Multistage Games with Payoff Uncertainty: Rational Planning

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Game Theory: Analysis of Strategic Thinking

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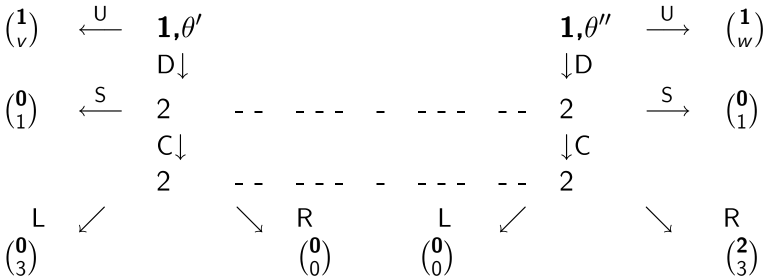
Abstract
This lecture extends the analysis of rational planning to multistage games with (observable actions and) payoff uncertainty.
[These slides summarize and, in part, complement Section 3 of Chapter 15 of GT-AST.]

Introduction

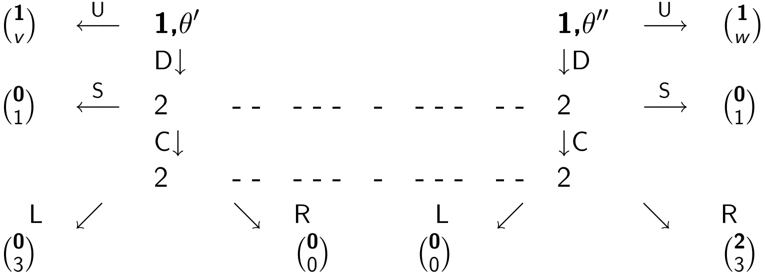
- We want to study rational planning in multistage games with observable actions and payoff uncertainty.
- With this aim, we extend our representation of i's beliefs about $-i$:
  - We start with conditional probability systems (CPSs) $\bar{\mu}^i \in \Delta^H(\Theta_{-i} \times S_{-i})$ over others' information types θ_{-i} and strategies (ways of behaving) s_{-i}, thus extending the analysis of beliefs used to study rationalizability in multistage games with complete information.
  - Next we derive pairs $\left(\beta^i, \mu_i\right)$ assigning conditional probabilities $\beta^i(a_{-i}|\theta_{-i}, h)$ to actions and conditional probabilities $\mu_i(\theta_{-i}|h)$ to types. [$\beta^i(\cdot|\theta_{-i}, h)$ is arbitrary if $\mu_i(\theta_{-i}|h) = 0$, but this is going to be innocuous.]
  - If $\left(\beta^i, \mu_i\right)$ is derived from a CPS, it must satisfy Bayes rule whenever possible and is called "**Bayes consistent** personal assessment".
- With this, we obtain results about *rational planning*.

Running Example: (Conditional) Beliefs, 1/2



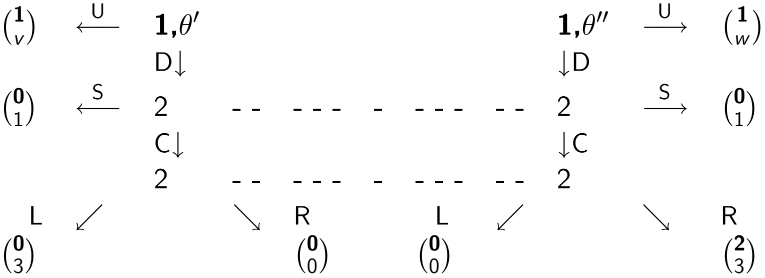
- Only player **1** (denoted in **bold**) is informed: $\Theta_1 \cong \Theta = \left\{\theta', \theta''\right\}$.
  - Payoffs v and w of player 2 do not matter. $H = \{\varnothing, (\mathrm{D}), (\mathrm{D}, \mathrm{C})\}$.
  - Consider CPS $\bar{\mu}^2 \in \Delta^H(\Theta \times S_1)$, with conditioning events $\Theta \times S_1(h)$ $(h \in H)$, where $S_1(\varnothing) = S_1 = \{\mathrm{U}, \mathrm{D}\}$,$S_1(\mathrm{D}) = S_1(\mathrm{D}, \mathrm{C}) = \{\mathrm{D}\}$ (C does not reveal anything about pl. 1).
  - **Abbreviations:** We often write $\bar{\mu}^2(\{(\theta, s_1)\} | \Theta \times S_1(h)) =: \bar{\mu}^2(\theta, s_1 | h)$, with $h = \varnothing$ omitted.

