Mixed Equilibrium

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Abstract

I start with a heuristic motivating example. The key result is the *Characterization Theorem* stating that a mixed equilibrium is a profile α^* of mixed actions such that, for each i, the support of α_i^* is included in the set of pure best replies to the co-players' mixed actions α_{-i}^*. The example clarifies the interpretation of this result: *a mixed equilibrium represents a stationary state of populations dynamics*, with *mixed actions describing statistical distributions of actions in populations rather than randomizations purposefully chosen by players*. It is shown that actions played with positive probability in some mixed equilibrium are rationalizable, hence, iteratively undominated. This and the Characterization Theorem yield an algorithm to compute mixed equilibria of finite 2-person games.

[These slides summarize Section 1 of Ch. 6 of "Game Theory: Analysis of Strategic Thinking" (GT-AST) devoted to the mixed (Nash) equilibrium concept. You should first read Section 4 of Ch. 5 on the interpretations of the Nash equilibrium concept.]

Heuristic example: owners and thieves

[See also alternative example and intepretation: "Hawk-Dove" game in Evolutionary Game Theory; e.g., Weibull (1995).]

- Interaction between n **owners** and n **thieves**, with *n large.* In each period, thieves are *matched at random* with owners: the probability that a particular thief (say, Mr. Lupin) finds himself near the home of a particular owner (say, Mr. Smith) is $1/n$.
- Each owner has objects that, upon burglary, can be stolen for a total value of V. He can activate his *Alarm* system at cost $c < V$, or *Not*, without knowing whether burglary will be attempted.
- Each thief can attempt *Burglary* or *Not*, without knowing if the alarm is on. If he does and the alarm is off, he steals the valuable objects and resells them for $V/2$; if the alarm is on, he cannot steal and incurs an expected penalty of $P/2$.

Owners and thieves: the game

- Assuming risk neutrality, each matched pair faces the following game (note, it turns out that it is not important to know the payoff function of the other):

$\mathbf{O} \backslash T$	*Burglary* (β)	*No* $(1-\beta)$
Alarm (α)	$\mathbf{V}-\mathbf{c}$, $-P/2$	$\mathbf{V}-\mathbf{c}$, 0
No $(1-\alpha)$	$\mathbf{0}$, $V/2$	$\mathbf{V}$, 0

- Let
  - α=*fraction* of owners activating the *Alarm* $(0 \leq \alpha \leq 1)$;
  - β=*fraction* of thieves attempting *Burglary* $(0 \leq \beta \leq 1)$.
- Previous-period statistics $\left(\alpha_{t-1}, \beta_{t-1}\right)$ are recorded and published.
- *Viscosity:* In each period t, only a few **active** owners and thieves determined at random look at the statistics $\left(\alpha_{t-1}, \beta_{t-1}\right)$ of the previous period and consider changing their choice; when they do and are indifferent, they keep the same choice. *Such statistics determine the current-period expectations of active agents.*

Owners and thieves: the dynamics

- An *active owner*
  - compares the safe choice (*Alarm*), which is worth $V - c$ to the risky one (*No*), whose expected payoff is $(1 - \beta) V$.
  - He chooses *Alarm* (resp. *No Alarm*) at t if $\beta_{t-1} > c/V$ (resp. $\beta_{t-1} < c/V$), and keeps the same choice of the previous period if $\beta_{t-1} = \beta^* := c/V$ (note, $0 < \beta^* < 1$).
  - Thus, $\alpha_t > \alpha_{t-1}$ if $\beta_{t-1} > \beta^*$, $\alpha_t < \alpha_{t-1}$ if $\beta_{t-1} < \beta^*$, and $\alpha_t = \alpha_{t-1}$ if $\beta_{t-1} = \beta^*$.
- An *active thief*
  - compares the safe choice (*No Burglary*), which is worth 0, with the risky one (*Burglary*), whose expected payoff is $(1 - \alpha) V/2 - \alpha P/2$.
  - He attempts *Burglary* (resp. does *Not*) at t if $\alpha_{t-1} < V/(V + P)$ (resp. $\alpha_{t-1} > V/(V + P)$), and keeps the same choice of the previous period if $\alpha_{t-1} = \alpha^* := V/(V + P)$ (note, $0 < \alpha^* < 1$).
  - Thus, $\beta_t > \beta_{t-1}$ if $\alpha_{t-1} < \alpha^*$, $\beta_t < \beta_{t-1}$ if $\alpha_{t-1} > \alpha^*$, and $\beta_t = \beta_{t-1}$ if $\alpha_{t-1} = \alpha^*$.

