Bargaining with Alternating Offers

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Abstract

We present three models of sequential bargaining: the **Ultimatum Game**, **Two-Period Bargaining with Alternating Offers**, and **Infinite-Horizon Bargaining with Alternating Offers**. The first two models can be analyzed with backward induction (of a kind). The subgame perfect equilibrium (SPE) of the infinite-horizon model is obtained *heuristically* relying on the analysis of the two-period model, and then using the OD principle to show that the solution so obtained is indeed an SPE. It turns out that this is the *unique SPE.*

[These slides summarize Chapter 14 of GT-AST]

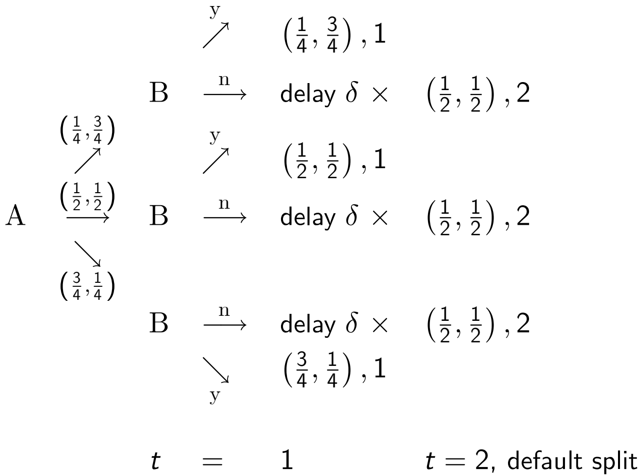
Introduction

- *Bargaining is ubiquitous in economic life:*
- Economic interaction (e.g., in production and/or exchange) generates a surplus, and often *agents can commit to surplus maximizing behavior and on how to split* the surplus.
- **Bargaining** is the process of offers, replies, and—possibly—counteroffers; this process may terminate, or not, with such an agreement.
- Focusing on the case of a fixed potential surplus, we study bargaining between *two agents* according to a precise **alternating-offers protocol** (agents discount future consumption):
  - An **offer** of how to split is made.
  - If the offer is *accepted* the split is implemented and agents consume immediately their shares. If it is *rejected*, the process *moves on to the next period*.
  - In the next period, *either* agents consume a *default* shares (exogenous end), *or* the *previous-period responder makes the next offer*.

Ultimatum Game: A Discrete Example

Example: size-1 surplus to split.

Game form of the Ultimatum Game with 0.25-increments



- Ann offers (x_A, x_B), Bob: y (immediate consumption), or n (delayed cons. of default shares). The BI-solution depends on δ.

Ultimatum Game with a Continuum of Offers: Model

- Set of **offers** $X = \left\{(x_{\mathrm{A}}, x_{\mathrm{B}}) \in \mathbb{R}_{+}^{\{\mathrm{A},\mathrm{B}\}} : x_{\mathrm{A}} + x_{\mathrm{B}} = 1\right\}$.
- **Default split** $(\bar{x}_{\mathrm{A}}, \bar{x}_{\mathrm{B}}) \in X$ in case of *disagreement*.
- Set of **outcomes** $Y = X \times \{1, 2\}$, (x, t)=split x in period t.
- **Period** $t = 1$:
  - Ann proposes $(x_{\mathrm{A}}, x_{\mathrm{B}}) \in X$;
  - Bob replies yes (y), or no (n);
  - if y, immediate consumption of $(x_{\mathrm{A}}, x_{\mathrm{B}})$ [outcome $((x_{\mathrm{A}}, x_{\mathrm{B}}), 1)$];
  - if n, go to period $t = 2$.
- **Period** $t = 2$: consumption of default split $(\bar{x}_{\mathrm{A}}, \bar{x}_{\mathrm{B}})$ [outcome $((\bar{x}_{\mathrm{A}}, \bar{x}_{\mathrm{B}}), 2)$].
- **Intertemporal preferences**: $((x_i, 1 - x_i), 2) \sim_i ((\delta x_i, 1 - \delta x_i), 1)$, with $\delta \in (0, 1)$ discount factor (common, just for simplicity).

