# Monotonicity and Robust Implementation under Forward Induction Reasoning

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# Premise

- We merge work on **solution concepts** for *sequential* games and their foundations with work on *robust* **mechanism design** (MD).
- **MD**=design rules of interaction (e.g., communication) and outcome function
  - so that each profile of agents' types (private information) $\theta = (\theta_i)_{i \in I}$ is associated with a desired social outcome as per some map (**social choice function**) $\theta \mapsto f(\theta)$ given agents' induced type-dependent strategic behavior.
  - **Robust**=the rules work as desired *independently of agents' interactive beliefs about each others' types* (Bergemann & Morris, e.g., TE 2009).
- Solution concept=**strong rationalizability** justified by *Rationality and Common Strong Belief in Rationality* (RCSBR, Battigalli & Siniscalchi 2002): players are *rational* (sequential SEU maximizers) and reason by **forward induction** (we say "**best rationalization principle"**):
  - they believe as long as possible (**strongly believe**) in co-players' rationality ⇒ exclude co-players' types for which observed behavior (even if unexpected) is clearly irrational;
  - furthermore, they ascribe to them (strongly believe) the highest degree of strategic sophistication consistent with observed behavior.

# Main motivation

- **Mueller (2016):** Assuming strong rationalizability, i.e., *under RCSBR*, there exist social choice functions that can be (virtually) implemented with dynamic mechanisms, although they cannot be with static mechanisms.
- **Key factor**: Learn about others' types by rationalizing their past moves/messages (*forward-induction reasoning*). No gain from dynamic mechanisms under the weaker assumption of Rationality and Common *Initial* Belief in Rationality (RCIBR, Mueller 2020; cf. Penta 2015: backward ind.).
- **Example:** Allocate *single good* to some $i \in I$ with *transfers*.
  - Finite sets of payoff types $\{0,1\} \subseteq \Theta_i \subseteq [0,1]$, *interdependent* valuations

$$v_i(\theta_i, \theta_{-i}) = \theta_i + \gamma \sum_{j \neq i} \theta_j \quad (\gamma \geq 0).$$

  - Efficient allocations can be v-implem. under RCSBR for almost all $\gamma \geq 0$, while only constant scf's can be v-implemented under RC(I)BR for $\gamma > \frac{1}{|I|-1}$.
  - Yet, only the latter form of implementation is known to be "robust."
- **Open question**: is RCSBR-implementation **robust** in the sense of Bergemann & Morris? That is, does it work for every Harsanyi type space?

# A more general problem

- In a game with payoff uncertainty (=incomplete information), does Strong Rationalizability (Battigalli 2003, Battigalli & Siniscalchi 2003) become more restrictive in terms of predicted outcomes as we introduce restrictions to players' initial beliefs (hierarchies) about the exogenous uncertainty?
- Hard to answer because of the *non-monotonicity of strong belief*: Fewer permitted beliefs entail fewer justifiable strategies, but strong belief in a smaller set of strategies is *not* more restrictive!
- This is because fewer observed moves can be rationalized.
- If after some steps of reasoning a move $a_j$ can be rationalized by player $i$ only without the restrictions, after further steps $i$'s possible reactions may end up being even disjoint from those with the restrictions [see example below].
- Conceivably, this might make some paths (hence, outcomes) *impossible without belief restrictions, but possible with restrictions.*
- Yet, *we prove this never happens.*

# Back to the implementation problem

- A social choice function $f$ from states to outcomes is virtually implementable when, for every $\delta > 0$, there exists a game form (mechanism) such that Strong Rationalizability yields, for each state $\theta$, an outcome (or set of outcomes) that is $\delta$-close to $f(\theta)$.
- In light of our result, if we introduce a(n) (incomplete) Harsanyi type space, we obtain a (nonempty) *subset* of outcomes for each state.
- Thus, *implementation under RCSBR is robust*.
- **Conclusion**: The use of dynamic mechanisms, under the assumption that players can engage in forward-induction reasoning, considerably expands the realm of *robustly* (virtually) implementable social choice functions. (IWD implementation?)

