Rational Planning in Multistage Games

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Game Theory: Analysis of Strategic Thinking

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Abstract

To make the exposition self-contained, we first summarily recall our definition of multistage games with observable actions. Next we move to **rational planning** in dynamic decision problems and games. We present the One-Deviation Principle from a decision-making perspective. Focusing on finite games with complete information and taking the perspective of a single player with a subjective probabilistic conjecture about the behavior of co-players, we analyze several dynamic optimality properties for strategies. In particular, we present (i) the *Folding-Back Principle:* Folding-Back Optimality is equivalent to One-Step Optimality, and (ii) the *Optimality Principle*: Sequential Optimality is equivalent to Folding-Back Optimality. These two results yield the *One-Deviation Principle:* Sequential Optimality is equivalent to One-Step Optimality. [These slides summarize and in part complement Chapter 10 and Section 9.4 of Chapter 9 of GT-AST.]

Preliminaries: Multistage Games

- We first consider a multistage game with observable actions $\left\langle I, (A_i, \mathcal{A}_i(\cdot), u_i)_{i\in I}\right\rangle$, where:
  - $i \in I$, **players**;
  - $a_i \in A_i$, potentially feasible **actions** of i;
  - $A = \times_{i\in I} A_i$, $A^t = \underbrace{A \times ... \times A}_{t \text{ times}}$, set of sequences of action profiles of length t; $A^0 := \{\varnothing\}$ where $\varnothing$ is the **empty sequence**;
  - $\mathcal{A}_i(\cdot) : \bigcup_{t\geq 0} A^t \rightrightarrows A_i$, **constraint correspondence** of i;
  - derive from $\left\langle I, (A_i, \mathcal{A}_i(\cdot))_{i\in I}\right\rangle$ the tree $\left(\bar{H}, \preceq\right)$ with root $\varnothing$; Z (resp., H), set of terminal (resp., non-terminal) histories;
  - $u_i : Z \to \mathbb{R}$, **payoff function** of i.

Preliminaries: Strategies

- Strategies are **rules of behavior** describing how actions are chosen as a function of the observed history. They may be interpreted as *descriptions* of how a player would behave at each $h \in H$, or *plans* in the mind of the players.
- $s_i \in S_i := \times_{h \in H} \mathcal{A}_i(h)$, **strategies** (pure).
- $s \in S := \times_{i \in I} S_i$, **strategy profiles**, $s(h) = (s_i(h))_{i \in I} \in \mathcal{A}(h)$ is the action profile selected by s at $h \in H$.
- **Path function:** $\zeta : S \to Z$

  $$\zeta(s) = (s(\varnothing), s(s(\varnothing)), s((s(\varnothing), s(s(\varnothing)))), \ldots) \text{ until termination.}$$

- **Strategies consistent with a history:** for each $h \in H$,
  $S(h) := \{s \in S : h \prec \zeta(s)\} = \times_{i \in I} S_i(h)$, with
  $S_i(h) := \mathrm{proj}_{S_i} S(h)$, strategies of i that **allow** (do not prevent) h.
- For $a_i \in \mathcal{A}_i(h)$, let $S_i(h, a_i) := \{s_i \in S_i(h) : s_i(h) = a_i\}$, strategies **allowing** h **and choosing** a_i **at** h.

Preliminaries: Randomized Strategies

- Although we are not going to assume that players truly randomize, randomized strategies are convenient theoretical concepts for two reasons (cf. mixed actions in static games):
  - (i) they can be used to characterize the justifiability of pure strategies,
  - (ii) with 2 players, a randomized strategy of the co-player can be interpreted as a probabilistic conjecture about the co-player.
- We consider two notions of randomization:
  - mixed strategies=global ex ante randomizations over pure strategies (not very intuitive),
  - behavior strategies=local randomizations over actions for each non-terminal history.
- $\sigma_i \in \Delta(S_i)$, **mixed strategies.**
- $\beta_i(\cdot|\cdot) \in B_i := \times_{h \in H} \Delta(\mathcal{A}_i(h))$, **behavior strategies:** $\beta_i(\cdot|h) \in \Delta(\mathcal{A}_i(h))$ is the *mixed action* planned conditional on reaching $h \in H$.

