Introduction to Game Theory

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Game Theory: Analysis of Strategic Thinking

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ABSTRACT: Game Theory (GT) is the *formal analysis* of interactive decision making, i.e., of situations with n individuals (called **players**), some or all of whom have to take actions, which affect the outcome for everybody. In one-stage, or **static** games, players move simultaneously once and for all. In **multistage** games, some or all moves are sequential. The **game tree** describes the rules of interaction (who can do what, when, etc.). The **game form** describes all the rules; hence, also how outcomes depend on actions. The **game** represents a *situation*, it describes both the rules and players' personal preferences (utility functions). There is **complete information** if the rules and players' preferences are common knowledge, and **in**complete information otherwise. There is **perfect information** if players move one at a time and perfectly observe previous moves, and **im**perfect information otherwise. The jargon of GT can be confusing, as it developed historically and therefore it was not perfectly designed.
[These slides summarize and complement Chapter 1 of "Game Theory: Analysis of Strategic Thinking" (GT-AST). For the mathematical formalism see also the note "Mathematical Language and Game Theory".]

Introduction: game theory

- **Game theory** is the formal analysis of interactive decision making, i.e., of situations with n individuals (called **players**), some or all of whom have to take actions, which affect the outcome (consequences) for everybody. We will often focus on monetary outcomes.
  - In one-stage, or **static** games, active players move simultaneously once and for all.
  - In **multistage** games, moves are sequential, although some of them may be simultaneous.
- Game theory describes interactive situations using a *mathematical language*. It is crucial to *distinguish*:
  - the *description* of the *"rules of the game"*, called **game form**,
  - from the *description of the exogenous personal features* of the participating individuals, such as their "tastes" (e.g., preferences over lotteries of outcomes).
  - By appending to a game form the description of players' exogenous features we obtain a **game.**

Example of static game form: Prisoners' Dilemma

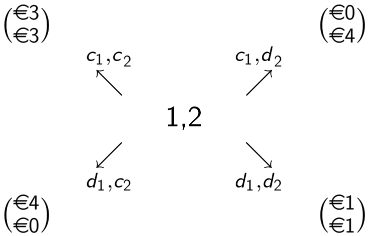
- Ann (pl. 1) and Bob (pl. 2) choose simultaneously between actions c (cooperate) and d (defect). Outcomes often are *monetary gains* $g = (g_1, g_2)$ in, say, €=euro. *Static* game forms admit a *tabular description*, the **monetary payoff matrix:**

PD
(not a "game"!)

g (in ECU)	c_2	d_2
c_1	€3, €3	€0, €4
d_1	€4, €0	€1, €1

- ... and also a *graphical description*, the **game tree** form:

PD
(not a "game"!)



Example of multistage game form: Ultimatum mini-Game

- There are €10 to split. Stage 1: Ann can implement the fair allocation (€5, €5) (action **f**) or make a greedy offer of only €1 to Bob (action **g**). *If* Ann makes the greedy offer, Bob can reject (**r**) or accept (**a**).
- The possible sequences of actions and the implied payoffs are described by the **game tree** form:

$$
\begin{array}{ccccccc}
 & & & & \text{Ann} & & \\
 & & & \text{f} \swarrow & \downarrow \text{g} & & \\
\begin{array}{c}\text{UmG} \\ \text{(not a "game"!)}\end{array} & & \binom{€5}{€5} & & \text{Bob} & & \\
 & & & \text{r} \swarrow & & \searrow \text{a} & \\
 & & \binom{€0}{€0} & & & & \binom{€9}{€1}
\end{array}
$$

Examples of "tastes" (utility functions)