# Sequential games with payoff uncertainty

Finite multistage game with payoff uncertainty and observed actions

$$\Gamma = \left\langle I, (\Theta_i, A_i, \mathcal{A}_i(\cdot), u_i)_{i \in I} \right\rangle .$$

- $i \in I$, players.
- $\theta_i \in \Theta_i$, information/payoff types of $i$. ($\Theta = \times_{i \in I} \Theta_i$, $\Theta_{-i} = \times_{j \neq i} \Theta_j$)
- $a_i \in A_i$, actions of player $i$. ($A = \times_{i \in I} A_i$)
- $\mathcal{A}_i(\cdot) : \bigcup_{t=0}^{T} A^t \rightrightarrows A_i$, feasibility correspondence of $i$.
- $h \in \bar{H} \subseteq \bigcup_{t=0}^{T} A^t$, possible histories [derived from $(\mathcal{A}_i(\cdot))_{i \in I}$].
  (initial h: $h^0$; non-terminal h's: $H$; terminal h's/paths: $Z$)
- $u_i : \Theta \times Z \to \mathbb{R}$, $i$'s parameterized payoff func. ($u_{i,\theta} : Z \to \mathbb{R}$, section at $\theta$)

# Beliefs

- $s_i \in S_i = \times_{h \in H} \mathcal{A}_i(h)$, strategies of $i$ ($S = \times_{i \in I} S_i$, $S_{-i} = \times_{j \neq i} S_j$).
- $S(h) = S_i(h) \times S_{-i}(h)$, strategy profiles inducing $h \in \bar{H}$.
- **Conditional probability systems** (CPSs) of $i$:

$$\mu_i = (\mu_i(\cdot|\Theta_{-i} \times S_{-i}(h)))_{h \in H} \in \Delta^H(\Theta_{-i} \times S_{-i})$$

  (s.t. the *chain rule* holds; abbreviation: $\mu_i(\cdot|\Theta_{-i} \times S_{-i}(h)) = \mu_i(\cdot|h)$).
- $\Delta^H_{\mathrm{sb}}(E_{-i})$ $[E_{-i} \subseteq \Theta_{-i} \times S_{-i}]$, set of CPSs $\mu_i$ that **strongly believe** $E_{-i}$:

$$\forall h \in H, \quad E_{-i} \cap (\Theta_{-i} \times S_{-i}(h)) \neq \emptyset \Rightarrow \mu_i(E_{-i}|h) = 1.$$

- $\Delta = (\Delta_{i,\theta_i})_{i \in I, \theta_i \in \Theta_i}$ $[\Delta_{i,\theta_i} \subseteq \Delta^H(\Theta_{-i} \times S_{-i})]$, (type-dependent) **belief restrictions.**

# Rationalizability: Heuristic Example

Signaling g.: $\Theta_1 = \{x, y, z\}$, $A_1 = \{\ell, r\}$, $\mathcal{A}_2(\ell) = \{a, b\}$, $\mathcal{A}_2(r) = \{\mathrm{c}, \mathrm{d}, \mathrm{e}\}$

Payoffs:

| after $\ell$ | a | b |
|---|---|---|
| $\theta_1 = x$ | *3* 1 | *1* 0 |
| $\theta_1 = y$ | *1* 0 | *1* 1 |
| $\theta_1 = z$ | *3* 1 | *1* 0 |

| after $r$ | c | d | e |
|---|---|---|---|
| $x$ | *0* 0 | *0* 0 | *0* 1 |
| $y$ | *0* 0 | *0* 1 | *3* 0 |
| $z$ | *0* 1 | *2* 0 | *2* 0 |

**Strong Rationalizability** (no belief restrictions; **FI**=Forward Induction):

1. $r$ is not justifiable (it is dominated) for type $x$.
2. By *FI*-reasoning, after $r$ the receiver rules out type $x$.
   Thus, strategies a.e and b.e (e if $r$) are not justifiable (e dominated by $\frac{1}{2}\mathrm{c}+\frac{1}{2}\mathrm{d}$ once $x$ is ruled out).
3. After eliminating (e if $r$), $r$ is not justifiable for type $y$.
4. By *FI*-reasoning, after $r$ the receiver becomes certain of $\theta_1 = z$.
   Given this, a.d and b.d (d if $r$) are not justifiable (d cond. dominated).
5. After eliminating (d if $r$), $r$ is not justifiable for type $z$.

**Strongly rationalizable strategies:** $\ell$ for all types; a.c and b.c for the receiver.
**Paths:** $(\ell, \mathrm{a})$ and $(\ell, \mathrm{b})$ under all types. **Reaction to $r$:** c.