Running Example: (Conditional) Beliefs, 2/2



- Derive from CPS $\bar{\mu}^2$ a corresponding personal assessment $\left(\beta^2, \mu_2\right)$ to obtain a *subjective decision tree* for pl. 2:
  - $\mu_2(\theta) = \bar{\mu}^2(\{\theta\} \times S_1)$ (prior exogenous belief of pl. 2, here it does not matter). Assume $0 < \mu_2(\theta') < 1$.
  - $\beta^2(\mathrm{D}|\theta) = \bar{\mu}^2(\theta, \mathrm{D}) / \bar{\mu}^2(\{\theta\} \times S_1) = \bar{\mu}^2(\theta, \mathrm{D}) / \mu_2(\theta)$.
  - $\mu_2(\theta|\mathrm{D}) = \bar{\mu}^2(\{(\theta, \mathrm{D})\} | \Theta \times S_1(\mathrm{D})) =$
    $= \bar{\mu}^2(\{(\theta, \mathrm{D})\} | \Theta \times S_1(\mathrm{D}, \mathrm{C})) = \mu_2(\theta| (\mathrm{D}, \mathrm{C}))$.

Running Example: Rational Planning by Folding Back



- Here, only part μ_2 of 2's personal assessment $\left(\beta^2, \mu_2\right)$ matters.
  - Let $q := \mu_2\left(\theta'|\mathrm{D}\right) = \mu_2\left(\theta'|\left(\mathrm{D},\mathrm{C}\right)\right)$; with this, $q < \frac{1}{2} \Rightarrow \hat{s}_2\left(\mathrm{D},\mathrm{C}\right) = \mathrm{R}$, $q > \frac{1}{2} \Rightarrow \hat{s}_2\left(\mathrm{D},\mathrm{C}\right) = \mathrm{L}$, $q = \frac{1}{2} \Rightarrow$indiff.
  - $\hat{V}_2^q\left(\left(\mathrm{D,C}\right)\right) = \max\left\{3q, 3\left(1-q\right)\right\} \geq \frac{3}{2}$; thus, $\hat{s}_2\left(\mathrm{D}\right) = \mathrm{C}$ for every q, i.e., for every $\bar{\mu}^2 \in \Delta^H\left(\Theta \times S_1\right)$.
  - **Key:** $\mu_2\left(\theta'|\mathrm{D}\right) = \mu_2\left(\theta'|\left(\mathrm{D},\mathrm{C}\right)\right)$, otherwise there may be no sequentially optimal strategy!

Beliefs in Multistage Games with Payoff Uncertainty

- Fix a (finite) **multistage game with payoff uncertainty** and observable actions $\hat{\Gamma} = \left\langle I, (\Theta_i, A_i, \mathcal{A}_i(\cdot), u_i)_{i \in I} \right\rangle$.
- To represent *strategic thinking* as rationalizability:
  - We will merge elements of Ch. 8 (static games with incomplete information) and Ch. 11 (rationalizability in multistage games with complete information).
  - With this goal, beliefs are conveniently represented as CPSs $\bar{\mu}^i = \left(\bar{\mu}^i\left(\cdot|h\right)\right)_{h \in H} \in \Delta^H\left(\Theta_{-i} \times S_{-i}\right)$, recalling that, for all $h', h'' \in H$,

$$\begin{aligned} S_{-i}\left(h'\right) &= S_{-i}\left(h''\right) \Rightarrow \\ \bar{\mu}^i\left(\cdot|h'\right) &= \bar{\mu}^i\left(\cdot|\Theta_{-i} \times S_{-i}\left(h'\right)\right) = \bar{\mu}^i\left(\cdot|\Theta_{-i} \times S_{-i}\left(h''\right)\right) = \bar{\mu}^i\left(\cdot|h''\right). \end{aligned}$$

- To represent *rational planning* (and later, for equilibrium analysis):
  - it is convenient to work with personal assessments $\left(\beta^i, \mu_i\right)$ satisfying *Bayes consistency*,
  - which—essentially—follows if $\left(\beta^i, \mu_i\right)$ is derived from a CPS $\bar{\mu}^i$.

