Rationalizability in Multistage Games with Payoff Uncertainty

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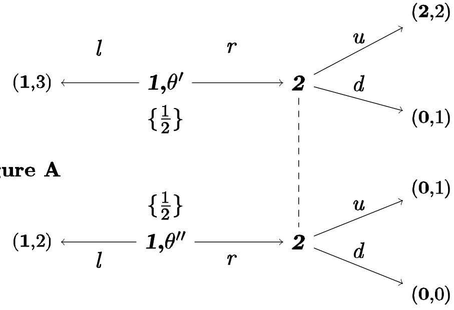
Abstract
This lecture extends the analysis of rationalizability from static games with payoff uncertainty and from multistage games with complete information to multistage games with payoff uncertainty.
[These slides summarize Section 4 of Chapter 15 of GT-AST.]

Introduction

- We want to study the behavioral implications of different versions of the idea of rationality and common belief in rationality in the context of multistage games with observable actions and payoff uncertainty. Such implications are characterized by versions of the rationalizability idea.
- With this aim, we represent beliefs about co-players as conditional probability systems (CPSs) $\bar{\mu}^i \in \Delta^H(\Theta_{-i} \times S_{-i})$ over their information types θ_{-i} and strategies (ways of behaving) s_{-i}, thus extending the analysis of beliefs used to study rationalizability in multistage games with complete information.
- Recall that it is possible to derive from a CPS $\bar{\mu}^i$ all the essential features of a Bayes consistent personal assessment (β^i, μ_i) (that is, everything but the conditional probabilities $\beta^i(a_{-i}|\theta_{-i}, h)$ for pairs (θ_{-i}, h) such that $\mu_i(\theta_{-i}|h) = 0$, which are irrelevant for expected utility calculations) $\Rightarrow$ connection with the analysis of rational planning.



- [The exogenous prior $\mu_2\left(\theta'\right)=\frac{1}{2}$ here is irrelevant.]
  - R: ℓ is dominant for θ'' (not for θ'), u is conditionally dominant given r. Delete $\left(\theta'', r\right)$ and d.
  - $R\cap \mathrm{B}_{\varnothing}\left(R\right)$: If pl. 1 of type θ' initially believes in 2's rationality, the best choice is r. Delete $\left(\theta', \ell\right)$.
  - **Solution** (initial rationalizability): r if θ', ℓ if θ'', u.

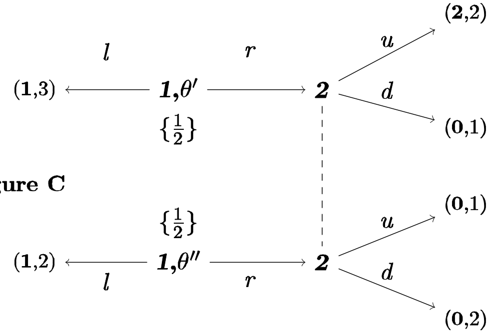


Figure C

- Here u is *not* conditionally dominant. [Prior: irrelevant.]
  - $R_1 \Rightarrow$ delete (θ'', r) [ℓ is dominant for θ'', not for θ'].
  - $\mathrm{SB}_2(R_1) \Rightarrow \bar{\mu}^2((\theta', r)\,|r) = 1 \Rightarrow$ b.r. is u. Delete d.
  - $\mathrm{SB}_1(R_2 \cap \mathrm{SB}_2(R_1)) \Rightarrow \bar{\mu}^1(u) = 1 \Rightarrow$ b.r. for θ' is r. Delete (θ', ℓ).
  - **Solution** (strong rationalizability): r if θ', ℓ if θ'', u. [Initial rationalizability only deletes (θ'', r). Why?]