# Rationalizability with beliefs restrictions: Example

| after $\ell$ | a | b |
|---|---|---|
| $\theta_1 = x$ | *3* 1 | *1* 0 |
| $\theta_1 = y$ | *1* 0 | *1* 1 |
| $\theta_1 = z$ | *3* 1 | *1* 0 |

| after $r$ | c | d | e |
|---|---|---|---|
| $x$ | *0* 0 | *0* 0 | *0* 1 |
| $y$ | *0* 0 | *0* 1 | *3* 0 |
| $z$ | *0* 1 | *2* 0 | *2* 0 |

| *1,z* | $\ell$ | $r$ |
|---|---|---|
| a.c | *3* | *0* |
| a.d | *3* | *2* |
| a.e | *3* | *2* |
| b.c | *1* | *0* |

**Strong $\Delta$-rat.** with $\Delta_2$: CPSs that *initially* assign (marg.) prob. 1 to $\theta_1 = z$.

1. **S:** $r$ is not justifiable for type $x$ (as before).
   **R:** b.d and b.e are not justifiable: every $\mu_2 \in \Delta_2$ assigns probability 1 to $z$ either after $\ell$ (inducing a) or after $r$ (inducing c).
2. **S:** After eliminating b.d & b.e, $r$ is not justifiable for type $z$ (see *1,z*-table).
   **R:** By *FI*-reasoning, after $r$ the receiver rules out type $x$, thus (e if $r$) is not justifiable (as before).
3. **S:** After eliminating (e if $r$), $r$ is not justifiable for type $y$ (as before).
   **R:** Every $\mu_2 \in \Delta_2$ that initially assigns prob. 0 to "type $z$ plays $r$" assigns prob. 1 to $\ell$, so the receiver after $\ell$ becomes certain of $\theta_1 = z$; after $r$, by *FI*-reasoning, he is certain of $\theta_1 = y$; then, *only* a.d is *rationalizable*.

**Path:** $(\ell, \text{a})$ under all types [fewer paths]. **Reaction to $r$:** d [instead of c, disjoint strategies].

# Rationalizability: Definitions

- $\zeta : S \to Z$, path function.
- $H_i(s_i) = \{h \in H : s_i \in S_i(h)\}$, (non-terminal) histories allowed by $s_i$.
- Set of (weak) **sequential best replies** to $\mu_i$ for $\theta_i$:

$$r_{i,\theta_i}(\mu_i) = \left\{\bar{s}_i : \forall h \in H_i(\bar{s}_i), \bar{s}_i \in \arg\max_{s_i \in S_i(h)} \mathbb{E}_{\mu_i(\cdot|h)}\left(u_i\left(\theta_i, \cdot, \zeta(s_i, \cdot)\right)\right)\right\}.$$

- Set of **strongly** $\Delta$-$n$-**rationalizable** $(\theta_i, s_i)$ pairs of $i$: $\Sigma_{i,\text{sb}}^{\Delta,0} = \Theta_i \times S_i$;

$$n > 0: \quad \Sigma_{i,\text{sb}}^{\Delta,n} = \left\{(\theta_i, s_i) : \exists \mu_i \in \cap_{m=0}^{n-1} \Delta_{\text{sb}}^{H}(\Sigma_{-i,\text{sb}}^{\Delta,m}) \cap \Delta_{i,\theta_i}, s_i \in r_{i,\theta_i}(\mu_i)\right\}.$$

- Set of strongly $\Delta$-$n$-rationalizable strategies for $\theta_i$:

$$S_i^{\Delta,n}(\theta_i) = \left(\Sigma_{i,\text{sb}}^{\Delta,n}\right)_{\theta_i} = \left\{s_i : (\theta_i, s_i) \in \Sigma_{i,\text{sb}}^{\Delta,n}\right\}.$$

- Set of strongly $\Delta$-$n$-rationalizable strategy profiles at $\theta$:

$$S^{\Delta,n}(\theta) = \times_{i \in I} S_i^{\Delta,n}(\theta_i).$$

- No restrictions: **Strong $n$-Rationalizability** $\left(S^n(\theta)\right)_{\theta \in \Theta}$.

