Multistage Bayesian Games and Equilibrium

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Abstract

As already argued for simultaneous-moves games with incomplete information, without assumptions about players' exogenous beliefs we can perform a rationalizability analysis [and also a self-confirming equilibrium analysis], but not an "orthodox" (i.e., traditional) equilibrium analysis: in general, we cannot ascertain whether a profile of decision rules about which players have correct conjectures is immune to deviations. Adding to a multistage game with payoff uncertainty a specification of players' exogenous beliefs, we obtain a multistage Bayesian game. The Bayesian equilibria of a multistage Bayesian game are the Nash equilibria of its ex ante strategic form. As in the case of games with complete information, some equilibria of the strategic form may be "imperfect".

[These slides summarize and complement Sections 15.5-15.6 in Chapter 15 of GT-AST.]

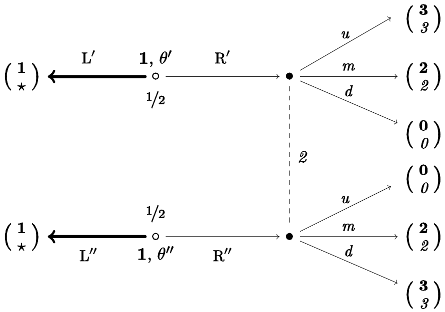
Introduction

- Recall: a **multistage game with** (observable actions and) **payoff uncertainty** is a structure

$$\hat{\Gamma} = \left\langle I, (\Theta_i, A_i, \mathcal{A}_i(\cdot), u_i)_{i\in I} \right\rangle$$

  where $\left\langle I, (A_i, \mathcal{A}_i(\cdot))_{i\in I} \right\rangle$ describes a game tree, Θ_i is the set of information types of i, and $u_i : \Theta \times Z \to \mathbb{R}$ is the parameterized payoff function of i (we neglect Θ_0 to ease notation).
- As in the case of static games, such description of the situation of strategic interaction with incomplete information is sufficient to analyze (versions of) rationalizability [and self-confirming equilibrium], but insufficient to define an "orthodox" notion of equilibrium: without a specification of players' exogenous beliefs about each others' private information, we cannot determine whether a profile of decision rules is immune to deviations.
- Appending to $\hat{\Gamma}$ such exogenous beliefs, we obtain a **multistage Bayesian game** (with observable actions).

Example 1



- The figure represents a Bayesian game. Temporarily disregard pl. 2's exogenous belief. Intuitively:
  - $\mathrm{L}'.\mathrm{L}''$ *cannot be an equilibrium* decision rule of player 1. (Why?)
  - $\left(\mathrm{L}'.\mathrm{R}'', d\right)$ and $\left(\mathrm{R}'.\mathrm{L}'', u\right)$ are "revealing" (separating) equilibria.
  - Is $\left(\mathrm{R}'.\mathrm{R}'', m\right)$ a "non-revealing" (pooling) equilibrium? It depends on $p_2 \in \Delta(\Theta)$: yes if $\frac{1}{3} \leq p_2\left(\theta'\right) \leq \frac{2}{3}$, as in the figure.

Example 2

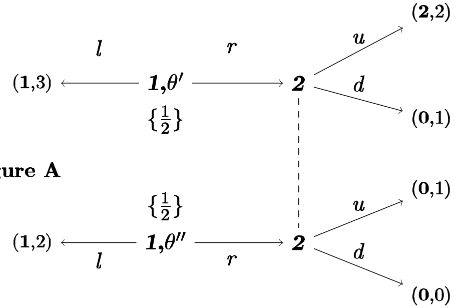


Figure A

- In this Bayesian game,
  - $(r'.\ell'', u)$ is the only reasonable equilibrium [it is the only initially rationalizable profile];
  - yet, also $(\ell'.\ell'', d)$ is a Bayesian equil.: pl. 2 is *ex ante* indifferent.
- We need to refine Bayesian equilibrium (topic of the next lectures).