Conditional Beliefs (rehearsal)

- Fix a (*finite*) **multistage game with payoff uncertainty** and observable actions $\hat{\Gamma} = \left\langle I, (\Theta_i, A_i, \mathcal{A}_i(\cdot), u_i)_{i\in I} \right\rangle$.
- We represent how players update/revise beliefs as the play unfolds with CPSs $\bar{\mu}^i \in \Delta^H (\Theta_{-i} \times S_{-i})$.
- We can derive from CPS $\bar{\mu}^i$ all the relevant elements of a corresponding Bayes consistent personal assessment $\left(\beta^i, \mu_i\right)$: recalling that $S_{-i}(h') = S_{-i}(h'') \Rightarrow \bar{\mu}^i(\cdot|h') = \bar{\mu}^i(\cdot|h'')$,
  - $\mu_i (\theta_{-i}|h) = \bar{\mu}^i (\{\theta_{-i}\} \times S_{-i}(h)\,|h)$,
  - if $\bar{\mu}^i (\{\theta_{-i}\} \times S_{-i}(h)\,|h) > 0$, $\beta^i (a_{-i}|\theta_{-i}, h) = \frac{\bar{\mu}^i(\{\theta_{-i}\}\times S_{-i}(h,a_{-i})|h)}{\bar{\mu}^i(\{\theta_{-i}\}\times S_{-i}(h)|h)}$,
  - if $\mu_i (\theta_{-i}|h) = 0$, $\beta^i (a_{-i}|\theta_{-i}, h)$ does not matter for EU calculations.
  - Write $r_i (\theta_i, \bar{\mu}^i) = r_i \left(\theta_i, \beta^i, \mu_i\right)$=set of weakly sequentially optimal strategies for θ_i given $\bar{\mu}^i$.

Conditional Dominance (rehearsal)

- Recall: we are interested in notions of optimality, justifiability, and dominance that are invariant to behavioral/realization equivalence, that is, notions that also apply to *reduced* strategies. Let us start with dominance.
- Write: $U_i(\theta, s) := u_i(\theta, \zeta(s))$, and $U_i(\theta, \sigma_i, s_{-i}) = \mathbb{E}_{\sigma_i}(U_i(\theta, \cdot, s_{-i}))$ for $\sigma_i \in \Delta(S_i)$.

Definition

Strategy s_i is **conditionally dominated for type** θ_i if there are $h \in H_i(s_i)$ and $\sigma_i \in \Delta(S_i(h))$ such that

$$\forall \theta_{-i}, \forall s_{-i} \in S_{-i}(h), U_i(\theta_i, \theta_{-i}, s_i, s_{-i}) < U_i(\theta_i, \theta_{-i}, \sigma_i, s_{-i}).$$

- **Exercise:** Find the conditionally dominated (reduced) strategies of the examples analyzed in this and previous lectures on multistage games with payoff uncertainty.

Justifiability and Conditional Dominance (rehearsal)

- We relate notions of dominance and optimality/justifiability that are invariant w.r.t. behavioral equivalence. Write values as $V_{\theta_i}^{s_i,\bar{\mu}^i}(h) = V_{\theta_i}^{\bar{s}_i,\beta^i,\mu_i}(h)$ where (β^i, μ_i) is derived from $\bar{\mu}^i$:

Definition

A strategy $\bar{s}_i$ is **weakly sequentially optimal for type** θ_i given CPS $\bar{\mu}^i$, written $\bar{s}_i \in r_i(\theta_i, \bar{\mu}^i)$, if $V_{\theta_i}^{\bar{s}_i,\bar{\mu}^i}(h) = \max_{s_i \in S_i(h)} V_{\theta_i}^{s_i,\bar{\mu}^i}(h)$ for all $h \in H_i(\bar{s}_i)$; $\bar{s}_i$ is **justifiable for type** θ_i if $\bar{s}_i \in r_i(\theta_i, \bar{\mu}^i)$ for some CPS $\bar{\mu}^i \in \Delta^H(\Theta_{-i} \times S_{-i})$.

- **Remark** *If $\bar{s}_i \in r_i(\theta_i, \bar{\mu}^i)$ and s_i is behaviorally equivalent to $\bar{s}_i$ then $s_i \in r_i(\theta_i, \bar{\mu}^i)$.*

Lemma

For every $s_i \in S_i$ and $\theta_i \in \Theta_i$, s_i is justifiable for θ_i IFF it is not conditionally dominated for θ_i.