# Monotonicity

A profile $\Delta = \left(\Delta_{i,\theta_i}\right)_{i\in I,\theta_i\in\Theta_i}$ represents **restrictions on exogenous beliefs** if, for every $i$ and $\theta_i$, there is a (nonempty) subset $\bar{\Delta}_{i,\theta_i} \subseteq \Delta\left(\Theta_{-i}\right)$ such that

$$\Delta_{i,\theta_i} = \left\{\mu_i \in \Delta^H\left(\Theta_{-i} \times S_{-i}\right) : \mathrm{marg}_{\Theta_{-i}}\mu_i\left(\cdot|h^0\right) \in \bar{\Delta}_{i,\theta_i}\right\}.$$

## Theorem

*Fix a profile $\Delta = (\Delta_{i,\theta_i})_{i,\in I,\theta_i\in\Theta_i}$ of restrictions on* **exogenous** *beliefs. Then, for all $n > 0$ and $\theta \in \Theta$,*

$$\emptyset \neq \zeta\left(S^{\Delta,n}\left(\theta\right)\right) \subseteq \zeta\left(S^n\left(\theta\right)\right),$$

*that is, for each $(\theta, s) \in \Sigma_{\mathrm{sb}}^{\Delta,\infty} \neq \emptyset$, there exists $s' \in S$ such that $(\theta, s') \in \Sigma_{\mathrm{sb}}^{\infty}$ and $\zeta(s) = \zeta(s')$.*

*Sketch of proof:* maybe later.

# Bayesian elaboration

An **elaboration** of $\Gamma = \left\langle I, (\Theta_i, A_i, \mathcal{A}_i(\cdot), u_i)_{i\in I}\right\rangle$ is a structure

$$\Gamma^{\mathrm{e}} = \left\langle I, (T_i, A_i, \mathcal{A}_i(\cdot), u_i^{\mathrm{e}})_{i\in I}\right\rangle$$

such that, for every player $i \in I$, $T_i = \Theta_i \times E_i$, where $E_i$ is a finite nonempty set, $u_i^{\mathrm{e}} : \left(\times_{j\in I} T_j\right) \times Z \to \mathbb{R}$, and

$$u_i^{\mathrm{e}}\left(\left(\theta_j, e_j\right)_{j\in I}, z\right) = u_i\left(\left(\theta_j\right)_{j\in I}, z\right)$$

for all $\left(\theta_j, e_j\right)_{j\in I} \in \times_{j\in I} T_j$ and $z \in Z$. [Kind of “duplication” of types.]

A **Bayesian elaboration** is obtained by adding belief maps $\beta_i : T_i \to \Delta\left(T_{-i}\right)$:

$$\Gamma^{\mathrm{b}} = \left\langle I, \left(T_i, A_i, \mathcal{A}_i(\cdot), u_i^{\mathrm{b}}, \beta_i\right)_{i\in I}\right\rangle$$

with $u_i^{\mathrm{b}} = u_i^{\mathrm{e}}$ for all $i \in I$ [“Bayesian” means only that beliefs are subjective (sic)].

# Strong Rationalizability for a Bayesian elaboration

- Fix a B-elaboration $\Gamma^{\mathrm{b}}$ of $\Gamma$. For each $i \in I$, $\Sigma_{i,\mathrm{sb}}^{\mathrm{b},0} = T_i \times S_i$.
- For each $n \in \mathbb{N}$,

$$\Sigma_{i,\mathrm{sb}}^{\mathrm{b},n} = \left\{ (t_i, s_i) : \begin{array}{c} \exists \mu_i \in \cap_{m=0}^{n-1} \Delta_{\mathrm{sb}}^H (\Sigma_{-i,\mathrm{sb}}^{\mathrm{b},m}), \\ \mathrm{marg}_{T_{-i}} \mu_i \left( \cdot | h^0 \right) = \beta_i \left( t_i \right), s_i \in r_{i,t_i}^{\mathrm{b}} (\mu_i) \end{array} \right\},$$

where

$$r_{i,t_i}^{\mathrm{b}} \left( \mu_i \right) = \left\{ \bar{s}_i : \forall h \in H_i \left( \bar{s}_i \right), \bar{s}_i \in \arg \max_{s_i \in S_i(h)} \mathbb{E}_{\mu_i(\cdot|h)} \left( u_i^{\mathrm{b}} \left( t_i, \cdot, \zeta \left( s_i, \cdot \right) \right) \right) \right\}$$

for every CPS $\mu_i \in \Delta^H \left( T_{-i} \times S_{-i} \right)$.