Multistage Bayesian Games

A **simple multistage Bayesian game** (with observable actions) is a *finite* structure

$$\Gamma = \left\langle I, \left(\Theta_i, A_i, \mathcal{A}_i(\cdot), u_i, (p_i(\cdot|\theta_i))_{\theta_i \in \Theta_i}\right)_{i \in I} \right\rangle \text{ where}$$

- $\left\langle I, (\Theta_i, A_i, \mathcal{A}_i(\cdot), u_i)_{i \in I} \right\rangle$ is a *finite* game with payoff uncertainty;
- $\forall i \in I, \forall \theta_i \in \Theta_i$, $p_i(\cdot|\theta_i) \in \Delta(\Theta_{-i})$ is the *initial exogenous* belief of type θ_i of player i, also called **interim belief**;
- Alternatively: **priors** $P_i \in \Delta(\Theta)$, with $\forall \theta_i \in \Theta_i$, $P_i(\theta_i) > 0$ and

$$\forall \theta_{-i} \in \Theta_{-i},\ p_i(\theta_{-i}|\theta_i) = \frac{P_i(\theta_i, \theta_{-i})}{P_i(\theta_i)}.$$

- One can verify that, for each i, any strictly convex combination

$$P_i(\cdot) = \sum_{\theta_i \in \Theta_i} p_i(\cdot|\theta_i)\lambda_i(\theta_i) \text{ [with } \lambda_i \in \Delta^\circ(\Theta_i) := \Delta(\Theta_i) \cap \mathbb{R}^{\Theta_i}_{++}]$$

of i's interim beliefs is a valid prior that generates $(p_i(\cdot|\theta_i))_{\theta_i \in \Theta_i}$.

Ex Ante Strategic Form

- [As we mentioned in the previous lecture, directed rationalizability can be used to analyze simple Bayesian games.]
- *If we define Γ using priors, we can characterize Bayesian equilibrium as the Nash equilibrium of the* **ex ante** *strategic form*. Let:
  - $\Sigma_i = S_i^{\Theta_i} = \left(\times_{h\in H}\mathcal{A}_i\left(h\right)\right)^{\Theta_i}$;
  - define $\bar{U}_i$ as follows:

$$\begin{array}{rccl} \bar{U}_i: & \times_{j\in I}\Sigma_j & \longrightarrow & \mathbb{R} \\ & \sigma & \longmapsto & \sum_{\theta\in\Theta} P_i(\theta)u_i\left(\theta,\zeta(\sigma\left(\theta\right))\right), \end{array}$$

with $\sigma\left(\theta\right) = \left(\sigma_j\left(\theta_j\right)\right)_{j\in I}$, P_i=*any* prior generating $\left(p_i\left(\cdot|\theta_i\right)\right)_{\theta_i\in\Theta_i}$.

Definition

The **ex ante strategic form** of Γ is the simultaneous-move game $\mathcal{AS}(\Gamma) = \left\langle I, \left(\Sigma_i, \bar{U}_i\right)_{i\in I}\right\rangle$.

Bayesian Equilibrium

Definition

A (randomized) **Bayesian equilibrium** of Γ is a (randomized) Nash equilibrium of $\mathcal{AS}(\Gamma)$.

- We will consider pure and randomized equilibria.
- In randomized equilibria players use **extended** (type-dependent) **behavior strategies** (randomized decision rules)

$$(\beta_i(\cdot|\theta_i,\cdot))_{\theta_i\in\Theta_i} = (\beta_i(\cdot|\theta_i,h))_{\theta_i\in\Theta_i,h\in H} \in B_i^{\Theta_i} := (\times_{h\in H}\Delta(\mathcal{A}_i(h)))^{\Theta_i},$$

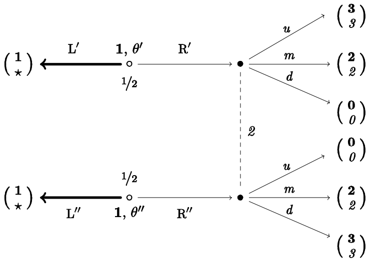
where $\beta_i(a_i|\theta_i,h)$ is the probability of choosing a_i given θ_i and h.