Initial Rationalization (Monotone) Operator

- We want to characterize the (type-dependent) behavioral implications of Rationality and Common Initial Belief in Rationality (RCIBR), by iteratively deleting pairs (θ_i, s_i) for each player i.
- Let $\mathcal{C}$ be the collection of Cartesian subsets $C = \times_{i\in I} C_i$ with $C_i \subseteq \Theta_i \times S_i$ (with $\mathrm{proj}_{\Theta_i} C_i = \Theta_i$, because types cannot be deleted as such).
- Let $\Delta^H_\varnothing (C_{-i}) := \left\{\bar{\mu}^i \in \Delta^H (\Theta_{-i} \times S_{-i}) : \bar{\mu}^i (C_{-i}|\varnothing) = 1\right\}$ denote the set of CPSs that *initially* assign probability 1 to C_{-i}.
- Define the **initial rationalization** (monotone) **operator** $\rho : \mathcal{C} \to \mathcal{C}$ as follows: for every $C \in \mathcal{C}$,

$$\rho (C) = \times_{i\in I} \left\{(\theta_i, s_i) : \exists \bar{\mu}^i \in \Delta^H_\varnothing (C_{-i}), s_i \in r_i \left(\theta_i, \bar{\mu}^i\right)\right\}.$$

Initial Rationalizability

[abbreviation: $\Theta \times S := \times_{i \in I} (\Theta_i \times S_i)$]	
Assumptions on Rationality & Interactive Beliefs	Behavioral Implications
R	$\rho(\Theta \times S)$ (justifiability)
$R \cap \mathrm{B}_{\varnothing}(R)$	$\rho^2(\Theta \times S)$
...	...
$R \cap \bigcap_{k=1}^{m} \mathrm{B}_{\varnothing}^{k}(R)$	$\rho^{m+1}(\Theta \times S)$
...	...
$RCIBR := R \cap \bigcap_{k=1}^{\infty} \mathrm{B}_{\varnothing}^{k}(R)$	$\rho^{\infty}(\Theta \times S)$

Definition

A profile of types and strategies $(\theta_i, s_i)_{i \in I}$ is **initially rationalizable** if $(\theta_i, s_i)_{i \in I} \in \rho^{\infty}(\Theta \times S)$.

Initial Rationalizability and Dominance

- For each $C \in \mathcal{C}$, let $\mathcal{N}\left(\hat{\Gamma}|_C\right) = \langle I, (U_i|_C, C_i)_{i \in I} \rangle$ denote the normal (or strategic) form of $\hat{\Gamma}$ restricted to C [where $U_i(\theta, s) = u_i(\theta, \zeta(s))$].
- The initially rationalizable profiles can be obtained as follows:
  - First, for each $i \in I$, delete all pairs (θ_i, s_i) such that s_i is conditionally dominated for θ_i, thus obtaining the set $C^1 = NCD = \times_{i \in I} NCD_i$. [Justifiability⇔"Conditional Undominance"]
  - Next, *iteratively delete*, for each k, all pairs $(\theta_i, s_i) \in C^k$ such that s_i is dominated for θ_i in the residual strategic form $\mathcal{N}\left(\hat{\Gamma}|_{C^k}\right)$:

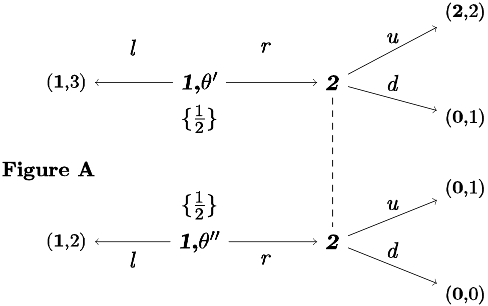
$$\forall (\theta_{-i}, s_{-i}) \in C^k_{-i}, U_i(\theta_i, \theta_{-i}, s_i, s_{-i}) < U_i(\theta_i, \theta_{-i}, \sigma_i, s_{-i}),$$

    for some $\sigma_i \in \Delta\left(\left\{s_i \in S_i : (\theta_i, s_i) \in C^k_i\right\}\right)$.