Owners and thieves: steady state

- From the previous intuitive description of the dynamics of (α_t, β_t), the state of the system, we see that the **steady state** (rest point), or **mixed equilibrium,** $(\alpha^*, \beta^*) = (V/(V+P), c/V)$ has the following features:
  - α^* is the (expected) fraction of owners activating the Alarm that makes (active) *thieves indifferent between the actions played by a positive fraction of them* (in this elementary example, all actions);
  - β^* is the fraction of thieves attempting Burglary that makes (active) *owners indifferent between the actions played by a positive fraction of them* (in this elementary example, all actions).
- **Take home message:**
  - *A mixed equilibrium describes steady-state statistics,* or frequency distributions, $(\alpha_i^*, \alpha_{-i}^*) \in \Delta(A_i) \times \Delta(A_{-i})$.
  - In 2-person games, if each action of each player/role is played by a positive fraction of agents in that role, then α_i^* (resp. α_{-i}^*) solves the indifference condition of $-i$ (resp. i). Thus, α_i^* (resp. α_{-i}^*) *is not chosen by any individual: agents do not randomize.*

Mixed extension of a game

Definition

The **mixed extension** of a *finite* game $G = \left\langle I, (A_i, u_i)_{i\in I} \right\rangle$ is the game $\bar{G} = \left\langle I, \left(\Delta\left(A_i\right), \bar{u}_i\right)_{i\in I} \right\rangle$ where, for all $i \in I$ and $\alpha \in \times_{j\in I}\Delta\left(A_j\right)$,

$$\bar{u}_i\left(\alpha\right) = \sum_{a\in A} u_i\left(a\right) \underbrace{\prod_{j\in I} \alpha_j\left(a_j\right)}_{\mathbb{P}_\alpha(a)}.$$

- **Note:**
    - Expected payoffs are computed under the assumption that the actions of different players are *statistically independent*.
    - Each function $\bar{u}_i : \times_{j\in I}\Delta\left(A_j\right) \to \mathbb{R}$ is *continuous* and *multi-affine*, that is, affine (concave and convex) in each variable α_j.
    - The definition can be extended to "non pathological" infinite games, e.g., compact-continuous games, letting $\bar{u}_i\left(\alpha\right) = \mathbb{E}_{\times_{j\in I}\alpha_j}\left(u_i\right)$, where $\times_{j\in I}\alpha_j$ is the product measure on A [$\times_{j\in I}\alpha_j$ is such that $\forall C \in \mathcal{C}$, $\left(\times_{j\in I}\alpha_j\right)\left(C\right) = \prod_{j\in I} \alpha_j\left(C_j\right)$].

Mixed equilibrium: definition and existence

Definition

A mixed action profile $\alpha = (\alpha_i)_{i \in I}$ is a **mixed** (Nash) **equilibrium** of a game G if it is a Nash equilibrium (NE) of the mixed extension $\bar{G}$.

- **Note:** Pure equilibria are a special, degenerate kind of mixed equilibria (check that you understand why).
- Many games have no pure equilibria (e.g., Matching Pennies, Rock-Scissor-Paper, ...), but all finite games have mixed equilibria.