Interpretation of Extended Behavior Strategies

- Possible interpretation: conditional distributions in heterogeneous populations of agents.
- Let
  - $\sigma_i(\theta_i, s_i)$ be the fraction of agents in pop. i of type θ_i who play s_i,
  - for $X_i \subseteq S_i$, $\sigma_i(\theta_i, X_i) = \sum_{s_i \in X_i} \sigma_i(\theta_i, s_i)$,
  - $S_i(h)$ and $S_i(h, a_i)$ have the usual meaning.
- Then

$$\beta_i(a_i|\theta_i, h) = \frac{\sigma_i(\theta_i, S_i(h, a_i))}{\sigma_i(\theta_i, S_i(h))} \text{ if } \sigma_i(\theta_i, S_i(h)) > 0.$$

Ex Ante Strategic Form of Example 1

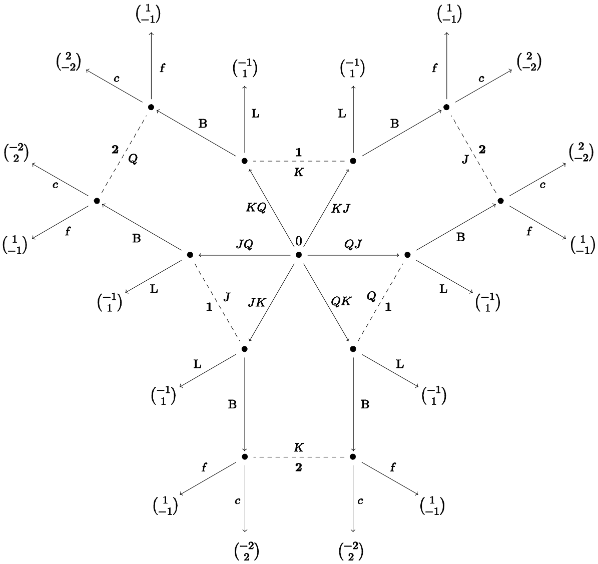


- *Ex ante strategic form*:

$2\backslash\mathbf{1}$	$L'.L''$	$L'.R''$	$R'.L''$	$R'.R''$
u	**1**, $\star$	**0.5**, $0.5\cdot\star$	**2**, $1.5+0.5\cdot\star$	**1.5**, 1.5
m	**1**, $\star$	**1.5**, $1+0.5\cdot\star$	**1.5**, $1+0.5\cdot\star$	**2**, 2
d	**1**, $\star$	**2**, $1.5+0.5\cdot\star$	**0.5**, $0.5\cdot\star$	**1.5**, 1.5

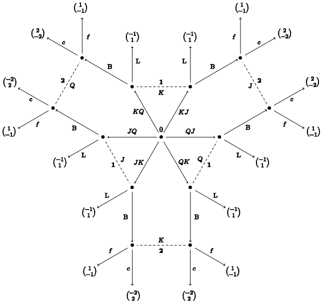
  - $L'.L''$ is **dominated**.
  - Pure equilibria: $(R'.R'', m)$, $(R'.L'', u)$, $(L'.R'', d)$.