- **Preliminaries**:
  - Fix an index set I of "individuals" (e.g., $I = \{\mathrm{Ann}, \mathrm{Bob}\}$) and some range X (a set, possibly a set of sets, or a set of functions).
  - We call **"profile"** a function with domain I and codomain X, typically denoted by $(x_i)_{i \in I}$, that is, individual i is associated with object $x_i \in X$.
  - The **set of profiles** is denoted X^I (if I and X are finite, its cardinality is $\left|X^I\right| = |X|^{|I|}$).
- **Standard economics** describes tastes by means of utilities of outcomes (e.g., monetary outcomes), $v_i : Y \to \mathbb{R}$, where typically $Y \subseteq \mathbb{R}^I$ with I=set of agents, for example (*non exhaustive* list, y_j=monetary payoff of $j \in I$):
  - $v_i\left((y_j)_{j \in I}\right) = y_i$ (selfish and risk neutral),
  - $v_i\left((y_j)_{j \in I}\right) = V_i(y_i)$ with $V_i' > 0$, $V_i'' < 0$ (selfish, risk averse),
  - $v_i\left((y_j)_{j \in I}\right) = y_i + \sum_{j \neq i} V_{ij}(y_j)$, $0 < V_{ij}' \leq 1$ (partial altruism).

s-b-miniG (form)

$$
\begin{tikzcd}
& & & s \arrow[dll, "1"'] \arrow[drr, "2"] & & & \\
& b \arrow[dl, "a"'] \arrow[dr, "r"] & & & & b \arrow[dl, "a"'] \arrow[dr, "r"] & \\
(o_b,1) & & (o_s,0) & & (o_b,2) & & (o_s,0)
\end{tikzcd}
$$

- The seller (**s**) owns an object, she can ask a price $p =$€1 or $p =$€2 for it.
- The Buyer (**b**) can accept (**a**), or reject (**r**).
- **Outcomes**: (o_i, t)=(i final owner,transfer €t from **b** to **s**).

Mathematical description: games tree (hints)

- **Game tree:** a mathematical structure $\langle I, (A_i)_{i\in I}, \mathcal{E}\rangle$, where
  - I is the finite **player set**, with generic element i (possibly also chance, typically denoted 0 or c);
  - A_i is the **set of feasible actions** of player i; $(A_i)_{i\in I}$ is the profile of action sets.
  - $\mathcal{E}$ is the **extensive form** representation of the *rules of interaction* (details are postponed to the 2nd part of the course).
- $\mathcal{E}$ yields a *tree structure* (see the figures), with a set Z of **terminal paths of play**, i.e., "legal" sequences of actions, or profiles of actions, leading to termination.
- In the **s-b-miniG**:
  - $I = \{\mathrm{s}, \mathrm{b}\}$;
  - $A_s = \{1, 2\}$, $A_b = \{\mathrm{a}, \mathrm{r}\}$;
  - $\mathcal{E}$: **s** starts, **b** follows upon observing the ask price;
  - $Z = \{(1, \mathrm{a}), (1, \mathrm{r}), (2, \mathrm{a}), (2, \mathrm{r})\}$.

Mathematical description: game form

To *complete the description of the rules*, we must specify *what outcomes are determined by players' actions*.

- **Game form:** a structure $\left\langle I, (A_i)_{i\in I}, \mathcal{E}, Y, g\right\rangle$ where
  - $\left\langle I, (A_i)_{i\in I}, \mathcal{E}\right\rangle$ is a game tree with a set Z of terminal paths of play;
  - Y is the set of possible **outcomes** (or consequences);
  - $g : Z \to Y$ is the **outcome** (or consequence) **function**.
- **Examples of outcome function:**
  - $Y \subset \mathbb{R}^I$ is a set of possible allocations of monetary payoffs, g determines the monetary payoff of each player i as a function of what players did.
  - $Y \subset (\mathbb{O} \times \mathbb{R})^I$ is a set of possible allocations of objects (elements of $\mathbb{O}$) and monetary transfers (positive or negative).
- **s-b-miniG**: $Y = \{o_b, o_s\} \times \{0, 1, 2\}$, $g(p, \mathrm{a}) = (o_b, p)$, $g(p, \mathrm{r}) = (o_s, 0)$ for each $p \in \{1, 2\}$.

Mathematical description: game

Adding a representation of players' preferences (personal features), we obtain a **"game"** in the jargon of Game Theory.