Conditional Probability Systems (CPSs)

- In the (rationalizability) analysis of static games with *incomplete* information, we considered conjectures $\mu^i \in \Delta(\Theta_{-i} \times A_{-i})$.
- In the (rationalizability) analysis of multistage games with *complete* information, we considered CPSs $\mu^i \in \Delta^H(S_{-i})$.
- In the (rationalizability) analysis of multistage games with *incomplete* information, we can use CPSs $\bar{\mu}^i \in \Delta^H(\Theta_{-i} \times S_{-i})$, where (as before) $S_{-i} = \times_{h \in H} \mathcal{A}_{-i}(h)$ are the co-players' pure strategies (we write $\bar{\mu}^i$ to distinguish from systems of beliefs $\mu_i \in (\Delta(\Theta_{-i}))^H$).
- We can *derive a personal assessment* (β^i, μ_i) from a CPS $\bar{\mu}^i$: for all $(\theta_{-i}, h) \in \Theta_{-i} \times H$ and $a_{-i} \in \mathcal{A}_{-i}(h)$, $\mu_i(\theta_{-i}|h) = \bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h)|h)$ and, if $\bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h)|h) > 0$, then

$$\beta^i(a_{-i}|\theta_{-i}, h) = \frac{\bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h, a_{-i})|h)}{\bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h)|h)}.$$

Bayes Consistency of Personal Assessments

- If $\left(\beta^i, \mu_i\right)$ is derived from a CPS $\bar{\mu}^i$, then it has to be **Bayes consistent:** For all $h \in H$, $a_{-i} \in \mathcal{A}_{-i}(h)$, θ_{-i}, write
  - $\mathbb{P}^{\beta^i}\left(a_{-i}|\theta_{-i}, h\right) := \beta^i\left(a_{-i}|\theta_{-i}, h\right)$, $\mathbb{P}^{\mu_i}\left(\theta_{-i}|h\right) := \mu_i\left(\theta_{-i}|h\right)$,
  - $\mathbb{P}^{\beta^i,\mu_i}\left(\theta_{-i}, a_{-i}|h\right) := \beta^i\left(a_{-i}|\theta_{-i}, h\right)\mu_i\left(\theta_{-i}|h\right)$,
  - $\mathbb{P}^{\beta^i,\mu_i}\left(a_{-i}|h\right) = \sum_{\theta'_{-i}} \mathbb{P}^{\beta^i,\mu_i}\left(\theta'_{-i}, a_{-i}|h\right) =$ $\sum_{\theta'_{-i}} \beta^i\left(a_{-i}|\theta'_{-i}, h\right)\mu_i\left(\theta'_{-i}|h\right)$.
  - If $\mathbb{P}^{\beta^i,\mu_i}\left(a_{-i}|h\right) > 0$, write $\mu_i\left(\theta_{-i}|h, a_{-i}\right) := \frac{\mathbb{P}^{\beta^i,\mu_i}(\theta_{-i}, a_{-i}|h)}{\mathbb{P}^{\beta^i,\mu_i}(a_{-i}|h)}$ $= \frac{\beta^i(a_{-i}|\theta_{-i},h)\mu_i(\theta_{-i}|h)}{\sum_{\theta'_{-i}} \beta^i\left(a_{-i}|\theta'_{-i},h\right)\mu_i\left(\theta'_{-i}|h\right)}$ (BR).
  - **Bayes consistency**: for all $h \in H$ s.t. $L\left(\hat{\Gamma}(h)\right) > 1$, $a_i \in \mathcal{A}_i(h)$, $a_{-i} \in \mathcal{A}_{-i}(h)$, and θ_{-i}

$$\mu_i\left(\theta_{-i}|h, (a_i, a_{-i})\right) = \mu_i\left(\theta_{-i}|h, a_{-i}\right),$$

    where $\mu_i\left(\theta_{-i}|h, a_{-i}\right)$ satisfies (BR) whenever possible. (Hence, $\mu_i\left(\cdot|h, (a_i, a_{-i})\right)$ is independent of own-action a_i.)
  - If i is the only active player at h, $\mu_i\left(\theta_{-i}|h, a_i\right) = \mu_i\left(\theta_{-i}|h\right)$.