- The set of strongly $n$-rationalizable strategies for type $t_i$ in $\Gamma^{\mathrm{b}}$ is the section

$$S_i^{\mathrm{b},n} \left( t_i \right) = \left( \Sigma_{i,\mathrm{sb}}^{\mathrm{b},n} \right)_{t_i} = \left\{ s_i : \left( t_i, s_i \right) \in \Sigma_{i,\mathrm{sb}}^{\mathrm{b},n} \right\}.$$

# Monotonicity for Bayesian elaborations

**Theorem**

*Fix any Bayesian elaboration* $\Gamma^{\mathrm{b}}$ *of* $\Gamma$*. Then, for all* $n > 0$ *and* $(\theta, e) \in T$*,*

$$\emptyset \neq \zeta\left(S^{\mathrm{b},n}(\theta, e)\right) \subseteq \zeta\left(S^{n}(\theta)\right).$$

- **Sketch of proof:**
  - For each $i \in I$, let $\hat{\Theta}_i = T_i$ and define $\hat{u}_i : \hat{\Theta} \times Z \to \mathbb{R}$ as $\hat{u}_i(t, z) = u_i^{\mathrm{b}}(t, z)$.
  - In the game with payoff uncertainty $\hat{\Gamma} = \left\langle I, \left(\hat{\Theta}_i, A_i, \mathcal{A}_i(\cdot), \hat{u}_i\right)_{i \in I} \right\rangle$, for each $t_i \in \hat{\Theta}_i$, let $\hat{\Delta}_{i,t_i}$ be the set of CPSs with (only) restrictions on exogenous beliefs derived from singleton $\bar{\Delta}_{i,t_i} = \{\beta_i(t_i)\}$.
  - For each $n > 0$ and $t = (\theta, e) \in \hat{\Theta} = T$, by construction $S^{\hat{\Delta},n}(t) = S^{\mathrm{b},n}(t)$.
  - In $\hat{\Gamma}$, by our main result (Theorem 1), $\zeta(S^{\hat{\Delta},n}(t)) \subseteq \zeta(S^{n}(t))$.
  - To conclude, we need to show that $S^{n}(\theta, e)$ in $\hat{\Gamma}$ coincides with $S^{n}(\theta)$ in $\Gamma$.
  - That is, Strong Rationalizability is invariant to duplications of types. ♡

# Economic environment

A finite **economic environment** is a structure

$$\mathcal{E} = \left\langle I, Y, (\Theta_i, v_i)_{i \in I} \right\rangle,$$

where $Y \subseteq \mathbb{R}^X$ [e.g., $Y = \Delta(X)$, $X$ finite] is an outcome space and each $v_i : \Theta \times Y \to \mathbb{R}$ is a param. vNM utility function ($v_{i,\theta} : Y \to \mathbb{R}$, section at $\theta$).

A **multistage mechanism** (with observed actions) is a game form

$$\mathcal{M} = \left\langle I, (A_i, \mathcal{A}_i(\cdot))_{i \in I}, g \right\rangle,$$

where $g : Z \to Y$ is an outcome function defined on the terminal histories determined by $(\mathcal{A}_i(\cdot))_{i \in I}$.

A pair $(\mathcal{E}, \mathcal{M})$ determines a game with payoff uncertainty

$$\Gamma(\mathcal{E}, \mathcal{M}) = \left\langle I, \left(\Theta_i, A_i, \mathcal{A}_i(\cdot), (u_{i,\theta} = v_{i,\theta} \circ g)_{\theta \in \Theta}\right)_{i \in I} \right\rangle$$

# Robust virtual implementation

**Definition**

A social choice function $f : \Theta \to Y$ is **virtually implementable under strong rationalizability** (in environment $\mathcal{E}$) if, for every $\delta > 0$, there exists a multistage mechanism $\mathcal{M}$ such that, in game with payoff uncertainty $\Gamma(\mathcal{E}, \mathcal{M})$, for all $\theta \in \Theta$ and $s \in S^{\infty}(\theta)$,

$$\|g(\zeta(s)) - f(\theta)\| < \delta.$$

**Definition**

A social choice function $f : \Theta \to Y$ is **robustly virtually implementable under strong rationalizability** (in environment $\mathcal{E}$) if, for every $\delta > 0$, there is a multistage mechanism $\mathcal{M}$ such that, in every Bayesian elaboration $\Gamma^{\mathrm{b}}$ of the game with payoff uncertainty $\Gamma(\mathcal{E}, \mathcal{M})$, for all $(\theta, e) \in T$ and $s \in S^{\mathrm{b},\infty}(\theta, e)$,

$$\|g(\zeta(s)) - f(\theta)\| < \delta.$$

# Strongly rationalizable v-implementation is robust

## Corollary

*Fix a finite economic environment $\mathcal{E}$ and a social choice function $f : \Theta \to Y$. If $f$ is virtually implementable under strong rationalizability, then $f$ is also robustly virtually implementable under strong rationalizability.*