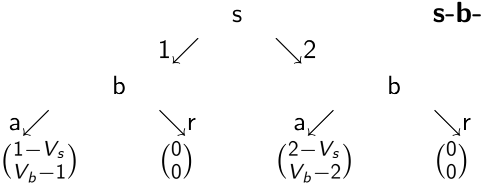
- **Game:** a mathematical structure $\left\langle I, (A_i)_{i\in I}, \mathcal{E}, Y, g, (v_i)_{i\in I}\right\rangle$ where
  - $\left\langle I, (A_i)_{i\in I}, \mathcal{E}, Y, g\right\rangle$ is a game form (description of the *rules of the game*);
  - for each $i \in I$, $v_i : Y \to \mathbb{R}$ is a von Neumann-Morgenstern **utility function**, representing preferences over lotteries of outcomes *via* expected utility calculations: for all $\lambda', \lambda'' \in \mathcal{L}(Y)$ (set of lotteries on Y with finitely many realizations)

$$\lambda' \succsim_i \lambda'' \Longleftrightarrow \mathbb{E}_{\lambda'}(v_i) \geq \mathbb{E}_{\lambda''}(v_i),$$

    with $\mathbb{E}_\lambda(v_i) = \sum_y \lambda(y) v_i(y)$ (expected utility of lottery λ).
- *Misleadingly*, the composite function $u_i = v_i \circ g : Z \to \mathbb{R}$ is called "**payoff function"** of i, because, when GT was born, players were assumed selfish and risk neutral: $Y \subset \mathbb{R}^I$, $v_i(y) = y_i$, and $u_i(z) = v_i(g(z))$=monetary payoff of i given z.

Example: seller-buyer mini-game

s-b-miniG



- Recall: according to EU theory, v_i is determined up to positive affine transformations (affine=convex & concave). Thus, we can arbitrarily fix, for each i, what outcome has 0-utility.
- Fix the *utility of the status quo* $(o_s, 0)$ to 0 for both.
- Let V_i the monetary value of the object for i.
- Let $v_b(o_b, p) = V_b - p$, $v_s(o_b, p) = p - V_s$ (risk neutrality).

Assumptions about knowledge: complete information

- **Common knowledge (CK) of some fact/event** E:
  - everybody knows that E is the case,
  - everybody knows that everybody knows that E is the case,
  - ... for each n, (everybody knows that)^n E is the case.
- We assume (for simplicity) that the *rules of interaction* represented by the game tree *are CK*.
- In many cases (e.g., in lab experiments), also the outcome function g is CK; hence, *the rules of the game* (represented by the game form) *are CK*. Since each i knows her vNM utility v_i, this implies that each i knows $u_i = v_i \circ g$ (and this fact is CK).
- Are the preferences (represented by $(v_i)_{i\in I}$) CK? In reality, it is unlikely. But *often we assume also CK of preferences as a simplification*. This is called **complete information:** *CK of every aspect of the game situation*, including personal preferences.
- E.g., outcomes are monetary payoffs and it is CK that players are selfish and risk neutral, as in the early GT.

Assumptions about knowledge: (in)complete information

- **s-b-mini-game:** complete information means CK of *everything*, including valuations V_s and V_b. Note, knowing V_b is key for **s**: ask $p = 1$ if $V_b < 2$, ask $p = 2$ if $V_b > 2$.
- **Incomplete information:** *some aspects of the game situation are not commonly known*, i.e., for some n it is not the case that (everybody knows that)^n the rules of the games and preferences are such and such.
- The most common situation of incomplete information is that players' personal preferences (e.g., whether they care about others, or are risk averse) are not commonly known.

Assumptions about knowledge: relevance

- *To model strategic thinking we must specify what players know.*
- E.g., *if* in the **s-b-mini-game** the rules (game form) are CK and **s** knows V_b, then we can "solve" (give a prediction for) the game by "backward induction":
  - start from the last (second) stage: if rational, **b** accepts (**a**) if $p < V_b$, and rejects (**r**) if $p > V_b$.
  - go back to the first stage: if **s** is rational and believes that **b** is rational, **s** asks $p < V_b$.
- But if **s** does not know V_b with sufficient precision, she has to form beliefs about V_b (e.g., based on statistics) and pick the ask price that maximizes her expected utility, comparing the opportunity cost of missing on a hefty sale with the risk of rejection of a "greedy" ask price.