Connection Between Mixed and Behavior Strategies, I

- Assuming "independent local randomization", $\beta_i \mapsto \sigma_i$ with

$$\forall s_i \in S_i, \, \sigma_i(s_i) = \prod_{h \in H} \beta_i(s_i(h)|h).$$

- If $\sigma_i(S_i(h)) > 0$ for each $h \in H$, computing conditional probabilities, $\sigma_i \mapsto \beta_i$ with

$$\forall h \in H, \forall a_i \in \mathcal{A}_i(h), \, \beta_i(a_i|h) = \frac{\sigma_i(S_i(h, a_i))}{\sigma_i(S_i(h))}.$$

- **Population interpretation**: Statistical distribution σ_i of (pure) strategies of agents in population i. If $\sigma_i \mapsto \beta_i$, $\beta_i(a_i|h)$ is the frequency of a_i conditional on the occurrence of h, that is, considering only agents whose (pure) strategies allow h.

Connection Between Mixed and Behavior Strategies, II

- Let $\mathbb{P}_{s_{-i},\mu_i}(z)$=prob. of z induced by s_{-i} and μ_i, with $\mu_i = \sigma_i \in \Delta(S_i)$ or $\mu_i = \beta_i \in B_i$; specifically (in finite games):
- $\mathbb{P}_{s_{-i},\sigma_i}(z) = \sum_{s_i:\zeta(s_{-i},s_i)=z} \sigma_i(s_i)$;
- let $z = \left(a^1, ..., a^{\ell(z)}\right)$ and $\beta_{-i}^{s_{-i}}(a^k|..., a^{k-1}) = 1$ if $s_{-i}\left(..., a^{k-1}\right) = a_{-i}^k$ and $\beta_{-i}^{s_{-i}}(a^k|..., a^{k-1}) = 0$ otherwise, then $\mathbb{P}_{s_{-i},\beta_i}(z) = \prod_{k=1}^{\ell(z)} \beta_{-i}^{s_{-i}}(a_{-i}^k|..., a^{k-1})\beta_i(a_i^k|..., a^{k-1})$.
- **Kuhn's Theorem**: *If $\sigma_i \mapsto \beta_i$ or $\beta_i \mapsto \sigma_i$, then σ_i and β_i induce the same probabilities of paths independently of the behavior of others, that is,*

$$\forall s_{-i} \in S_{-i}, \forall z \in Z,\ \mathbb{P}_{s_{-i},\sigma_i}(z) = \mathbb{P}_{s_{-i},\beta_i}(z).$$

Preliminaries: Conjectures

- Start with 2 players:
  - **initial conjecture** $\mu^i \in \Delta(S_{-i})$ (same as a mixed strategy of $-i$), if $\mu^i(S_{-i}(h)) > 0$, updated conjecture $\mu^i(\cdot|S_{-i}(h)) \in \Delta(S_{-i}(h))$, with
  $$\forall s_{-i} \in S_{-i}(h), \; \mu^i(s_{-i}|S_{-i}(h)) = \frac{\mu^i(s_{-i})}{\mu^i(S_{-i}(h))}.$$
  But, what if $\mu^i(S_{-i}(h)) = 0$? Pl. i is "surprised" and needs a "brand new conjecture" (we will come back to this).
  - At first, we bypass this problem considering **conjectures** $\beta^i(\cdot|\cdot) \in \times_{h \in H} \Delta(\mathcal{A}_{-i}(h))$ (same as behavior strategies of $-i$); $\beta^i(\cdot|h) \in \Delta(\mathcal{A}_{-i}(h))$ is i's conjecture on $-i$'s actions conditional on h.
- With multiple co-players we let conjectures allow for *correlation*. Thus, if $|I| > 2$ conjectures are not like profiles of co-players' randomized strategies (cf. static games).
- **Connection:** If $\mu^i(S_{-i}(h)) > 0$, $\beta^i(a_{-i}|h) = \frac{\mu^i(S_{-i}(h,a_{-i}))}{\mu^i(S_{-i}(h))}$.