One-Step and Sequential Optimality

- Fix $\left(\beta^{i},\mu_{i}\right)$, θ_{i} and $\beta_{i}\in\times_{h\in H}\Delta\left(\mathcal{A}_{i}\left(h\right)\right)$.
  - For all $h\in H$, $z\in Z\left(h\right)$, $a_{i}\in\mathcal{A}_{i}\left(h\right)$, $a_{-i}\in\mathcal{A}_{-i}\left(h\right)$, θ_{-i} let
  - $\mathbb{P}^{\beta_{i},\beta^{i}}\left(z|\theta_{-i},h\right)$=prob. of z conditional on h given θ_{-i},
  - $V_{\theta_{i}}^{\beta_{i},\beta^{i}}\left(\theta_{-i},h\right)=\sum_{z\in Z(h)}u_{i}\left(\theta_{i},\theta_{-i},z\right)\mathbb{P}^{\beta_{i},\beta^{i}}\left(z|\theta_{-i},h\right)$,
  - $V_{\theta_{i}}^{\beta_{i},\beta^{i},\mu_{i}}\left(h\right)=\sum_{\theta'_{-i}}V_{\theta_{i}}^{\beta_{i},\beta^{i}}\left(\theta'_{-i},h\right)\mu_{i}\left(\theta'_{-i}|h\right)$,
  - $V_{\theta_{i}}^{\beta_{i},\beta^{i},\mu_{i}}\left(h,a_{i}\right)=$ $\sum_{\theta'_{-i},a'_{-i}}V_{\theta_{i}}^{\beta_{i},\beta^{i}}\left(\theta'_{-i},\left(h,\left(a_{i},a'_{-i}\right)\right)\right)\beta^{i}\left(a'_{-i}|\theta'_{-i},h\right)\mu_{i}\left(\theta'_{-i}|h\right)$.

Definition

Behavior strategy β_{i} is **one-step optimal** given $\left(\beta^{i},\mu_{i}\right)$ if, for all $h\in H$, $\mathrm{supp}\beta_{i}\left(\cdot|h\right)\subseteq\arg\max_{a_{i}\in\mathcal{A}_{i}(h)}V_{\theta_{i}}^{\beta_{i},\beta^{i},\mu_{i}}\left(h,a_{i}\right)$;
β_{i} is **sequentially optimal** given $\left(\beta^{i},\mu_{i}\right)$ if, for all $h\in H$, $V_{\theta_{i}}^{\beta_{i},\beta^{i},\mu_{i}}\left(h\right)=\max_{s_{i}\in S_{i}(h)}V_{\theta_{i}}^{s_{i},\beta^{i},\mu_{i}}\left(h\right)$.

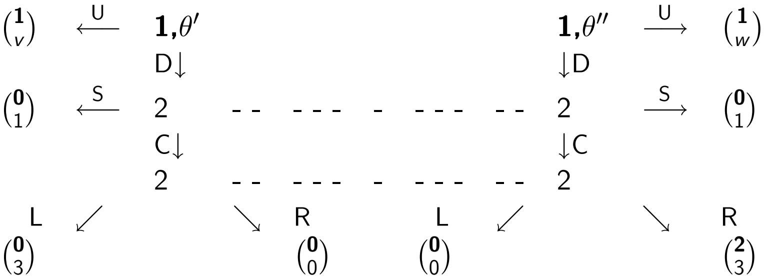
The One-Deviation Principle

- The results about rational planning can be extended to allow for incomplete information (payoff uncertainty). In particular, one can prove a version of the OD Principle:

Theorem

For all [behavior] strategies s_i [β_i] and **Bayes consistent** *personal assessments $\left(\beta^i, \mu_i\right)$, s_i [β_i] is one-step optimal given $\left(\beta^i, \mu_i\right)$ IFF it is sequentially optimal given $\left(\beta^i, \mu_i\right)$.*

- The proof is similar to the complete-information case. The novelty is that we also need a system of beliefs $\mu_i \in \left(\Delta\left(\Theta_{-i}\right)\right)^H$ and that the personal assessment $\left(\beta^i, \mu_i\right)$ has to be Bayes consistent.