Theorem

$\rho(\Theta \times S) = C^1 = NCD$ and $\rho^k(\Theta \times S) = C^k$ for all $k \in \mathbb{N} \cup \{\infty\}$.

Example 1: Formal Analysis of Initial Rationalizability



1. $\rho^1 (\Theta \times S) = \left\{ (\theta', \ell), (\theta', r), (\theta'', \ell) \right\} \times \{u\}$,
2. $\rho^2 (\Theta \times S) = \left\{ (\theta', r), (\theta'', \ell) \right\} \times \{u\}$ END.

Strong Rationalizability: Definition

- As we did for the complete-information case, we want to capture the *best rationalization principle*. Let $\Delta^H_{\mathrm{sb}}(C_{-i})$ denote the set of CPSs of i that **strongly believe** $C_{-i} \subseteq \Theta_{-i} \times S_{-i}$, that is,

$$\left\{ \bar{\mu}^i : \forall h \in H, (\Theta_{-i} \times S_{-i}(h)) \cap C_{-i} \neq \emptyset \Rightarrow \bar{\mu}^i(C_{-i}|h) = 1 \right\}.$$

- We define the sequence $(C^n_{\mathrm{sb}})_{n=1}^{\infty}$ as follows:
  - $C^1_{\mathrm{sb}} = \rho(\Theta \times S)$ is just the set justifiable profiles,
  - given the first n subsets $(C^m_{\mathrm{sb}})_{m=1}^n$,

$$C^{n+1}_{\mathrm{sb}} = \times_{i\in I} \left\{ (\theta_i, s_i) : \exists \bar{\mu}^i \in \bigcap_{m=1}^{n} \Delta^H_{\mathrm{sb}}\left(C^m_{-i}\right), s_i \in r_i\left(\theta_i, \bar{\mu}^i\right) \right\}.$$

Definition

A profile of types and strategies $(\theta_i, s_i)_{i\in I}$ is **strongly rationalizable** if $(\theta_i, s_i)_{i\in I} \in C^{\infty}_{\mathrm{sb}} = \cap_n C^n_{\mathrm{sb}}$.

Strong Rationalizability: Interpretation

[Abbreviation: $\Theta \times S := \times_{i \in I} (\Theta_i \times S_i)$]	
Assumptions on Rationality & Interactive Beliefs	Behavioral Implications
R	C^1_{sb} (justifiability)
$R \cap \text{SB}(R)$	C^2_{sb}
$R \cap \text{SB}(R) \cap \text{SB}(R \cap \text{SB}(R))$	C^3_{sb}
...	...
$RCSBR$ (Rat & Common Strong Belief in Rat)	C^∞_{sb}

- **Preliminaries** *for the next frame*: Fix $i \in I$, $\theta_i \in \Theta_i$, and $C \in \mathcal{C}$. Let $C_j(h) := C_j \cap (\Theta_j \times S_j(h))$ for $j = i$, $j = -i$. Let $H(C) := \{h \in H : C_i(h) \times C_{-i}(h) \neq \emptyset\}$. Finally, let $C_{i,\theta_i} := \{s_i : (\theta_i, s_i) \in C_i\}$ and $C_{i,\theta_i}(h) := \{s_i : (\theta_i, s_i) \in C_i(h)\}$ respectively denote the sections of C_i and $C_i(h)$ at θ_i.

Strong Rationalizability and Iterated Cond. Dominance

- Recall: $C_{i,\theta_i} \subseteq S_i$ and $C_{i,\theta_i}(h) \subseteq S_i(h)$ are, respectively, the sections of $C_i \subseteq \Theta_i \times S_i$ and $C_{i,\theta_i}(h) \subseteq \Theta_i \times S_i(h)$ at θ_i.

Definition

Strategy $\bar{s}_i \in C_{i,\theta_i}$ is **conditionally dominated in C for type θ_i** if there are $h \in H_i(\bar{s}_i) \cap H(C)$ and $\sigma_i \in \Delta(C_{i,\theta_i}(h))$, such that

$$\forall (\theta_{-i}, s_{-i}) \in C_{-i}(h),\ U_i(\theta_i, \theta_{-i}, \bar{s}_i, s_{-i}) < U_i(\theta_i, \theta_{-i}, \sigma_i, s_{-i}).$$

We say that $\bar{s}_i \in C_{i,\theta_i}$ is **conditionally undominated in C for type θ_i** if it is *not* conditionally dominated in C. The set of pairs (θ_i, s_i) that are *conditionally undominated in C* is denoted by $\mathrm{NCD}_i(C)$ and $\mathrm{NCD}(C) = \times_{i \in I} \mathrm{NCD}_i(C)$.