Sequential Optimality in Finite Games, Values

- Assume that Γ is *finite* (hence, $\max = \sup$). *Fix* strategy $s_i \in S_i$ and conjecture $\beta^i \in \times_{h \in H} \Delta(\mathcal{A}_{-i}(h))$. Then:
  - Let $\mathbb{P}^{s_i,\beta^i}(h'|h)$ be the prob. of reaching h' from $h \prec h'$.
  - Let $Z(h) := \{z \in Z : h \preceq z\}$. With this,
- the **value** of reaching h is

$$V_i^{s_i,\beta^i}(h) = \sum_{z \in Z(h)} \mathbb{P}^{s_i,\beta^i}(z|h) u_i(z),$$

- the **value of** taking **action** a_i **given** h is

$$V_i^{s_i,\beta^i}(h, a_i) = \sum_{a_{-i} \in \mathcal{A}_{-i}(h)} \beta^i(a_{-i}|h) V_i^{s_i,\beta^i}(h, (a_i, a_{-i})).$$

Sequential and One-Step Optimality, Definition

Definition

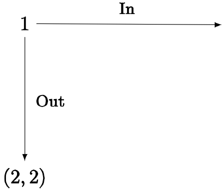
Fix $\bar{s}_i$ and β^i. Strategy $\bar{s}_i$ is **sequentially optimal** given β^i IF

$$\forall h \in H,\ V_i^{\bar{s}_i,\beta^i}(h) = \sup_{s_i \in S_i(h)} V_i^{s_i,\beta^i}(h);$$

$\bar{s}_i$ is **one-step optimal** given β^i IF

$$\forall h \in H,\ \bar{s}_i(h) \in \arg\sup_{a_i \in \mathcal{A}_i(h)} V_i^{\bar{s}_i,\beta^i}(h, a_i).$$

Example: BoS with Outside Option



$1 \backslash 2$	B_2	S_2
B_1	$3, 1$	$0, 0$
S_1	$0, 0$	$1, 3$

- Suppose $\beta^1(\mathrm{B}_2|\mathrm{In}) = \frac{1}{2}$. What is the best plan $\bar{s}_i$ for pl. 1? Find B.R. in BoS and value of In, then compare with Out:
- *Algorithm*: Obtain values $\hat{V}_1^{\beta^1}(h, a_1)$, $\hat{V}_1^{\beta^1}(h)$ for $h \in H$, $a_1 \in \mathcal{A}_1(h)$ and $\bar{s}_i$ as follows:
    - $\hat{V}_1^{\beta^1}(\mathrm{In}, \mathrm{B}_1) = 3 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{3}{2} > \frac{1}{2} = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \hat{V}_1^{\beta^1}(\mathrm{In}, \mathrm{S}_1)$ $\Rightarrow \bar{s}_1(\mathrm{In}) = \mathrm{B}_1$.
    - $\hat{V}_1^{\beta^1}(\mathrm{In}) = \max\left\{\hat{V}_1^{\beta^1}(\mathrm{In}, \mathrm{B}_1), \hat{V}_1^{\beta^1}(\mathrm{In}, \mathrm{S}_1)\right\} = \frac{3}{2} < 2 =$ $= \hat{V}_1^{\beta^1}(\mathrm{Out}) \Rightarrow \bar{s}_1(\varnothing) = \mathrm{Out}$. **Note:** $\bar{s}_1$ *satisfies SO and OSO.*