- If $\left(\beta^2, \mu_2\right)$ is derived from a CPS, then it is Bayes consistent, $\mu_2\left(\theta'|\mathrm{D}\right) = \mu_2\left(\theta'|\left(\mathrm{D},\mathrm{C}\right)\right)$, one-step optimality is equivalent to sequential optimality, and the optimal strategies select C if D.
- Suppose $\left(\beta^2, \mu_2\right)$ is *not* derived from a CPS and

$$\mu_2\left(\theta'|\mathrm{D}\right) < \frac{1}{3},\ \mu_2\left(\theta'|\left(\mathrm{D},\mathrm{C}\right)\right) > \frac{1}{2}.$$

Then, one-step optimality yields L if $\left(\mathrm{D},\mathrm{C}\right)$ and S if D.

Conditional Dominance

- We can extend the definition of conditional dominance to this incomplete-information environment.
- Write: $U_i(\theta, s) := u_i(\theta, \zeta(s))$, and $U_i(\theta, \sigma_i, s_{-i}) = \mathbb{E}_{\sigma_i}(U_i(\theta, \cdot, s_{-i}))$ for $\sigma_i \in \Delta(S_i)$.

Definition

Strategy s_i is **conditionally dominated for type** θ_i if there are $h \in H_i(s_i)$ and $\sigma_i \in \Delta(S_i(h))$ such that

$$\forall \theta_{-i}, \forall s_{-i} \in S_{-i}(h), U_i(\theta_i, \theta_{-i}, s_i, s_{-i}) < U_i(\theta_i, \theta_{-i}, \sigma_i, s_{-i}).$$

- **Exercise:** Show that (reduced) strategy S of the running example is conditionally dominated.

Justifiability and Conditional Dominance

- As for the complete-information case, we use notions of optimality and justifiability that are invariant w.r.t. behavioral equivalence:

Definition

A strategy $\bar{s}_i$ is **weakly sequentially optimal for type** θ_i given $\left(\beta^i, \mu_i\right)$, written $\bar{s}_i \in r_i\left(\theta_i, \beta^i, \mu_i\right)$, if $V_{\theta_i}^{\bar{s}_i, \beta^i, \mu_i}(h) = \max_{s_i \in S_i(h)} V_{\theta_i}^{s_i, \beta^i, \mu_i}(h)$ for all $h \in H_i(\bar{s}_i)$; $\bar{s}_i$ is **justifiable for type** θ_i if $\bar{s}_i \in r_i\left(\theta_i, \beta^i, \mu_i\right)$ for some **Bayes consistent** $\left(\beta^i, \mu_i\right)$.

- **Remark** *If* $\bar{s}_i \in r_i\left(\theta_i, \beta^i, \mu_i\right)$ *and* s_i *is behaviorally equivalent to* $\bar{s}_i$ *then* $s_i \in r_i\left(\theta_i, \beta^i, \mu_i\right)$. *Hence,* $\bar{s}_i$ *is justifiable for* θ_i *IFF every behaviorally equivalent* s_i *is justifiable for* θ_i.

Lemma

For every $s_i \in S_i$ *and* $\theta_i \in \Theta_i$, s_i *is justifiable for* θ_i *IFF it is not conditionally dominated for* θ_i.

References

- Battigalli, P., E. Catonini, and N. De Vito (2023): *Game Theory: Analysis of Strategic Thinking*. Typescript, Bocconi University.
- Battigalli, P. (2023): *Mathematical Language and Game Theory*. Typescript, Bocconi University.