- Bob:
  - accepts $(x_{\mathrm{A}}, x_{\mathrm{B}})$ if $x_{\mathrm{B}} > \delta \bar{x}_{\mathrm{B}}$,
  - rejects $(x_{\mathrm{A}}, x_{\mathrm{B}})$ if $x_{\mathrm{B}} < \delta \bar{x}_{\mathrm{B}}$,
  - indifferent if $x_{\mathrm{B}} = \delta \bar{x}_{\mathrm{B}}$, 2 cases: y and n.
- Ann:
  - **case**-y: Ann chooses $(x_{\mathrm{A}}, x_{\mathrm{B}})$ to maximize the value function

$$V_{\mathrm{A}}^{*}(x_{\mathrm{A}}, x_{\mathrm{B}}) = \begin{cases} x_{\mathrm{A}} & \text{if } x_{\mathrm{A}} \leq 1 - \delta \bar{x}_{\mathrm{B}} \\ \delta \bar{x}_{\mathrm{A}} & \text{if } x_{\mathrm{A}} > 1 - \delta \bar{x}_{\mathrm{B}} \end{cases}$$

    since $\delta \bar{x}_{\mathrm{A}} = \delta(1 - \bar{x}_{\mathrm{B}}) < 1 - \delta \bar{x}_{\mathrm{B}}$, Ann offers $(x_{\mathrm{A}}, x_{\mathrm{B}}) = (1 - \delta \bar{x}_{\mathrm{B}}, \delta \bar{x}_{\mathrm{B}})$.
  - **case**-n: the value function is as above with $\leq$ (resp. $>$) replaced by $<$ (resp. $\geq$); hence, no maximum! *No SPE in this case!*
- **Comments**:
  - *Non-existence* in **case**-n is just a technicality due to the continuum, it *disappears with discrete offers.*
  - *To solve by BI, break responder's indifference* (relevant tie) *with* yes.

Alternating Offers with 2 Periods of Bargaining

- Set of **Outcomes** $Y = X \times \{1, 2, 3\}$, $((x_i, 1 - x_i), t + 1) \sim_i ((\delta x_i, 1 - \delta x_i), t)$.
- **Period** $t = 1$:
  - Ann proposes $x^1 = (x^1_{\mathrm{A}}, x^1_{\mathrm{B}}) \in X$;
  - Bob replies yes (y), or no (n);
  - if y, immediate consumption of x^1 [outcome $(x^1, 1)$];
  - if n, go to period $t = 2$.
- **Period** $t = 2$:
  - *Bob* proposes $x^2 = (x^2_{\mathrm{A}}, x^2_{\mathrm{B}}) \in X$;
  - Ann replies yes (y), or no (n);
  - if y, immediate consumption of x^2 [outcome $(x^2, 2)$];
  - if n, go to period $t = 3$.
- **Period** $t = 3$: consumption of default split $\bar{x} = (\bar{x}_{\mathrm{A}}, \bar{x}_{\mathrm{B}})$ [outcome $(\bar{x}, 3)$].

Alternating Offers with 2 Periods: BI-Solution

- **Period** 2:
  - Ann accepts iff $x_{\mathrm{A}}^{2} \geq \delta \bar{x}_{\mathrm{A}}$;
  - Bob chooses $\left(x_{\mathrm{A}}^{2}, x_{\mathrm{B}}^{2}\right)$ to maximize

$$V_{\mathrm{B}}^{*}\left(\left(x_{\mathrm{A}}^{1}, x_{\mathrm{B}}^{1}\right), \mathrm{n}, \left(x_{\mathrm{A}}^{2}, x_{\mathrm{B}}^{2}\right)\right) = \begin{cases} x_{\mathrm{B}}^{2} & \text{if } x_{\mathrm{B}}^{2} \leq 1 - \delta \bar{x}_{\mathrm{A}} \\ \delta \bar{x}_{\mathrm{B}} & \text{if } x_{\mathrm{B}}^{2} > 1 - \delta \bar{x}_{\mathrm{A}} \end{cases}$$