Folding-Back Optimality 1: Preliminaries

- We defined values (expected utilities) for pl. i of histories and actions, taking as given that i would choose in future stages (if any) according to a strategy s_i. Hence, such values depend on conjecture β^i and also on strategy s_i: $V_i^{s_i,\beta^i}(h)$, $V_i^{s_i,\beta^i}(h, a_i)$.
- If h is "pre-terminal" [if $(h, a) \in Z$ for each $a \in \mathcal{A}(h)$] the dependence on s_i is vacuous, because there is no further choice to make later on.
- Given β^i, we find **optimal values** $\hat{V}$ with a **backward calculation**, starting from the last stage, as we did in the BoSOO.
- We define *recursively* the **folding-back** (optimal) **value** $\hat{V}_i^{\beta^i}(h)$ of reaching h:
- $\ell(h) =$ **length** of h [thus, $\ell(\varnothing) := 0$, $\forall (h, a)$, $\ell(h, a) = \ell(h) + 1$)].
- $\Gamma(h) = \left\langle I, (A_i, \mathcal{A}_{i,h}(\cdot), u_{i,h})_{i\in I} \right\rangle =$ **subgame** starting at h: $\mathcal{A}_{i,h}(h') = \mathcal{A}_i(h, h')$, $u_{i,h}(h') = u_i(h, h')$ if $(h, h') \in Z$.
- $L(\Gamma(h)) = \max_{z\in Z(h)} \ell(z) - \ell(h) =$ **height** of $\Gamma(h)$.
  - [Recall: $Z(h) := \{z \in Z : h \preceq z\}$; in particular, $Z(z) = \{z\}$.]

Folding-Back Optimality 2: Algorithm

- Define a recursive computation based on the height $L(\Gamma(h))$:
- *Basis step:* $L(\Gamma(h)) = 0$ $(h \in Z)$, $\hat{V}_i^{\beta^i}(h) := u_i(h)$.
- *Recursive step:* suppose $\hat{V}_i^{\beta^i}(h')$ is defined for every h' with $L(\Gamma(h')) \leq k$. If $L(\Gamma(h)) = k+1$, then $L(\Gamma(h,a)) \leq k$ for each $a \in \mathcal{A}(h)$; with this, for every $a_i \in \mathcal{A}_i(h)$,

$$\hat{V}_i^{\beta^i}(h, a_i) := \sum_{a_{-i} \in \mathcal{A}_{-i}(h)} \beta^i(a_{-i}|h)\hat{V}_i^{\beta^i}(h,(a_i,a_{-i})),$$

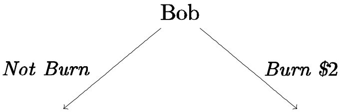
$$\hat{V}_i^{\beta^i}(h) := \sup_{a_i \in \mathcal{A}_i(h)} \hat{V}_i^{\beta^i}(h, a_i).$$

Definition

$\bar{s}_i$ is **folding-back optimal** given β^i IF, for all $h \in H$,

$$\bar{s}_i(h) \in \arg \sup_{a_i \in \mathcal{A}_i(h)} \hat{V}_i^{\beta^i}(h, a_i).$$

Folding Back in the BoS With Dissipative Action



Ann\Bob	L	R
U	$4, 1$	$0, 0$
D	$0, 0$	$1, 4$

Ann\Bob	l	r
u	$4, -1$	$0, -2$
d	$0, -2$	$1, 2$

- Conjecture of Bob: $\beta^b(D|N) = p < \frac{1}{5}$, $\beta^b(d|B) = q > \frac{1}{5}$.
- $\hat{V}_b^{p,q}(N) = \max\{(1-p), 4p\} = 1 - p \Rightarrow L\ (p < \frac{1}{5})$.
- $\hat{V}_b^{p,q}(B) = \max\{(1-q), 4q\} - 2 = 4q - 2 \Rightarrow r\ (q > \frac{1}{5})$.
- $\hat{V}_b^{p,q}(\varnothing) = \max\left\{\hat{V}_b^{p,q}(N), \hat{V}_b^{p,q}(B)\right\} = \max\{1-p, 4q-2\} \Rightarrow$ $[N \text{ if } 1 - p > 4q - 2]$.

