Cognitive Hierarchies and the Strategy Method

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Introduction

- The **strategy method** [Selten 1967, Mitzkewitz & Nagel 1993] makes subjects in an experiment play the normal form, or reduced normal form of a sequential game to elicit off-path choices. Its *empirical* validity was studied by Brands & Charness (2013).
- The *theoretical* validity of the strategy method rests on the adopted theory of choice under uncertainty and strategic reasoning. [For example, iterated admissibility—based on lexicographic EU maximization—is reduced-normal-form invariant, cf. Brandenburger (2007).]
- It is known that the strategy method is not valid if subjects have (or are believed by others to have) dynamically inconsistent preferences, e.g., for psychological reasons. [See, e.g., Section 7 of Battigalli & Dufwenberg 2022, and Aina et al. 2020.]
- Lin & Palfrey (2024) showed that—even assuming dynamically consistent preferences—a prominent behavioral theory of strategic reasoning, the Cognitive-Hierarchies (CH) model, does not support the most common form of the strategy method. *I clarify why.*

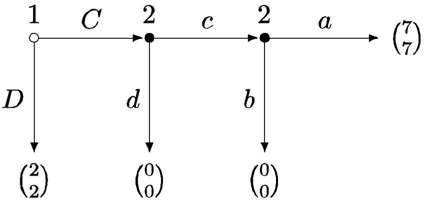
Cognitive hierarchies

- The **CH model** for *simultaneous* games [Camerer et al. 2004] posits a distribution $p_i = (p_{i\ell})_{\ell \in \mathbb{N}_0} \in \Delta(\mathbb{N}_0)$ of level-types for each player-role i. **Level**-0 **types** *uniformly randomize*; **level**-k **types** best reply to the $(k-1)$-truncated $p_{-i} = \times_{j \neq i} p_j$ mixture of the (possibly mixed) actions of the co-players, whereby level-1 types best reply to the uniform distribution, and so on.
- Lin & Palfrey (L&P) extend the CH model to *sequential games* (seq-CH), assuming that **level**-0 **types** play the uniform behavior strategy, i.e., *uniformly randomize over actions at each decision node*, and **level**-k **types** play *sequential best replies* to the $(k-1)$-truncated $p_{-i} = \times_{j \neq i} p_j$ behavior strategy mixture of the co-players. [Natural extension of the CH model.]
- L&P note that the *seq-CH model is not* reduced-*normal-form invariant* (see examples below) and comment on suggestive experimental evidence showing that, in the Centipede, subjects behave in different ways with the direct method of play and the reduced-strategy method (whereby subjects choose either at which node to take, or to always pass).

My contribution

- (Focusing for simplicity on games with perfect information) I show that *the seq-CH model is normal-form invariant* (it gives the same prediction for all games with the same normal form) and I explain the difference with the reduced normal form: it depends on a simple *counting argument*.
- I also comment on related *(in)variances w.r.t. transformations of the game* (see Battigalli, Leonetti & Maccheroni 2020) the seq-CH model
  - is *invariant* to *interchanging* essentially simultaneous moves,
  - is *not invariant* to *coalescing sequential moves* (and sequential-agent splitting).

Heuristic examples: Game 1



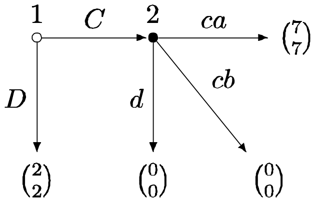
- Uniform behavior strategy: each action has 50% (conditional) probability. Thus,
  - for the level-1 type of pl. 1, C yields 7 utils with prob. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ and 0 with prob. $\frac{3}{4}$, i.e., $\frac{7}{4} < 2$ in expectation;
  - the *level*-1 type of *pl.* 1 plays D, the BR to the uniform behav. strat. of pl. 2;
  - *level-k* > 0 types of *pl.* 2 play $c.a$; the CH solution for *level-k* > 1 of *pl.* 1 *depends* on the fraction of level-0 types of pl. 2.