Theorem

$C^n_{\mathrm{sb}} = \mathrm{NCD}^n(\Theta \times S)$ *for all* $n \in \mathbb{N} \cup \{\infty\}$.

Example 2: Formal Analysis of Strong Rationalizability

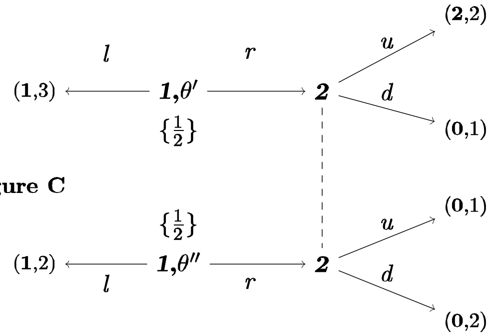
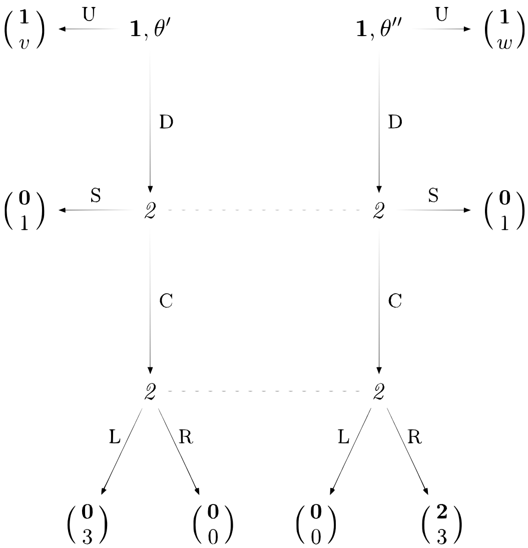


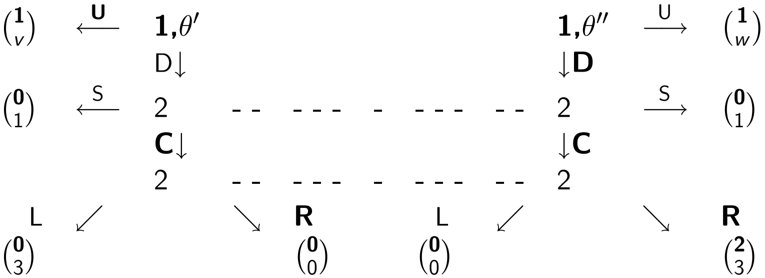
Figure C

1. $C^1_{\mathrm{sb}} = \left\{ \left(\theta', \ell\right), \left(\theta', r\right), \left(\theta'', \ell\right) \right\} \times \{u, d\}$,
2. $C^2_{\mathrm{sb}} = \left\{ \left(\theta', \ell\right), \left(\theta', r\right), \left(\theta'', \ell\right) \right\} \times \{u\}$,
3. $C^3_{\mathrm{sb}} = \left\{ \left(\theta', r\right), \left(\theta'', \ell\right) \right\} \times \{u\}$ END.

Example 3: A 3-Stage Game



Example 3: Strong Rationalizability in a 3-Stage Game



- Here $C_{\mathrm{sb}}^{\infty} = C_{\mathrm{sb}}^{3} = \left\{ \left(\theta', \mathrm{U}\right), \left(\theta'', \mathrm{D}\right) \right\} \times \{\mathrm{C.R}\}$:
  1. U dominates D for θ': delete $\left(\theta', \mathrm{D}\right)$. Also, delete S, conditionally dominated by $\frac{1}{2}\delta_{\mathrm{C.L}} + \frac{1}{2}\delta_{\mathrm{C.R}}$ given D.
  2. By strong belief in rationality, $\bar{\mu}^2 \left(\theta'' | \mathrm{D}\right) = 1$, hence delete $\mathrm{C.L}$.
  3. Pl. 1 is certain of $\mathrm{C.R}$, hence D if θ''.