Theorem

(**Existence**) *Every finite game G has at least one mixed equilibrium.*

- **Proof:**
  - We must show that the mixed extension $\bar{G}$ has an NE.
  - For each i, (1) $\Delta(A_i) \subseteq \mathbb{R}^{A_i}$ is (nonempty) compact and convex; (2) $\bar{u}_i(\alpha)$ is continuous in α and affine (hence concave) in α_i. Thus, $\bar{G}$ satisfies the sufficient conditions for existence of an NE. ∎

Mixed equilibrium: characterization

- **Recall**: $A_i \subseteq \Delta(A_i)$, $r_i(\mu^i) := A_i \cap \arg\max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i)$, and $\alpha_i^* \in \arg\max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i)$ IFF $\mathrm{supp}\alpha_i^* \subseteq r_i(\mu^i)$ (**Lemma 1** in GT-AST).
- In two-person games, conjectures and mixed actions of the co-players are represented by the same mathematical objects, the elements of $\Delta(A_{-i})$.
- It follows that, in two-person games, α^* is a mixed equilibrium IFF $\mathrm{supp}\alpha_i^* \subseteq r_i(\alpha_{-i}^*)$ for each i.
- More generally, write $r_i(\alpha_{-i}) = r_i(\times_{j \neq i}\alpha_j)$, where $\times_{j \neq i}\alpha_j \in \Delta(A_{-i})$ is the product measure on A_{-i} obtained from α_{-i} under statistical independence. *Lemma 1* (GT-AST) yields:

Theorem

*(**Characterization**) In every finite (or compact-continuous) game G, for every mixed action profile $\alpha^* = (\alpha_i^*)_{i \in I}$, α^* is a mixed equilibrium of G IFF $\mathrm{supp}\alpha_i^* \subseteq r_i(\alpha_{-i}^*)$ for each $i \in I$.*

Mixed equilibrium and rationalizability

- The **Characterization Theorem** and **Theorem 3** of GT-AST imply that *the actions profiles played with positive probability in a mixed Nash equilibrium are rationalizable, hence iteratively undominated*:

Theorem

In every finite (or compact-continuous) game G, for every mixed equilibrium α^* of G, $\times_{i \in I} supp\alpha_i^* \subseteq \rho^\infty(A) = \mathrm{ND}^\infty(A)$.

- **Proof:**
  - Recall, by Theorem 3 (GT-AST), for every $C \in \mathcal{C}$, $C \subseteq \rho(C) \Rightarrow C \subseteq \rho^\infty(A)$.
  - Fix a mixed equilibrium α^* and let $C = \times_{i \in I}\mathrm{supp}\alpha_i^*$, we prove $C \subseteq \rho(C)$. Indeed, by Lemma 1 (GT-AST) $C_i = \mathrm{supp}\alpha_i^* \subseteq r_i(\alpha_{-i}^*)$. Also, $\mathrm{supp}(\times_{j \neq i}\alpha_j^*) = \times_{j \neq i}\mathrm{supp}(\alpha_j^*) = C_{-i}$. Thus, $r_i(\alpha_{-i}^*) \subseteq r_i(\Delta(C_{-i}))$ for each $i \in I$, and $C \subseteq \times_{i \in I} r_i(\Delta(C_{-i})) = \rho(C)$. ■

Computation of the set of mixed equilibria: 2 players

- The set of mixed equilibria of *finite 2-person games* can be computed by solving a sequence of linear programming (LP) problems.
- **First.** Iteratively delete (in any order) the dominated actions (checking whether an action in a finite game is dominated is an LP problem) and obtain $A^* := \mathrm{ND}^\infty(A) = \rho^\infty(A)$. By Theorem 3 (GT-AST), the search for mixed equilibria of G can be limited to the restricted game $G^* = \left\langle I, (A_i^*, u_i^*)_{i\in I} \right\rangle$, where $u_i^* = u_i|_{A^*}$ is the restriction of u_i to A^*.
- **Second.** By the Characterization Theorem, for every nonempty Cartesian subset $C \subseteq A^*$, α^C is a mixed equilibrium of G^* (and G) with $\mathrm{supp}\alpha_i^C = C_i$ for each $i \in I$ IFF for some $\left(y_i^C\right)_{i\in I} \in \mathbb{R}^I$ and each i, α_{-i}^C solves the following *system of* **linear** *equalities and inequalities*:
  - $\forall a_i \in C_i, \sum_{a_{-i}\in C_{-i}} \alpha_{-i}(a_{-i})\, u_i(a_i, a_{-i}) = y_i^C$ (**indifference** cond.)
  - $\forall a_i' \in A_i^* \backslash C_i, \sum_{a_{-i}\in C_{-i}} \alpha_{-i}(a_{-i})\, u_i(a_i', a_{-i}) \le y_i^C$ (**incentive** cond.)