Heuristic examples: normal form of Game 1

$1\backslash 2$	$c.a$	$c.b$	$d.a$	$d.b$
C	7, 7	0, 0	0, 0	0, 0
D	2, 2	2, 2	2, 2	2, 2

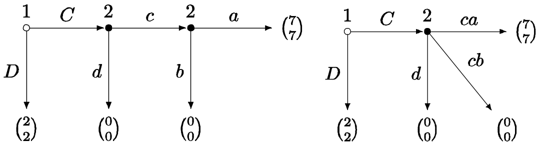
- The uniform mixed strategy of pl. 2 assigns prob. $\frac{1}{4}$ to each pure strategy. Thus,
  - for the *level*-1 type of *pl.* 1, C yields 7 utils with prob. $\frac{1}{4}$ and 0 with prob. $\frac{3}{4}$, i.e., $\frac{7}{4} < 2$ in expectation;
  - the *level*-1 type *of pl.* 1 plays D, the BR to the uniform mixed strat. of pl. 2;
  - *level-k* > 0 types of *pl.* 2 play the *weakly dominant* strat. $c.a$; the CH solution for *level-k* > 1 of *pl.* 1 *depends* on the fraction of level-0 types of pl. 2, *as above.*

Heuristic examples: Game 2



- Game 2 is obtained from Game 1 by *coalescing* the sequential moves of *pl.* 2. The uniform randomized strategy of pl. 2 assigns prob. $\frac{1}{3}$ to each action. Thus,
  - for the *level*-1 type of *pl.* 1, C yields 7 utils with prob. $\frac{1}{3}$ and 0 with prob. $\frac{2}{3}$, i.e., $\frac{7}{3} > 2$ in expectation;
  - the *level*-1 type of *pl.* 1 plays C, the BR to the uniform randomized strategy of pl. 2;
  - *level-k* > 0 types of *pl.* 2 play action/strategy *ca*; *level-k* > 1 types of *pl.* 1 play C; *same solution in the normal form* of Game 2.

Heuristic examples: comparison



- The second game is obtained from the first by coalescing the sequential moves of pl. 2 (the first is obtained from the second by sequential-agent splitting).
    - This *transformation does not change the* (structurally) *reduced normal form*, that aggregates the realization-equivalent strategies of each player (see Battigalli *et al.* 2020).
    - *Although the CH model is normal-form invariant* (hence, also invariant to interchanging essentially simultaneous moves), it is *not reduced-normal-form invariant*, not is it invariant to coalescing/sequential-agent splitting.
    - Thus, *the usual strategy method*—which makes subjects choose reduced strategies—is *not theoretically justified by the CH model.*

Games (with perfect information)

- I focus on (finite) games with *perfect information* to simplify notation (cf. Ch. 6 of Osborne & Rubinstein 1994, and Ch. 9 of Battigalli et al. 2023):
  - *Finite* **set of actions** A.
  - *Finite* **set of histories** H (finite sequences of actions). Given the **prefix-of** relation $\preceq$, H is a *tree* with root $\varnothing$ **(empty sequence).**
  - $A(h) := \{a \in A : (h, a) \in H\}$ is the set of feasible actions given h, and $Z := \{h \in H : A(h) = \varnothing\}$ is the **set of terminal histories**. To avoid trivialities, I assume that *there are at least 2 feasible actions at each non terminal history*.
  - **Player set** I, **player function** $P : H \backslash Z \to I$. Thus, $H_i := P^{-1}(i)$ the set of non-terminal histories where $i \in I$ plays.
  - Profile of "payoff functions" $u = (u_i : Z \to \mathbb{R})_{i \in I}$.

Pure and randomized strategies

- **Pure strategies** of i: $s_i = (s_{ih})_{h\in H_i} \in \times_{h\in H_i} A(h) =: S_i$ (s_{ih}=action selected by s_i at h).
- Probability simplex on finite set X:

$$\Delta(X) := \left\{ \mu \in \mathbb{R}_+^X : \sum_{x\in X} \mu(x) = 1 \right\}.$$

  Number of elements (cardinality) of X: $|X|$.
- **Behavior strategies** of i: $\sigma_i = (\sigma_{ih})_{h\in H_i} \in \Sigma_i := \times_{h\in H_i} \Delta(A(h))$.
  Uniform: $\sigma^0_{ih}(a) = 1/|A(h)|$ for all $h \in H_i$, $a \in A(h)$.
- **Mixed strategies** of i: $\mu_i \in \Delta(S_i)$.
  Uniform: $\mu^0_i(s_i) = 1/|S_i|$ for all $s_i \in S_i$.