- Recall:
  - $C_{i,\theta_i} = \{s_i : (\theta_i, s_i) \in C_i\}$ denotes the section of $C_i \subseteq \Theta_i \times S_i$ at θ_i;
  - $(\theta, s) \mapsto u_i(\theta, \zeta(s)) =: U_i(\theta, s)$ denotes the parameterized normal-(i.e., strategic-)form payoff function of i.

Definition

Fix a nonempty $C \in \mathcal{C}$ and $(\bar{s}_i, \theta_i) \in C_i$. Strategy $\bar{s}_i$ is **weakly dominated for type θ_i in** C if there is a mixed strategy $\sigma_i \in \Delta(C_{i,\theta_i})$ such that

$$\begin{aligned} \forall (\theta_{-i}, s_{-i}) &\in C_{-i}, U_i(\theta_i, \theta_{-i}, \bar{s}_i, s_{-i}) \leq U_i(\theta_i, \theta_{-i}, \sigma_i, s_{-i}) \text{ and} \\ \exists (\bar{\theta}_{-i}, \bar{s}_{-i}) &\in C_{-i}, U_i(\theta_i, \bar{\theta}_{-i}, \bar{s}_i, \bar{s}_{-i}) < U_i(\theta_i, \bar{\theta}_{-i}, \sigma_i, \bar{s}_{-i}); \end{aligned}$$

$\bar{s}_i$ is **admissible for** θ_i in C if it is not weakly dominated for θ_i in C.

Iterated Admissibility: Generic Equivalence

- Let $\mathrm{NWD}(C)$ denote the set of profiles $(\theta_i, s_i)_{i\in I} \in C$ such that s_i is not weakly dominated for θ_i in C, and $NWD := \mathrm{NWD}(\Theta \times S)$. Note: appending to $\left\langle I, (\Theta_i, A_i, \mathcal{A}_i(\cdot))_{i\in I}\right\rangle$ a profile of parameterized payoff functions $(u_i)_{i\in I} \in \mathbb{R}^{\Theta\times Z\times I}$, we obtain a game with payoff uncertainty.

Lemma

Fix a finite structure $\left\langle I, (\Theta_i, A_i, \mathcal{A}_i(\cdot))_{i\in I}\right\rangle$ and a nonempty $C \in \mathcal{C}$. For all payoff function profiles $u = (u_i : \Theta \times Z \to \mathbb{R})_{i\in I} \in \mathbb{R}^{\Theta\times Z\times I}$ except at most a negligible set, $\mathrm{NCD}(C) = \mathrm{NWD}(C)$.

Theorem

Fix a finite structure $\left\langle I, (\Theta_i, A_i, \mathcal{A}_i(\cdot))_{i\in I}\right\rangle$. For all profiles $u \in \mathbb{R}^{\Theta\times Z\times I}$ except at most a negligible set,

$$\forall n \in \mathbb{N}, C^n_{\mathrm{sb}} = \mathrm{NWD}^n(\Theta \times S), \ \rho^n(\Theta \times S) = \mathrm{ND}^{n-1}(NWD).$$