Proposition

*(**Folding-Back Principle**) $\bar{s}_i$ is folding-back optimal (given β^i) IFF $\bar{s}_i$ is one-step optimal (given β^i).*

Theorem

*(**Optimality Principle**) $\bar{s}_i$ is sequentially optimal (given β^i) IFF $\bar{s}_i$ is folding-back optimal (given β^i).*

Corollary

*(**One-Deviation Principle**) $\bar{s}_i$ is sequentially optimal (given β^i) IFF $\bar{s}_i$ is one-step optimal (given β^i).*

Intuition

- The OD-Principle is obviously implied by the conjunction of the FB-Principle and the Optimality Principle.
  - **Folding-Back Principle** By inspection the recursive definition of folding-back optimality, it is quite easy to see that it implies one-step optimality. The converse can be proved by induction: The respective maximization conditions are equivalent by definition at histories of height 1 (last stage, basis step). Assuming that the equivalence holds for histories of height k or less (inductive hypothesis), it must hold also for histories of height $k+1$ (inductive step).
  - **Optimality Principle** Sequential optimality (by definition) implies one-step optimality, which implies folding-back optimality as argued above. As above, the converse can be proved by induction: The respective maximization conditions are equivalent by definition at histories of height 1 (last stage, basis step). Assuming that the equivalence holds for histories of height k or less (inductive hypothesis), it must hold also for histories of height $k+1$ (inductive step).

Perspective

- *Folding-back optimality* (equivalent to one-step optimality) *is* the conceptually primitive notion of *rational planning:* it is a kind of "intra-personal equilibrium" justified by the assumption that player i is introspective, hence able to predict his future behavior, conditional on the realization of every history. (More generally, i.e., for infinite-horizon games, we take the one-step optimality as the definition of rational planning.)
- *Sequential optimality* is just a *characterization* of rational planning that holds when i has dynamically consistent preferences, hence with the subjective EU criterion. **This is our interpretation of the Optimality Principle.**
- The *OD Principle* (equivalence between the one-step and sequential optimality) also holds for most infinite-horizon games of interest (e.g., infinitely repeated games and bargaining games with standard discounting).

Justifiability 1/2

- We want to understand *whether a description s_i of i's behavior* is *consistent with rationality*.
- **Possible answer**: there is some conjecture β^i such that s_i is sequentially (folding-back) optimal given β^i.
- **Problem:** two *behaviorally equivalent* strategies $s_i' \approx s_i''$ are *indistinguishable* from the perspective of i's co-players (or of an external observer), because—by the Equivalence Lemma—$\zeta(s_i', s_{-i}) = \zeta(s_i'', s_{-i})$ for all $s_{-i} \in S_{-i}$.
- **Solution:** Use a *notion of justifiability* that is *invariant under behavioral equivalence* (and hence also applies to reduced strategies).

Justifiability 2/2

- Recall that
  - $H_i(s_i) = \{h \in H : s_i \in S_i(h)\}$ is the set of non-terminal histories allowed by s_i.
  - (**behavioral equivalence**) $s_i \approx \bar{s}_i$ if $(H_i(s_i) = H_i(\bar{s}_i))$ and $(\forall h \in H_i(s_i), s_i(h) = \bar{s}_i(h))$.