Assumptions about information: perfect vs imperfect

- *Another strange feature of the GT jargon*: Assumptions about *knowledge* of the game situation are classified as complete/incomplete *information*. But there are very different assumptions about *information concerning previous moves* for which a similar language is used.
- A game tree features **perfect information** if (1)-(2) below holds, and **im**perfect information otherwise:
  - (1) there are *no simultaneous moves,*
  - (2) *earlier moves* (including chance moves) *are perfectly observed.*
- If (2) holds (whereas (1) may or may not hold), we say that there are **observed actions** (or **perfect monitoring**).
- For example:
  - The **PD** has simultaneous moves; hence, it does not feature perfect information, i.e., it is a game with *imperfect information*. Also the **PD played repeatedly**, with *perfect monitoring of actions of previous rounds* is a game with *im*perfect information.
  - The **UmG** and **s-b-miniG** are games with *perfect information*.

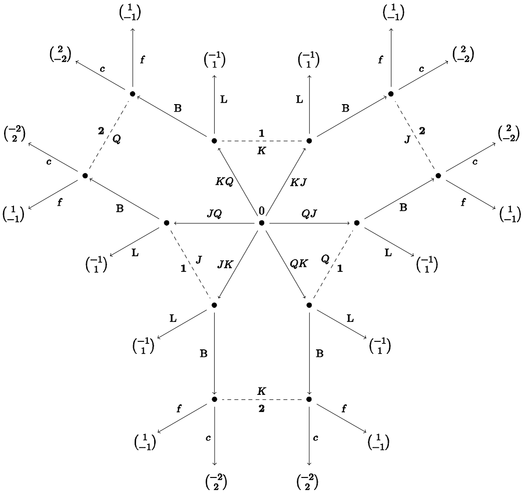
Example: Poker

- Consider the following situation:
  - n players play *version X of Poker* (e.g., Texas hold'em);
  - of course, *the rules of X-Poker are commonly known*;
  - furthermore, it is CK that each player cares only about how much he gains (i.e., is selfish) and is risk neutral, that is, it is CK that $v_i(y) = y_i$ for each i.
- This is a *game with imperfect but complete information* (it is imperfect because players do not perfectly observe the random distribution of cards, a chance move).

Example: mini-Poker, the rules

- 2 players, plus chance (player 0); 3 cards: Jack, Queen, and King. Each player puts €1 in the pot. Chance determines the order of cards, player 1 gets the 1st, player 2 the 2nd; each player sees only her/his card (*imperfect information*).
- Player 1 can **B**et (+€1 in the pot) or **L**eave; player 2 can respond to **B** by **f**olding, or **c**alling (+€1 in the pot).
- Outcome function: if
  - P1 **B**ets and P2 **c**alls, the one with the highest card gets the pot;
  - P1 **L**eaves, P2 gets the pot;
  - P1 **B**ets and P2 **f**olds, P1 gets the pot.

Example: mini-Poker, graphical representation



Chance (pl. 0) determines “hands”. Nodes joined by dotted lines represent the information of the active player, who does not know the opponent's “hand”.

Static and dynamic games

- In the games analyzed here, the play unfolds through a sequence of stages. (This assumption is quite innocuous for our purposes.)
- In every stage, either only one player moves, or more players move simultaneously; chance is a (pseudo)player.
- **Games with simultaneous moves**, or **static games** have only one stage; e.g., PD, Matching Pennies ("Pari o Dispari"), Rock-Scissor-Paper ("Morra Cinese"), first and second-price sealed-bid auctions.
- Other games, i.e., **multistage games** are **"dynamic"**; e.g., Tic-Tac-Toe ("Tris"), Poker, Chess, ascending and descending auctions.
- Many game theorists call "dynamic" only those games where a state variable changes according to a (deterministic or stochastic) transition function. Therefore, we speak of *static* (one-stage, simultaneous) *games* and *multistage* (sequential) *games*.
- We start with static games (Part I) because they are simpler.

References

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