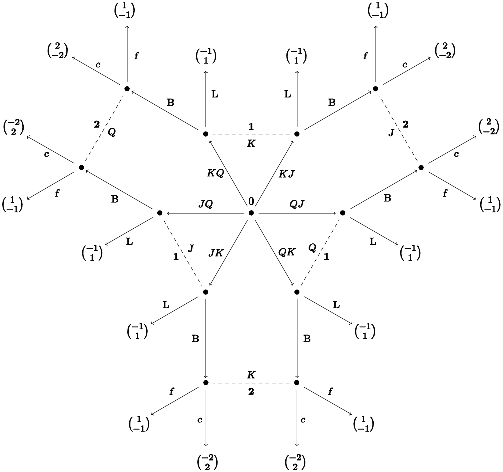
Exercise

- For all the foregoing games $\hat{\Gamma}$ with payoff uncertainty,
  - derive the normal form $\mathcal{N}\left(\hat{\Gamma}\right)$ (it is given by a pair of payoff matrices, labelled θ' and θ'', with the understanding that only pl. 1 knows θ);
  - verify that initial rationalizability coincides with $\left(\mathrm{ND}^{n-1}\left(NWD\right)\right)_{n\in\mathbb{N}}$;
  - verify that strong rationalizability coincides with $\left(\mathrm{NWD}^{n}\left(\Theta\times S\right)\right)_{n\in\mathbb{N}}$.

Directed Rationalizability (Optional)

- In some situations of strategic interaction, the context makes some features of players' beliefs "transparent."
- For example, in the **MiniPoker** game (see previous lectures) it is *true and commonly believed to be true* that if the first mover has the Queen, she assigns probability $\frac{1}{2}$ to the second mover having the King, and probability $\frac{1}{2}$ to the second mover having the Jack.
- Such contextual assumptions about beliefs give a "direction" to (initial or strong) rationalizability analysis: **directed** (initial or strong) **rationalizability** modifies the previously explained rationalizability solutions to take into account contextual restrictions on beliefs that the modeler takes as given.
- **This part is optional.** The interested student can consult GT-AST 15.4.3. Here we only give *hints* and *examples*.

Directed Rationalizability in Mini-Poker



Derive exogenous beliefs given information types from uniform prior on $\{JQ, JK, QJ, QK, KJ, KQ\}$: $\mu_i(Q|J) = \mu_i(K|J) = \frac{1}{2}$, etc.

Directed Rationalizability in Mini-Poker: Solution

- Behavior is pinned down only for two types out of three, leaving room for *bluffing*.

1. *B*etting is dominant for pl. 1 with K. *C*alling is conditionally dominant for pl. 2 with K given B, while *f*olding is conditionally dominant for pl. 2 with J given B.
2. Pl. 1 with Q can predict the type contingent reply of pl. 2 and deems B optimal: L yields -1, B yields $-\frac{1}{2}$ in expectation:

$$\begin{array}{ccccccc} -1 & \overset{L}{\longleftarrow} & \overset{1/2}{QJ} & \overset{1.Q}{- - -} & \overset{1/2}{QK} & \overset{L}{\longrightarrow} & -1 \\ & & B\downarrow & & \downarrow B & & \\ & & 2 & & 2 & & \\ & f\swarrow & & & \downarrow c & & \\ & 1 & & & -2 & & \end{array}$$

3. With this, both B and L are rationalizable for $1.J$ and both *c*all and *f*old are rationalizable for $2.Q$ given B. Thus, *bluffing is rationalizable.*

Directed Rationalizability and Iterated Admissibility

- Suppose that *beliefs restrictions pin down the initial exogenous belief* $p_i(\cdot|\theta_i) \in \Delta(\Theta_{-i})$ of each type θ_i of each pl. i.
- Get "**simple Bayesian game**" BG with (somewhat arbitrary) exogenous prior belief $p_i \in \Delta(\Theta)$ for each i as strict convex combination of the beliefs of types:

$$p_i(\theta_i, \theta_{-i}) = p_i(\theta_{-i}|\theta_i)\lambda_i(\theta_i) \in \Delta(\Theta) \ [\lambda_i \in \Delta^\circ(\Theta_i), \text{ cf. Ch.8.4}].$$

- With this,
  - analyze BG from the *ex ante perspective*, considering initial conjectures of the form $\mu^i = p_i \times \mu_{-i}$ where $p_i \in \Delta(\Theta)$ is i's prior exogenous belief and $\mu_{-i} \in \Delta\left(S_{-i}^{\Theta_{-i}}\right)$ (cf. Appendix of Ch. 8);
  - (ex ante) strong directed rationalizability is *generically equivalent* to iterated admissibility on the ex ante strategic form of BG;
  - (ex ante) initial directed rationalizability is *generically equivalent* to one round of admissibility followed by iterated strict dominance on the ex ante strategic form of BG.

References

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