    and, thus, offers $\left(\hat{x}_{\mathrm{A}}^{2}, \hat{x}_{\mathrm{B}}^{2}\right) = \left(\delta \bar{x}_{\mathrm{A}}, 1 - \delta \bar{x}_{\mathrm{A}}\right)$.
- **Period** 1:
  - Bob accepts iff $x_{\mathrm{B}}^{1} \geq \delta \hat{x}_{\mathrm{B}}^{2} = \delta\left(1 - \delta \bar{x}_{\mathrm{A}}\right)$, which is the present value of going to $t = 2$;
  - Ann chooses $\left(x_{\mathrm{A}}^{1}, x_{\mathrm{B}}^{1}\right)$ to maximize $V_{\mathrm{A}}^{*}\left(x_{\mathrm{A}}^{1}, x_{\mathrm{B}}^{1}\right)$; thus, she offers $\left(\hat{x}_{\mathrm{A}}^{1}, \hat{x}_{\mathrm{B}}^{1}\right) = \left(1 - \delta\left(1 - \delta \bar{x}_{\mathrm{A}}\right), \delta\left(1 - \delta \bar{x}_{\mathrm{A}}\right)\right)$.
- **Comments**
  - In the unique SPE *agreement is reached immediately*, due to impatience and complete information (knowledge of the stingiest acceptable offer).
  - *Impatience* (low δ) *yields a first-mover advantage*.

Comparison Between SPE and Rationalizability

- We defined strong rationalizability for *finite* multistage games. Thus, consider a *finite*, but *fine-grained grid of offers*:

$$X_k = \left\{ x \in X : x_{\mathrm{A}} \in \left\{ 0, \frac{1}{k}, ..., \frac{k-1}{k}, 1 \right\} \right\} \text{ (e.g., } k = 100).$$

  There may be *multiple SPEs if* responders may be indifferent (*relevant ties*), which depends of δ and k; but they are close to each other: *offers of different SPEs can differ by* $1/k$ *at most.*
- For *every SPE path* the *first offer is accepted* and the *responder accepts if indifferent.*
- If there are *no relevant ties* (NRT), there is a unique SPE obtained by BI with path $\hat{z} = \left(\left(\hat{x}^1_{\mathrm{A}}, \hat{x}^1_{\mathrm{B}} \right), \mathrm{y} \right)$, and *this is also the unique strongly rationalizable path* (because PI & NRT $\Rightarrow \zeta \left(S^{\infty} \right) = \{ \hat{z} \}$).
- If there are *relevant ties* and there is an SPE path $\left(\left(\hat{x}^1_{\mathrm{A}}, \hat{x}^1_{\mathrm{B}} \right), \mathrm{y} \right)$ so that Bob is *indifferent*, then strong *rationalizability allows* for *delay*, i.e., also $\left(\left(\hat{x}^1_{\mathrm{A}}, \hat{x}^1_{\mathrm{B}} \right), \mathrm{n}, ... \right)$, because *rationalizability does not require an indifferent responder to accept.*

Alternating Offers with Infinite Horizon: Model

- Set of **outcomes** $Y = \{d\} \cup (X \times \mathbb{N})$, d=**permanent disagreement**, $((x_{\mathrm{A}}, x_{\mathrm{B}}), t) = ((0,1), t) \sim_{\mathrm{A}} d$ (similarly for B).
- **Period** t odd ($t = 1, 3, ...$):
  - Ann proposes $x^t = (x^t_{\mathrm{A}}, x^t_{\mathrm{B}}) \in X$;
  - Bob replies yes (y), or no (n);
  - if y, immediate consumption of x^t [outcome (x^t, t)];
  - if n, go to period $t+1$ (even).
- **Period** $t+1$ even ($t+1 = 2, 4, ...$):
  - Bob proposes $x^{t+1} = \left(x^{t+1}_{\mathrm{A}}, x^{t+1}_{\mathrm{B}}\right) \in X$;
  - Ann replies yes (y), or no (n);
  - if y, immediate consumption of x^{t+1} [outcome $\left(x^{t+1}, t+1\right)$];
  - if n, go to period $t+2$ (odd).
- **Period** $t+2$ (odd): the bargaining protocol re-starts with Ann proposing (no default split in case of disagreement, keep on bargaining if n).