Kuhn's transformation and uniform randomization

Kuhn's map from behavior to mixed strategies preserves the probabilities of paths of play: for all $s_i \in S_i$,

$$\mu_i^{\sigma_i}(s_i) := \prod_{h \in H_i} \sigma_{ih}(s_{ih}).$$

Remark (Unif) *For each player $i \in I$, the cardinality of i's strategy set is $|S_i| = \prod_{h \in H_i} |A(h)|$; therefore, the uniform behavior strategy σ_i^0 of i yields the uniform mixed strategy μ_i^0 under Kuhn's map.*

Proof. Using Kuhn's map, the mixed strategy obtained from the uniform behavior strategy σ_i^0 satisfies, for every $s_i \in S_i$,

$$\mu_i^{\sigma_i^0}(s_i) = \prod_{h \in H_i} \sigma_{ih}^0(s_{ih}) = \prod_{h \in H_i} \frac{1}{|A(h)|} = \frac{1}{\prod_{h \in H_i} |A(h)|} = \frac{1}{|S_i|} = \mu_i^0(s_i).$$

∎

Normal-form invariance of the CH model: preliminaries I

- **Outcome (path) function:** $O : \times_{i\in I} S_i \to Z$, $O(s)$=path (term. hist.) induced by $s = (s_i)_{i\in I}$.
- **Normal-form payoffs:** $U_i = u_i \circ O : \times_{i\in I} S_i \to \mathbb{R}$ [that is, $U_i(s) = u_i(O(s))$].
- Fix behav. strat. profile $\sigma = (\sigma_i)_{i\in I}$; $\mu^\sigma = \left(\mu_i^{\sigma_i}\right)_{i\in I}$=mixed strat. profile induced by Kuhn's maps, which preserves the probabilities of paths. Thus

$$\mathbb{E}_{\mu^\sigma}(U_i) = \sum_{s\in\times_{i\in I} S_i} U_i(s) \prod_{j\in I} \mu_j^\sigma(s_j) = \sum_{z\in Z} u_i(z)\,\mathbb{P}_\sigma(z) = \mathbb{E}_\sigma(u_i).$$

- **Ex ante best reply:** $\mu_i^* = \arg\max_{\mu_i\in\Delta(S_i)} \mathbb{E}_{\mu_i,\mu_{-i}}(U_i)$ (uniformly randomizing at the top, for definiteness and in the CH spirit).
- $\sigma_i^* = \overline{\mathrm{BR}}_i(\sigma_{-i})$=**weakly sequential best reply** to $\sigma_{-i} = \left(\sigma_j\right)_{j\neq i}$: it maximizes i's expected payoff $\mathbb{E}_{\sigma_i,\sigma_{-i}}(u_i)$ *in each subgame reachable under* σ_i^* (uniformly randomizing at the top and off-the-σ_i^*-paths, for definiteness and in the CH spirit). It is realization-equivalent to the **sequential best reply** that maximizes continuation expected payoff in every subgame (unif. at top).

Normal-form invariance of the CH model: preliminaries II

Relatively standard arguments based on *dynamic consistency of subjective expected utility maximization* yield the following:

Lemma

(Ex Ante) *If conjecture σ_{-i} (equivalently $\mu_{-i}^{\sigma_{-i}}$) is strictly positive, the ex ante best reply μ_i^* to σ_{-i} (or $\mu_{-i}^{\sigma_{-i}}$) is the Kuhn's transformation of the weakly sequential best reply to σ_{-i}: $\mu_i^* = \mu_i^{\overline{\mathrm{BR}}_i(\sigma_{-i})}$.*

To state the main theorem, let $\widetilde{\sigma}_{-i}^{\ell}$ denote **"mixture"** of the behavior strategy profiles $\left(\sigma_{-i}^{k}\right)_{k=0}^{\ell}$ of the seq-CH model, using the ℓ*-truncated distributions* $\left(p_j^{\ell}\right)_{j\neq i}$ of co-players' level-types [a kind of Bayesian (product) prior].

Normal-form invariance

Main result: the *CH model is normal-form invariant.*

Theorem

Consider the CH models applied to the normal-form and extensive-form representations of a finite game (with perfect information). For every player $i \in I$ and every level $\ell \geq 0$, the level-$(\ell+1)$ mixed best reply $\mu_i^{\ell+1}$ to conjecture $\widetilde{\mu}_{-i}^{\ell} = \times_{j \neq i} \left(\sum_{k=0}^{\ell} p_{jk}^{\ell} \mu_j^k \right)$ in the normal form is the Kuhn's transformation of the weakly sequential best reply $\bar{\sigma}_i^{\ell+1} = \overline{\mathrm{BR}}_i \left(\widetilde{\sigma}_{-i}^{\ell} \right)$.

