \forall a_i &\in A_i, \forall n \in \mathbb{N}, u_i\left(a_i^n, a_{-i}^n\right) \geq u_i\left(a_i, a_{-i}^n\right); \text{ hence,} \\ \forall a_i &\in A_i, u_i(\bar{a}_i, \bar{a}_{-i}) \geq u_i(a_i, \bar{a}_{-i}) \ (\text{as } n \to \infty). \end{aligned}$$

The second set of inequalities says that $\bar{a}_i \in r_i(\bar{a}_{-i})$. ∎

Fixed Point Theorems

- **Brouwer** proved the following important **Theorem**: *Let $X \subseteq \mathbb{R}^n$ be (nonempty) convex and compact, and let $f : X \to X$ be continuous; then f has a fixed point, i.e., $x^* = f(x^*)$ for some $x^* \in X$.*
- This implies that *every nice game has an NE*: Define the function $a \mapsto r(a) := (r_i(a_{-i}))_{i \in I}$; r is continuous because each r_i is continuous; A is (nonempty) compact and convex; thus, by Brouwer's Theorem, $a^* = r(a^*)$ [that is, $\forall i \in I, a_i^* = r_i\left(a_{-i}^*\right)$] for some a^*, which is an NE. To prove existence for games with (possibly) non-unique best replies, we use the following result:

Theorem

(Kakutani) *Let $X \subseteq \mathbb{R}^n$ be (nonempty) compact and convex, and let $\varphi : X \rightrightarrows X$ be such that (i) φ is nonempty valued and the graph $\mathrm{Gr}(\varphi)$ is closed, and (ii) $\varphi(x)$ is convex for each $x \in X$; then, $x^* \in \varphi(x^*)$ for some $x^* \in X$.*

Proof of the Existence Theorem

- For each $a \in A$, let $r(a) = \times_{i \in I} r_i(a_{-i})$. This defines a correspondence $r : A \rightrightarrows A$, where A is (nonempty) *compact* and *convex*.
- Note: for all $(a, a') \in A^2$, $a' \in r(a)$ IFF $\forall i \in I$, $a'_i \in r_i(a_{-i})$. Thus, an action profile a^* is an *NE IFF it is a fixed point* of r, i.e., $a^* \in r(a^*)$.
- We prove below that r satisfies the hypotheses (i)-(ii) of the Kakutani Theorem. Hence, $a^* \in r(a^*)$ for some a^*, which is an NE.
  - (i) By the Closed Graph Lemma, for every $i \in I$, r_i is nonempty valued and has a *closed graph*; hence, so does r.
  - (ii) For every $i \in I$ and $a_{-i} \in A_{-i}$, let $u_i^*(a_{-i}) = \max_{a_i \in A_i} u_i(a_i, a_{-i})$. Note, $r_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i^*(a_{-i})\}$, which is convex by quasi-concavity of u_i. ■

Equilibrium in Nice Symmetric Games

- We already noticed that Brouwer Fixed Point theorem implies that all nice games have at least one NE.
- Yet, Brouwer's theorem is hard to prove. We would like to use more elementary results in real analysis.
- Furthermore, we are going to prove a different theorem with stronger hypotheses and thesis:
  - Say that a game is **symmetric** if "it looks the same from the perspective of every player." We prove with elementary methods that *symmetric nice games have symmetric equilibria*, i.e., NEs where each player chooses the same action.

Symmetric games

- We mostly focus on two-person games for simplicity.

Definition

A two-person game $\langle \{1,2\}, A_1, A_2, u_1, u_2 \rangle$ is **symmetric** if for some set $\hat{A}$ and function $\hat{u} : \hat{A} \times \hat{A} \to \mathbb{R}$, (i) $A_1 = \hat{A} = A_2$, and (ii) for all $(a_1, a_2) \in A_1 \times A_2$, $u_1(a_1, a_2) = \hat{u}(a_1, a_2)$ and $u_2(a_1, a_2) = \hat{u}(a_2, a_1)$.