- **Proof:** Let scf $f$ be $\delta$-implemented by $\mathcal{M}$ under strong rationalizability in $\Gamma = \Gamma(\mathcal{E}, \mathcal{M})$. Fix any B-elaboration $\Gamma^{\mathrm{b}}$ of $\Gamma$.
  - By Theorem 2, $\emptyset \neq \zeta\left(S^{\infty,\mathrm{b}}(\theta, e)\right) \subseteq \zeta\left(S^{\infty}(\theta)\right)$ for all $(\theta, e) \in T = \Theta \times E$.
  - Since $\mathcal{M}$ $\delta$-implements $f$, $\|g(\zeta(s)) - f(\theta)\| < \delta$ for all $\theta$ and $s \in S^{\infty}(\theta)$.
  - Thus, $\|g(\zeta(s)) - f(\theta)\| < \delta$ for all $(\theta, e) \in T$ and $s \in S^{\infty,\mathrm{b}}(\theta, e)$. ■

# Sketch of proof of the main result

Comparing strong rationalizability and strong $\Delta$-rationalizability "directly" is hard.

**Idea:** create a *finite sequence of elimination procedures that gradually transform strong $\Delta$-rationalizability into strong rationalizability* and prove step-by-step path-inclusion between each pair of consecutive procedures.

Procedure $k$ performs the *first $k$* steps of elimination *without* belief *restrictions* and the *following* steps *with* the belief *restrictions*.

The sequence of procedures can be seen as a *slower and slower order of elimination than strong $\Delta$-rationalizability*.

But if strong rationalizability ends in $K$ steps, the first $K$ steps of the $K$-th procedure coincide with it, and since all path-inclusions hold at all steps, we are done.
(We can generalize to the case in which strong rationalizability does not end in finitely many steps.)

Formally:

- For $k = 0$, the $k$-procedure is strong $\Delta$-rationalizability, that is,

$$(\mathrm{X}_0^n)_{n=0}^{\infty} = \left(\Sigma_{\mathrm{sb}}^{\Delta,n}\right)_{n=0}^{\infty}.$$

- For each $k = 1, ..., K$, define the $k$-procedure $((\mathrm{X}_{k,i}^n)_{i\in I})_{n=0}^{\infty}$ as follows. Let $\mathrm{X}_k^0 = \Theta \times S$.

  Steps 1 through k: for each $1 \leq n \leq k$ and $i \in I$,

$$\mathrm{X}_{k,i}^n = \left\{ \begin{array}{c} (\theta_i, s_i) \in \Theta_i \times S_i : \\ \exists \mu_i \in \cap_{m=0}^{n-1} \Delta_{\mathrm{sb}}^H(\mathrm{X}_{k,-i}^m), s_i \in r_{i,\theta_i}(\mu_i) \end{array} \right\}. \quad (1)$$

  Steps k+1 onwards: for each $n > k$ and $i \in I$,

$$\mathrm{X}_{k,i}^n = \left\{ \begin{array}{c} (\theta_i, s_i) \in \Theta_i \times S_i : \\ \exists \mu_i \in \cap_{m=0}^{n-1} \Delta_{\mathrm{sb}}^H(\mathrm{X}_{k,-i}^m) \cap \Delta_{i,\theta_i}, s_i \in r_{i,\theta_i}(\mu_i) \end{array} \right\}. \quad (2)$$

Let us focus now on procedure $k-1$ and procedure $k$ for some $k>0$: call them $P$ [procedure $((\mathrm{X}^n_{k-1,i})_{i\in I})^\infty_{n=0}$] and $Q$ [procedure $((\mathrm{X}^n_{k,i})_{i\in I})^\infty_{n=0}$].
$P$ and $Q$ *coincide* with Strong Rationalizability *for* steps $n\in\{1,...,k-1\}$ and *depart at step* $n=k$.
At step $n=k$, $P$ *adopts the belief restrictions* and $Q$ does not, so:

$$\mathrm{P}^n\subseteq\mathrm{Q}^n \text{ for } n=k.$$

At step $n+1=k+1$ both $P$ and $Q$ adopt the restrictions, but $P$ *imposes strong belief in smaller strategy sets* and therefore, *along the paths consistent with these sets*, it remains *more restrictive*:

$$\mathrm{P}^{n+1}|_{H(\mathrm{P}^n)}\subseteq\mathrm{Q}^{n+1}|_{H(\mathrm{P}^n)} \text{ for } n=k. \quad (3)$$

At step $n+2=k+2$ it gets complicated.