Mini-Poker



0
KQ
KJ
JQ
QJ
JK
QK
1
K
2
Q
J
L
B
f
c
$\binom{1}{-1}$
$\binom{2}{-2}$
$\binom{-1}{1}$
$\binom{-2}{2}$

Mini-Poker: Equilibrium



- With $p(\theta_1, \theta_2) = \frac{1}{6}$, *unique partially randomized equilibrium*:
  - pl. 1: **B**idding with the King is dominant, $\beta_1(B|K_1) = 1$;
  - pl. 2: **f**olding with the Jack and **c**alling with the King is dominant conditional on **B**, $\beta_2(f|J_2, B) = \beta_2(c|K_2, B) = 1$;
  - with this, $\mathbb{E}(u_{1,B}|Q_1) = -\frac{1}{2} > -1 = \mathbb{E}(u_{1,L}|Q_1) \Rightarrow \beta_1(B|Q_1) = 1$;
  - in eq. $\beta_1(B|J_1) = \frac{1}{3} \Rightarrow \mu_2(J_1|Q_2, B) = \frac{\frac{1}{3}\frac{1}{6}}{\frac{1}{3}\frac{1}{6} + \frac{1}{6}} = \frac{1}{4} \Rightarrow$ pl. 2 indifferent; $\beta_2(c|Q_2, B) = \frac{1}{3} \Rightarrow$ pl. 1 indifferent.

(Im-)perfect Bayesian equilibrium: an example

- Like Nash equilibrium, Bayesian equilibrium allows for non-maximizing choices at histories that are not supposed to occur in equilibrium.

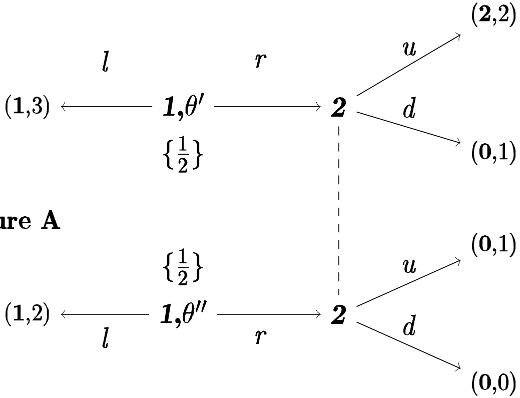


Figure A

- Profile $(\ell'.\ell'', d)$ *is an "imperfect" Bayesian equilibrium*, because d is a "non-credible threat" (u dominates d *given* r).

Ex Ante Strategic Form of Game in Fig. A

- The *ex ante strategic form* (assuming, w.l.o.g. in this case, a common prior) of the game depicted in Fig. A is:

$\mathbf{1}\backslash 2$	u	d
$\ell'.\ell''$	$\mathbf{1},\quad 2.5$	$\mathbf{1},\quad 2.5$
$\ell'.r''$	$\mathbf{0.5},\quad 2$	$\mathbf{0.5},\quad 1.5$
$r'.\ell''$	$\mathbf{1.5},\quad 2$	$\mathbf{0.5},\quad 1.5$
$r'.r''$	$\mathbf{1},\quad 1.5$	$\mathbf{0},\quad 0.5$

- **Note**: u dominates d conditional on r in the multistage game, hence u *weakly* dominates d.
- Furthermore, ℓ dominates r for θ'', hence $\ell'.\ell''$ dominates $\ell'.r''$, and $r'.\ell''$ dominates $r'.r''$.
- Iterated admissibility yields $(r'.\ell'', u)$ in 2 steps ($r'.\ell''$ is the unique best reply to u). This—the intuitive solution found earlier—is also the unique "perfect" Bayesian equilibrium of this game.

Bayesian equilibrium and the problem of (im)perfection

- In multistage games with *complete* information the "imperfect-equilibrium" problem has been addressed using the notion of *subgame perfect equilibrium (SPE).*
- While we do not necessarily subscribe to the orthodox view that SPE is the "right" way to deal with these problems, we will adopt the orthodox view as a working hypothesis and extend the SPE idea to multistage Bayesian games, thus defining some kind of **"perfect" Bayesian equilibrium (PBE).**

References

Battigalli, P., E. Catonini, and N. De Vito (2023): *Game Theory: Analysis of Strategic Thinking*. Typescript, Bocconi University.

Battigalli, P. (2023): *Mathematical Language and Game Theory*. Typescript, Bocconi University.