- In words, players have the same action set and *payoffs depend on the own action and the other's action in the same way*. The definition for n-person games is similar, with the additional requirement that $\hat{u} : \hat{A} \times \hat{A}^{|I|-1} \to \mathbb{R}$ is invariant to permutations of co-players actions (the second argument). Thus, the game looks exactly the same from the perspective of every player.
- **Examples.** A Cournot (quantity-setting) oligopoly in which each firm has the same capacity constraint and cost function is a symmetric game; or a beauty contest with $\theta_i = \hat{\theta}$ for each i.

Best replies in symmetric games

- **Observation.** In a symmetric two-person game, each player has the same best reply correspondence

$$\begin{array}{rccl} \hat{r}: & \hat{A} & \rightrightarrows & \hat{A}, \\ & b & \mapsto & \arg\max_{a\in\hat{A}} \hat{u}(a,b). \end{array}$$

- For n-person symmetric games, we interpret $\hat{r}(b)$ as the set of best replies to **symmetric** (deterministic) **conjectures,** according to which each co-player is expected to play the same action $b \in \hat{A}$:

$$\hat{r}(b) = \arg\max_{a\in\hat{A}} \hat{u}(a, b, ..., b).$$

Symmetric equilibria in symmetric nice games

Definition

An *action profile* in a symmetric game is **symmetric** if it associates the *same* action with each player.

Theorem

Every symmetric nice game has a symmetric Nash equilibrium.

- **Proof:**
  - Let $[\underline{a}, \bar{a}] \subseteq \mathbb{R}$ denote the common action set. By known results about nice games, $\hat{r} : [\underline{a}, \bar{a}] \to [\underline{a}, \bar{a}]$ is a *continuous* self-map. We show that is has a fixed point, which must be a symmetric Nash equilibrium.
  - Define the *continuous* function $b \mapsto f(b) := b - \hat{r}(b)$. With this, $f(\underline{a}) \leq 0$ and $f(\bar{a}) \geq 0$.
  - By the **Intermediate Value Theorem**, there is $a^* \in [\underline{a}, \bar{a}]$ such that $f(a^*) = 0$, that is $a^* = \hat{r}(a^*)$. ■

(Nice) Games with Strategic Complements

- We are going to consider concepts based on *monotonicity.* When we say "*increasing*" (of a function, or a sequence) we mean it in the *weak* sense, otherwise we explicitly say "*strictly increasing*".
- Actions of different players in a game with **strategic complements** are like complementary inputs of a firm, or complementary goods for a consumer: the incentive to increase an action of a player is increasing w.r.t. the (conjectured) action(s) of the co-player(s).
- To simplify the analysis, we focus here on *symmetric nice games*. To further simplify, here we only consider *two-person* games in our formulas. But *the analysis extends seamlessly to all nice games with strategic complements* (see GT-AST).

Games with strategic complements, definition

Definition

(*Strat. Complements*) A two-person symmetric nice game has (strict) **strategic complements** if, for all $a', a'' \in \hat{A}$ with $a' < a''$, function

$$\begin{array}{rccl} MU_{a',a''}: & [\underline{a}, \bar{a}] & \rightarrow & \mathbb{R}, \\ & b & \mapsto & \hat{u}(a'', b) - \hat{u}(a', b), \end{array}$$

is (strictly) increasing.

Observation *Suppose that $\hat{u}$ is twice continuously differentiable, then there are (strict) strategic complements if the cross-partial second derivative of $\hat{u}$ is (strictly) positive:*

$$\frac{\partial^2 \hat{u}(a, b)}{\partial a \partial b} \geq 0 \ \left(\frac{\partial^2 \hat{u}(a, b)}{\partial a \partial b} > 0 \right).$$

Monotonic best replies

By intuitive use of the Implicit Function Theorem (IFT):

Lemma

*(**Monotonicity**) In every (two-person, symmetric) nice game with strategic complements the best reply function is increasing.*