Alternating Offers with Infinite Horizon: Heuristic Solution

- The model is **stationary**: from every history h at which i has to propose, the "same" infinite-horizon bargaining game starts (similarly, from every history (h, x) at which i has to respond to x, the "same" infinite-horizon subgame starts).
- With this, it is reasonable to *look for a* **stationary** solution $(\hat{s}_{\mathrm{A}}, \hat{s}_{\mathrm{B}})$ such that, if i has to propose at h, s/he demands $\hat{x}_i$ for her/himself and the offer is accepted. (In this case, the value for the responder $-i$ of saying no is $\delta\hat{x}_{-i}$, since s/he is the next period proposer and expects to get $\hat{x}_{-i}$.)
- Thus, replace the default split $(\bar{x}_{\mathrm{A}}, 1 - \bar{x}_{\mathrm{A}})$ of the 2-period model with the yet *unknown* (to us) solution split $(\hat{x}_{\mathrm{A}}, 1 - \hat{x}_{\mathrm{A}})$.
- Obtain as a 2-period solution for A: $\xi_{\mathrm{A}}(\hat{x}_{\mathrm{A}}) := 1 - \delta(1 - \delta\hat{x}_{\mathrm{A}})$ and solve the fixed-point equation $\hat{x}_{\mathrm{A}} = \xi_{\mathrm{A}}(\hat{x}_{\mathrm{A}})$ to obtain $\hat{x}_{\mathrm{A}} = 1 - \delta(1 - \delta\hat{x}_{\mathrm{A}}) = 1 - \delta + \delta^2\hat{x}_{\mathrm{A}}$.
- **Candidate stationary offer**: $\hat{x}_{\mathrm{A}} = \frac{1-\delta}{1-\delta^2} = \frac{1-\delta}{(1-\delta)(1+\delta)} = \frac{1}{1+\delta}$.

Infinite-Horizon Model: Verification, Uniqueness

- The **candidate stationary solution** $(\hat{s}_{\mathrm{A}}, \hat{s}_{\mathrm{B}})$ is:
  - for each h at which A [respectively, B] *proposes*, $\hat{s}_{\mathrm{A}}(h) = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ [respectively, $\hat{s}_{\mathrm{B}}(h) = \left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\right)$];
  - for each $h' = (h, (x_{\mathrm{A}}, x_{\mathrm{B}}))$ at which A [respectively, B] *responds*, $\hat{s}_{\mathrm{A}}(h, (x_{\mathrm{A}}, x_{\mathrm{B}})) = \mathrm{y}$ if $x_{\mathrm{A}} \geq \frac{\delta}{1+\delta}$, and $\hat{s}_{\mathrm{A}}(h, (x_{\mathrm{A}}, x_{\mathrm{B}})) = \mathrm{n}$ otherwise [respectively, $\hat{s}_{\mathrm{B}}(h, (x_{\mathrm{A}}, x_{\mathrm{B}})) = \mathrm{y}$ if $x_{\mathrm{B}} \geq \frac{\delta}{1+\delta}$, and $\hat{s}_{\mathrm{B}}(h, (x_{\mathrm{A}}, x_{\mathrm{B}})) = \mathrm{n}$ otherwise].
- Is this an SPE? *Yes!* The *OD principle applies* to this game (it is compact-continuous). To prove that $(\hat{s}_{\mathrm{A}}, \hat{s}_{\mathrm{B}})$ is a SPE it is *enough to verify that each $\hat{s}_i$ is One-Step Optimal given $\hat{s}_{-i}$*, which is quite easy :-) [Do it!]
- **Proposition:** $(\hat{s}_{\mathrm{A}}, \hat{s}_{\mathrm{B}})$ *is the unique SPE.* [We skip the proof.]
- **Comment:** Differently from repeated games, the long-but-finite-horizon SPE-solution approximates the infinite-horizon SPE-solution.

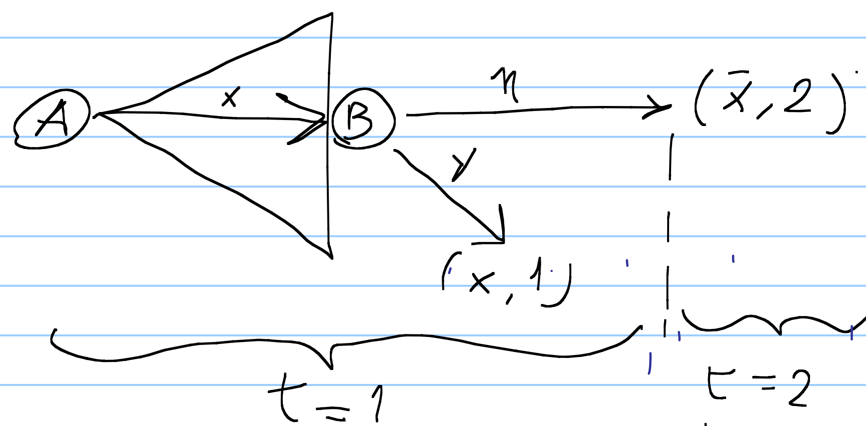
References

BATTIGALLI, P., E. CATONINI, AND N. DE VITO (2023): *Game Theory: Analysis of Strategic Thinking*. Typescript, Bocconi University.