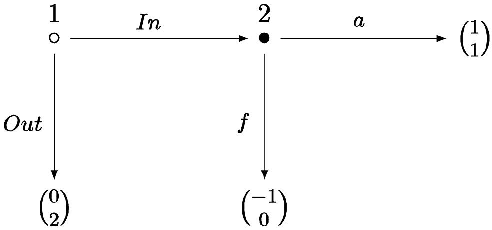
Definition

Strategy $\bar{s}_i$ is **weakly sequentially optimal** given β^i, written $\bar{s}_i \in r_i(\beta^i)$, if $\forall h \in H_i(\bar{s}_i)$, $V_i^{\bar{s}_i,\beta^i}(h) = \sup_{s_i \in S_i(h)} V_i^{s_i,\beta^i}(h)$; $\bar{s}_i$ is **justifiable** if $\exists \beta^i$, $\bar{s}_i \in r_i(\beta^i)$.

- **Remark** *For all* $s_i, \bar{s}_i$ *and* β^i, **if** $s_i \approx \bar{s}_i$ *and*
  - s_i *is sequentially optimal given* β^i, *then* $\bar{s}_i \in r_i(\beta^i)$;
  - $\bar{s}_i \in r_i(\beta^i)$, *then* $s_i \in r_i(\beta^i)$.

Dominance

- In static games an action is justifiable IFF it is undominated (by a mixed). In dynamic games undominated strategies may be unjustifiable, e.g., the fighting strategy **f**=(f if In) in the Entry Game.



- Yet, **f** is dominated conditional on history $h = (In)$, which is allowed by **f** $[h \in H_2(\mathbf{f})]$.

Conditional Dominance

- Recall: $U_i(s) = u_i(\zeta(s))$. With this, the EU of σ_i given s_{-i} is: $U_i(\sigma_i, s_{-i}) = \sum_{s_i \in S_i} U_i(s_i, s_{-i})\, \sigma_i(s_i)$.

Definition

Strategy $\bar{s}_i$ is **conditionally dominated** if there are a history $h \in H_i(\bar{s}_i)$ and a mixed strategy $\sigma_i \in \Delta(S_i(h))$ s.t.

$$\forall s_{-i} \in S_{-i}(h),\ U_i(\sigma_i, s_{-i}) > U_i(\bar{s}_i, s_{-i}).$$

- **Remark** *If a strategy $\bar{s}_i$ is dominated, then $\bar{s}_i$ is also conditionally dominated, but the converse does not hold* (see the Entry Game).

Proposition

If a strategy $\bar{s}_i$ is conditionally dominated, then $\bar{s}_i$ is also weakly dominated.

Justifiability and Conditional Dominance

Lemma

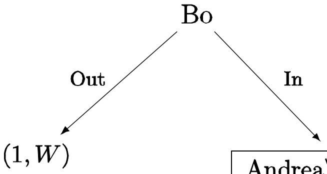
A strategy is justifiable **if and only if** *it is not conditionally dominated.*

- **Intuition**
  - (**Only if**) Let $\bar{s}_i \in r_i\left(\beta^i\right)$, fix any $\bar{h} \in H_i(\bar{s}_i)$. Then $\bar{s}_i$ is a B.R. in $S_i(\bar{h})$ to $\mu^i_{\bar{h}} \in \Delta\left(S_{-i}(\bar{h})\right)$ derived from β^i as follows

    $$\forall s_{-i} \in S_{-i}(\bar{h}),\ \mu^i_{\bar{h}}(s_{-i}) = \prod_{h \in H : h \not\prec \bar{h}} \beta^i\left(s_{-i}(h)\,|h\right)$$

    [$\forall z \in Z(\bar{h})$,$\forall s_i \in S_i(h)$, $\mathbb{P}_{s_i,\mu^i_{\bar{h}}}(z|\bar{h}) = \mathbb{P}_{s_i,\beta^i}(z|\bar{h})$]. By (easy part of) W-P Lemma, $\bar{s}_i$ is not dominated conditional on $\bar{h}$. Thus, $\bar{s}_i$ is not conditionally dominated.
  - (**If**) If $\bar{s}_i$ is not conditionally dominated, by (hard part of) W-P Lemma, there is array $\left(\mu^i_h\right)_{h \in H_i(\bar{s}_i)} \in \times_{h \in H_i(\bar{s}_i)} \Delta\left(S_{-i}(h)\right)$ s.t., for every $h \in H_i(\bar{s}_i)$, $\bar{s}_i$ is a B.R. in $S_i(h)$ to μ^i_h. One can derive (with quite a bit of work) β^i s.t. $\bar{s}_i \in r_i\left(\beta^i\right)$. ♡