- **Intuition** (Twice-differentiable case, SOC satisfied). Let $\hat{u}_1(a,b) := \partial \hat{u}(a,b)/\partial a$ denote the partial derivative w.r.t. the 1st argument (own action). Similarly, $\hat{u}_{11} = \partial^2 \hat{u}/\partial^2 a$ and $\hat{u}_{12} = \partial^2 \hat{u}/(\partial a \partial b)$ denote the 2nd own and cross partial derivative, respectively.
  - By the FOC, the B.R. function $\hat{r}$ satisfies: $\forall b \in \hat{A}$, $\hat{u}_1(\hat{r}(b), b) = 0$.
  - Since $\frac{\mathrm{d}\hat{u}_1(r(b),b)}{\mathrm{d}b} = \hat{u}_{11}(\hat{r}(b), b)\hat{r}'(b) + \hat{u}_{12}(\hat{r}(b), b)$, it follows that

$$\hat{r}'(b) \overset{\text{(IFT)}}{=} -\frac{\hat{u}_{12}(\hat{r}(b), b)}{\hat{u}_{11}(\hat{r}(b), b)} \geq (>)\ 0,$$

    where $\geq (>)$ follows from $\hat{u}_{11}(\hat{r}(b), b) < 0$ (SOC) and $\hat{u}_{12} \geq 0$ ($\hat{u}_{12} > 0$) by (strict) complementarity.

Strategic complements, rationalizability, and NE

- Recall: $\underline{a}^k$, $\bar{a}^k$ denote the **lowest** and **highest** k**-rationalizable** actions in a symmetric nice game ($k \in \mathbb{N} \cup \{\infty\}$).

Theorem

(Sandwich) *In every symmetric nice game with strategic complements, for every* $k \in \mathbb{N}$, $\underline{a}^k = \hat{r}\left(\underline{a}^{k-1}\right)$ *and* $\bar{a}^k = \hat{r}\left(\bar{a}^{k-1}\right)$. *Furthermore,* $\underline{a}^\infty$ *and* $\bar{a}^\infty$ *are, respectively, the lowest and highest Nash equilibrium actions.*

Corollary

If a symmetric nice game with strategic complements has only one Nash equilibrium, then this equilibrium is symmetric and the equilibrium action is the unique rationalizable action $\underline{a}^\infty = \bar{a}^\infty$.

- **Comment:** *In* (symmetric, nice) *games with strategic complements and a unique NE, Rationality and Common Belief in Rationality imply NE behavior.*

Proof of the Sandwich Theorem for strategic complements

- By the results on rationalizability in nice games $\left(\rho^k(A)\right)_{k=1}^{\infty} = \left(\left[\underline{a}^k, \bar{a}^k\right]^{\{1,2\}}\right)_{k=1}^{\infty}$.
- Since $\left(\rho^k(A)\right)_{k=1}^{\infty}$ is decreasing (w.r.t. $\subseteq$), then $\left(\underline{a}^k\right)_{k=1}^{\infty}$ is increasing and $\left(\bar{a}^k\right)_{k=1}^{\infty}$ is decreasing.
- Thus, the two sequences, which are bounded, must have limits: $\lim_{k\to\infty} \underline{a}^k = \underline{a}^\infty$ and $\lim_{k\to\infty} \bar{a}^k = \bar{a}^\infty$, with $[\underline{a}^\infty, \bar{a}^\infty]^{\{1,2\}} = \rho^\infty(A)$.
- Since $\hat{r}$ is increasing, $\underline{a}^k = \hat{r}\left(\underline{a}^{k-1}\right)$ and $\bar{a}^k = \hat{r}\left(\bar{a}^{k-1}\right)$. Letting $k \to \infty$, we get $\underline{a}^\infty = \hat{r}\left(\underline{a}^\infty\right)$ and $\bar{a}^\infty = \hat{r}\left(\bar{a}^\infty\right)$. Thus, $\underline{a}^\infty$ and $\bar{a}^\infty$ are symmetric *NE actions*.
- Since every NE action is rationalizable, $\underline{a}^\infty$ and $\bar{a}^\infty$ must be *the lowest and highest NE action*, respectively. ■

References

- Battigalli, P., E. Catonini, and N. De Vito (2023): *Game Theory: Analysis of Strategic Thinking*. Typescript, Bocconi University.
- Battigalli, P. (2023): *Mathematical Language and Game Theory*. Typescript, Bocconi University.
- Battigalli, P. (2021): *Two-Person, Symmetric Nice Games*. Typescript, Bocconi University.
- Nash, J. (1950): *Non-cooperative games*. Ph.D. thesis, Princeton University.