BATTIGALLI, P. (2023): *Mathematical Language and Game Theory*. Typescript, Bocconi University.

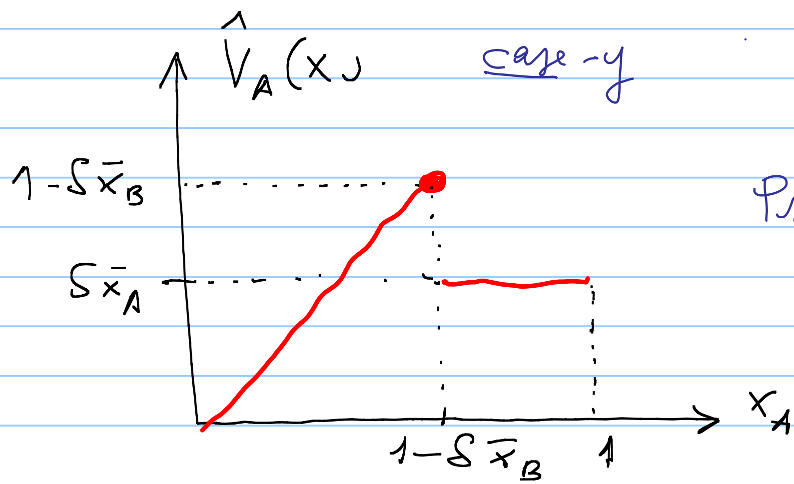
PICTURES BARGAINING

1-period model : Ultimatum Game (with "fan of offers")



Picture of the UG game form

default split $\bar{x} = (x_A, x_B)$ if Ann's offer is rejected

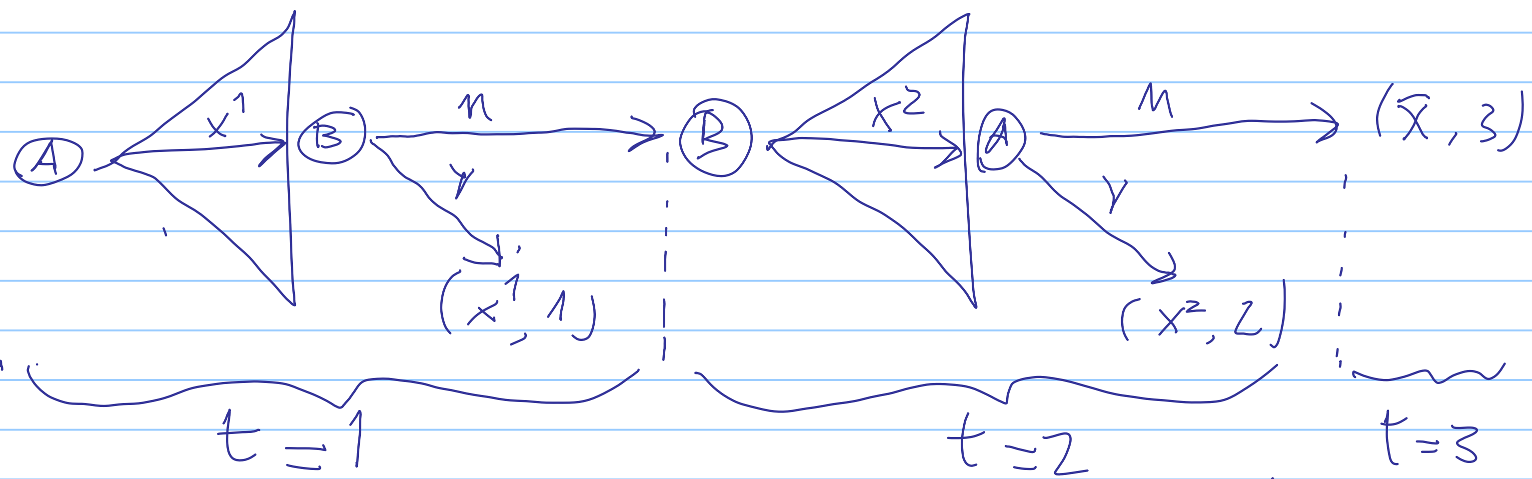


Problem of Ann if Bob is expected to accept when indifferent: "innocuous" discontinuity at indifference point