Example of Conditional Dominance



Andrea\Bo	C	M
home	1, 0	1, 0
c	2, 4	0, 0
m	0, 0	4, 2

- *home* is dominated for Andrea in the subgame by mixed action $\frac{1}{2}\delta_c + \frac{1}{2}\delta_m$. Thus, $s_a =$**home**$=$ (home if In) is conditionally dominated.
- If $W = 1$, Bo knows u_a and Bo believes that Andrea is rational, Bo goes In, because $\beta^b(\text{home}|\text{In}) = 0$ implies $V_b^{\beta^b}(\text{In}) > 1 = W$.

Infinite games 1: continuity

- Suppose that $A \subseteq \mathbb{R}^n$ is *bounded*. Fix $\delta \in (0,1)$. For each $T \in \mathbb{N} \cup \{\infty\}$, endow A^T with the following "**discounting metric**":

$$d_T\left(\left(a^t\right)_{t=1}^T, \left(\bar{a}^t\right)_{t=1}^T\right) = \sum_{t=1}^{T} \delta^{t-1} d\left(a^t, \bar{a}^t\right)$$

(d is the metric in $\mathbb{R}^n$; by boundedness and $0 < \delta < 1$, d_T is a metric even if $T = \infty$). Thus, $\left(A^T, d_T\right)$ is a *metric space*. Let $Z_T := Z \cap A^T$ be the set of terminal histories of length T.

Definition

Game Γ is **compact-continuous** if Z_T is compact in metric space $\left(A^T, d_T\right)$ for each $T \in \mathbb{N} \cup \{\infty\}$ and u_i is continuous on Z_T for each $T \in \mathbb{N} \cup \{\infty\}$ and $i \in I$.

[A subset K of a metric space is compact if, for every cover of K with open sets, there is a finite sub-cover of K. For $T < \infty$, compact is equivalent to closed and bounded.]

Infinite games 2: Folding Back and One-Step Optimality

- We take folding-back (FB) optimality as our basic notion of rational planning. But, by definition, the *FB algorithm cannot be applied to infinite-horizon games.*
- If the game has *finite horizon*, but it is infinite (because some feasible actions set $\mathcal{A}_i(h)$ is infinite), then maximizations may be impossible (we will study a prominent example concerning bargaining).
- But the definitions (with sup) still apply (as written, if each $\beta^i(\cdot|h)$ has finite/countable support) and versions of the *FB, Optimality, and OD principles hold.*
- With this, we take the *one-step optimality* as our *general* characterization of *rational planning*.

Infinite games 3: OD principle

- The following result extends the OD principle (equivalence between one-step and sequential optimality) to compact-continuous games.

Theorem

(**Generalized OD principle**) *In every compact-continuous game the OD principle holds, that is, for every i, s_i, and β^i, strategy s_i is seq.lly optimal given conjecture β^i IFF s_i is one-step optimal given β^i.*

- **Intuition** (by *contraposition*): If s_i is not sequentially optimal given β^i in the compact-continuous game Γ, then we can find a finite-horizon approximation of Γ, viz. $\bar{\Gamma}$, such that the restriction of s_i to $\bar{\Gamma}$ is not sequentially optimal in $\bar{\Gamma}$ given (the restriction of) β^i; hence (by the OD principle for finite-horizon games), it fails one-step optimality in $\bar{\Gamma}$. Given that $\bar{\Gamma}$ is a sufficiently good approximation of Γ, s_i must fail one-step optimality (given β^i) in Γ. ♡

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