By (3), *strong belief in* $Q_{-i}^{n+1}$ *is less restrictive than in* $P_{-i}^{n+1}$ *about behavior on the paths consistent with* $P^n$.

- Role of the assumption of exogenous restrictions:
  - The statement above remains true after taking the belief restrictions into account:
  - The restrictions could potentially allow to believe in some $(\theta_{-i}, s_{-i}) \in \mathrm{P}_{-i}^{n+1}$, but not in its counterpart $(\theta_{-i}, s'_{-i}) \in \mathrm{Q}_{-i}^{n+1}$ with $s_{-i}|_{H(\mathrm{P}^n)} = s'_{-i}|_{H(\mathrm{P}^n)}$.
  - But they do not because they only concern initial beliefs about the *exogenous* uncertainty.
  - The fact that the restrictions are only on the *initial* beliefs also avoids that, at some point of the game where strong belief in $\mathrm{P}^{n+1}$ and in $\mathrm{Q}^{n+1}$ may induce different conditional beliefs about $\theta_{-i}$, only those derived from $\mathrm{P}^{n+1}$ are compatible with the restrictions.

However, *strong belief* (SB) *in* $Q_{-i}^{n+1}$ *may be more restrictive than SB in* $P_{-i}^{n+1}$ *about reactions to deviations from* $H(P^n)$*.* Then, if $H(\mathrm{Q}^{n+1}) \backslash H(\mathrm{P}^n) \neq \emptyset$, there could be a deviation from the paths consistent with $\mathrm{P}^n$ that is always profitable for $i$ under SB in $Q_{-i}^{n+1}$ but not in $P_{-i}^{n+1}$. This makes it hard to prove that

$$\mathrm{Q}^{n+2}|_{H(\mathrm{P}^{n+1})} \supseteq \mathrm{P}^{n+2}|_{H(\mathrm{P}^{n+1})} \text{ for } n = k. \quad (4)$$

What guarantees that *such a deviation does not exist*? That $H(\mathrm{P}^n) \supseteq H(\mathrm{Q}^{n+1})$.

To see that $H(\mathrm{P}^n) \supseteq H(\mathrm{Q}^{n+1})$, let us repeat the same reasoning with the *roles* of $P$ and $Q$ *flipped*, and leaving $P$ *one step behind*.
Since $\mathrm{Q}^n \subseteq \mathrm{Q}^{n-1} = \mathrm{P}^{n-1}$ for each $n \leq k$,

$$\mathrm{Q}^n \subseteq \mathrm{P}^{n-1} \text{ for } n = k.$$

Hence,

$$\mathrm{Q}^{n+1}|_{H(\mathrm{Q}^n)} \subseteq \mathrm{P}^n|_{H(\mathrm{Q}^n)} \text{ for } n = k.$$

As $H(\mathrm{Q}^n) \supseteq H(\mathrm{Q}^{n+1})$, this yields

$$H(\mathrm{P}^n) \supseteq H(\mathrm{Q}^{n+1}) \text{ for } n = k,$$

as we wanted to show.

To continue, proving $\mathrm{Q}^{n+2}|_{H(\mathrm{Q}^{n+1})} \subseteq \mathrm{P}^{n+1}|_{H(\mathrm{Q}^{n+1})}$ entails the same complications as proving (4), but we can solve them in the same way because we have already shown $H(\mathrm{Q}^n) \supseteq H(\mathrm{P}^n)$.

*Induction hypothesis*: $H(\mathrm{Q}^n) \supseteq H(\mathrm{P}^n)$ and $H(\mathrm{P}^n) \supseteq H(\mathrm{Q}^{n+1})$. ♡

# Why dynamic mechanisms...

...instead of their strategic form?

- Easier to describe and to play.
- The ties in the strategic form can only be solved with some form of cautiousness.
- And even if players iteratively eliminate the weakly dominated strategies...it need not yield the same outcomes as Strong Rationalizability in the dynamic mechanism! (Catonini, 2024)
  (go back)

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## Solution Concepts with Epistemic Foundations


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