In case-n, "bad discontinuity"

2-period model

(stylized picture with "fans of offers"



default split $\bar{x} = (\bar{x}_A, \bar{x}_B)$ in case of disagreement

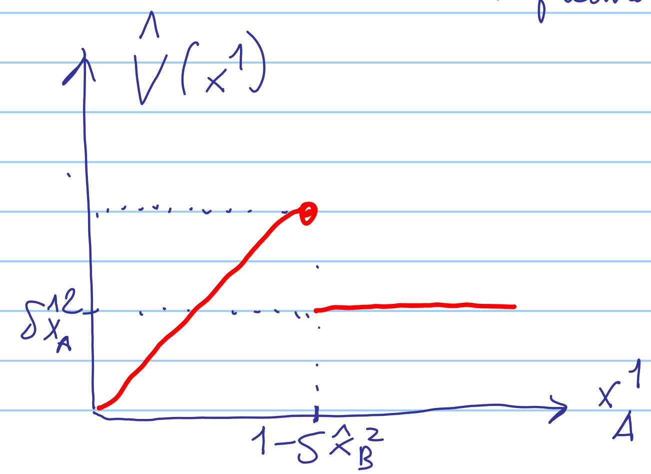
Solution of period 2:

$\hat{x}^2 = (\delta \bar{x}_A, 1 - \delta \bar{x}_A)$

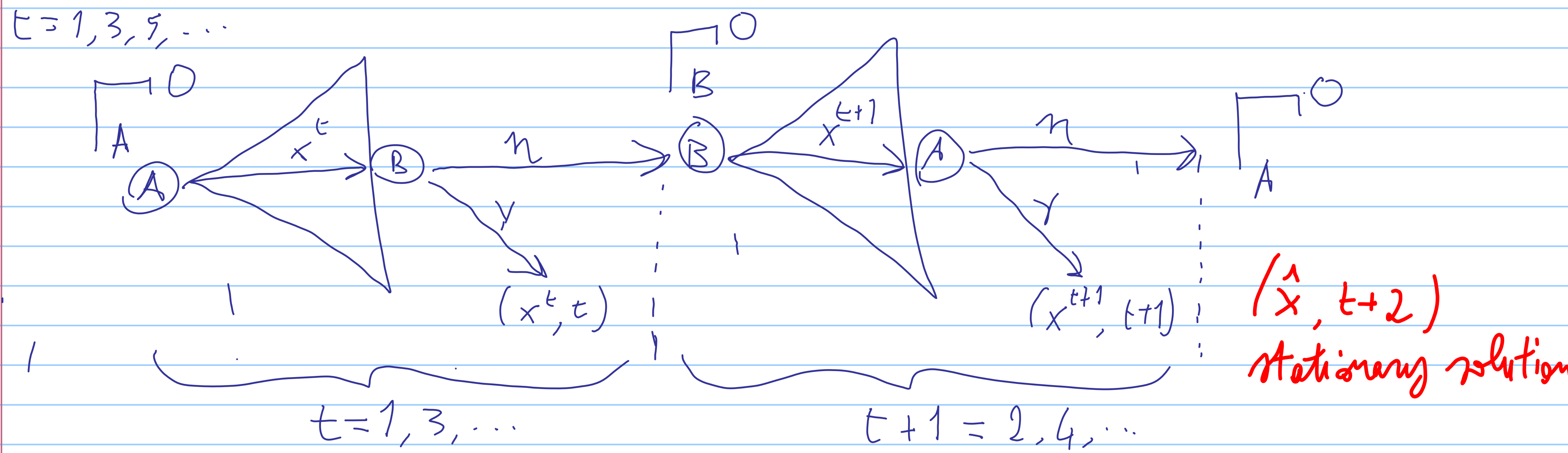
BI-problem of Ann in period 1

$\delta \hat{x}_B^2$ indifference for B

$\hat{x}_B^2 = 1 - \delta \bar{x}_A$



∞-horizon model (fragment of game form, with "fans")



After two rejections, the bargaining protocol starts all over again

to find a stationary SPE, posit a stationary solution with shares $\hat{x} = (\hat{x}_A, \hat{x}_B)$. Use it as if it were the default split of the 2-period model. Then solve $\hat{x}_A = 1 - \delta(1 - \delta \hat{x}_A)$