Computation of the set of mixed equilibria: comments

- **Note:**
  - (**The co-player does not care!**) The indifference and incentive conditions of i are solved w.r.t. the mixed action α_{-i} of the *co-player* $-i$. But the co-player could not care less about choosing a mixed action satisfying such conditions! Go back to the heuristic example to interpret this.
  - (**Number of equilibria**) Let $I = \{1, 2\}$. There are at most $\left(2^{|A_1^*|} - 1\right) \times \left(2^{|A_2^*|} - 1\right)$ Cartesian sets C supporting mixed equilibria. Except for "non-generic" games, this is an upper bound on the number of mixed equilibria (an indifference between payoffs, or linear combinations of payoffs, may yield a continuum of equilibria; think of examples).
  - (**Complexity**) The search for mixed equilibria is exponentially complex in the number of (rationalizable) actions.

Numerical example: two pure equilibria, one mixed

Payoffs of the **row pl.** (*column pl.*) in **bold** (*Italics*). First delete b, then delete r, then compute equilibria.

	ℓ	c	r
t	**4**, *2*	**1**, *1*	**4**, *1*
m	**1**, *1*	**2**, *4*	**1**, *2*
b	**2**, *0*	**1**, *0*	**0**, *1*

$\longrightarrow$

	ℓ	c	r
t	**4**, *2*	**1**, *1*	**4**, *1*
m	**1**, *1*	**2**, *4*	**1**, *2*

$\longrightarrow$

	ℓ (λ)	c
t (τ)	**4**, *2*	**1**, *1*
m	**1**, *1*	**2**, *4*

$\longrightarrow$

mixed	Nash	eq.
$\{(\tau^*, \lambda^*) = \left(\frac{1}{4}, \frac{3}{4}\right)$,	(t, ℓ),	$(m, c)\}$

indifference of row: $4\lambda + (1 - \lambda) = \lambda + 2(1 - \lambda) \Rightarrow \lambda^* = 1/4$.

indifference of col.: $2\tau + (1 - \tau) = \tau + 4(1 - \tau) \Rightarrow \tau^* = 3/4$.

Numerical example: one mixed equilibrium, no pure NE

Payoffs of the **row pl.** (*column pl.*) in **bold** (*Italics*). First delete b, then delete r, then compute equilibria.

	ℓ	c	r
t	**4**, *1*	**1**, *2*	**4**, *1*
m	**1**, *4*	**2**, *1*	**1**, *2*
b	**2**, *0*	**1**, *0*	**0**, *1*

$\longrightarrow$

	ℓ	c	r
t	**4**, *1*	**1**, *2*	**4**, *1*
m	**1**, *4*	**2**, *1*	**1**, *2*

$\longrightarrow$

	ℓ (λ)	c
t (τ)	**4**, *1*	**1**, *2*
m	**1**, *4*	**2**, *1*

$\longrightarrow$ mixed eq. $(\tau^*, \lambda^*) = \left(\frac{1}{4}, \frac{3}{4}\right)$

No pure NE.

indifference of row: $4\lambda + (1-\lambda) = \lambda + 2(1-\lambda) \Rightarrow \lambda^* = 1/4.$

indifference of col.: $\tau + 4(1-\tau) = 2\tau + (1-\tau) \Rightarrow \tau^* = 3/4.$

References

BATTIGALLI, P., E. CATONINI, AND N. DE VITO (2023): *Game Theory: Analysis of Strategic Thinking*. Typescript, Bocconi University.

BATTIGALLI, P. (2023): *Mathematical Language and Game Theory*. Typescript, Bocconi University.

WEIBULL, J. (1995): *Evolutionary Game Theory*. Cambridge MA: MIT Press.