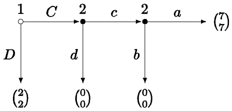
Intuition The proof is by induction on ℓ. The *basis step* $\ell = 0$ follows from *Remark (Unif)* and *Lemma (Ex Ante).* Suppose by way of induction that the result holds for each $k \in \{0, ..., \ell\}$ and fix any $i \in I$. One can *show* that the *strictly positive conjecture* $\widetilde{\sigma}_{-i}^{\ell}$ is *realization-equivalent to* $\widetilde{\mu}_{-i}^{\ell}$. Thus, *Lemma (Ex Ante)* yields the result. ♡

Lack of Reduced-NF invariance

- Two strategies s_i' and s_i'' are **realization-equivalent**, written $s_i' \approx_i s_i''$, if—for every behavior of co-players—they induce the same outcome/path:

$$\forall s_{-i} \in \times_{j \neq i} S_j,\ O\left(s_i', s_{-i}\right) = O\left(s_i'', s_{-i}\right).$$

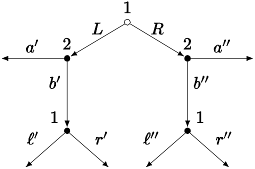
- **(Structurally) Reduced strategies:** $\mathbf{r}_i \in \mathbf{R}_i := S_i | \approx_i$, cells of the realization-equivalence partition $S_i | \approx_i$ (cf. Battigalli et al. 2020, 2023).



- In **Game 1**, 3 reduced strat.: $\mathbf{R}_2 = \{\mathbf{d}, \mathbf{c.a}, \mathbf{c.b}\}$, with $\mathbf{d} = \{d.a, d.b\} \subset S_2$, $\mathbf{c.x} = \{c.x\}$, with $x \in \{a, b\}$ (singleton).
- Uniform $\mu_i^0 \in \Delta(S_i)$ need not induce uniform $\boldsymbol{\mu}_2^0 \in \Delta(\mathbf{R}_2)$. In Game 1 $\mu_2^0(\{d.a, d.b\}) = \mu_2^0(d.a) + \mu_2^0(d.b) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq \frac{1}{3} = \boldsymbol{\mu}_2^0(\mathbf{d})$.

Conditions for R-NF invariance, I

- A game is **equi-reducible** if, for each player $i \in I$, all the reduced strategies (equivalence classes) $\mathbf{r}_i \in \mathbf{R}_i = S_i | \approx_i$ have the same cardinality.
- **Remark** *All one-move games (where each player moves at most once on each path) are—trivially—equi-reducible: In one-move games reduced strategies are singletons and* $|S_i| = |\mathbf{R}_i|$ *for each* $i \in I$.
- The following is not a one-move game tree, but it is equi-reducible:



Conditions for R-NF invariance, II

- **Remark** *The uniform* $\mu_i^0 \in \Delta(S_i)$ *induces the uniform on reduced strategies* $\boldsymbol{\mu}_i^0 \in \Delta(\mathbf{R}_i)$ *for each* i if and only if the game is equi-reducible.

Corollary

Fix a game tree (with perfect information). The seq-CH model is equivalent to the CH model in the reduced normal form for every profile of payoff functions $(u_i)_{i \in I} \in \mathbb{R}^{I \times Z}$ *if and only if the game is equi-reducible.*

Level-k thinking, I

- Just like the CH model, the level-k thinking model (LKT) model assumes that *level-*0 *types randomize uniformly*, and level-1 types best reply to the uniform randomization of the co-players.
- Thus, *the foregoing considerations and results apply to level-*1 *types of the LKT model.*
- Unlike the CH model, the LKT model assumes that *level-*$k > 1$ *types best reply to the strategies of the level-*$(k-1)$ *types of the co-players.*

Level-k thinking, II

- *Some nodes/histories $h \in H_i$ of i may be unreachable under the level-$(k-1) > 0$ strategy profile σ_{-i}^{k-1} of the co-players.*
- Hence, any reasonable extension of the LKT model from simultaneous-move to sequential games must posit a meaningful theory of how i of level-$k > 1$ type thinks about the co-players if such unexpected nodes are reached; e.g. believe in the highest $\ell < k-1$ making the node reachable (see Schipper & Zhou 2024).
- Thus, the *analogs of the foregoing (NF and R-NF invariance) Theorems do not hold for the LKT model*. But *invariance to INTERCHANGE and lack of invariance to COALESCE hold also for the LKT model.*

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