

Nice Games: Dominance and Rationalizability

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Abstract

A *compact-continuous* game is **nice** if, for each player, the action set is an interval and the payoff function is strictly quasi-concave in his own action given the actions of others. Many economic models satisfy these assumptions, e.g., oligopoly, network, and beauty-contest games. The noteworthy feature of nice games is that, in the analysis of strategic thinking, *we can restrict our attention to deterministic conjectures*. Therefore, if the outcome function is deterministic, results are independent of players' risk attitudes and the analysis is simplified. [These slides summarize and complement material about nice games contained in Chapters 3 (3.3.2) and 4 (4.5) of "Game Theory: Analysis of Strategic Thinking."]

Example: Cournot oligopoly

Consider a Cournot (*quantity-setting*) oligopoly with n firms ($|I| = n$):

- inverse **market demand** $P(Q) = \max\{0, \bar{P} - \beta Q\}$, $Q = \sum_{i \in I} q_i$, $\bar{P}, \beta > 0$;
- **cost functions** $C_i(q_i) = cq_i$, $c < \bar{P}$ (*constant marginal cost*);
- and *large capacity* $\bar{q}$ for each firm: $(n-1)\bar{q} > (\bar{P} - c)/\beta$, i.e., $P((n-1)\bar{q}) < c$.
- Firms are expected profit maximizers (hence, *risk neutral*).
- All of the above is common knowledge (*complete information*).

Example: Cournot oligopoly (cont.)

- Let $Q_{-i} = \sum_{j \neq i} q_j$ denote i 's competitors' total output. The payoff (profit) function is

$$u_i(q_i, q_{-i}) = \begin{cases} q_i (\bar{P} - \beta q_i - \beta Q_{-i}) - cq_i & \text{if } \beta(q_i + Q_{-i}) < \bar{P}, \\ -cq_i & \text{if } \beta(q_i + Q_{-i}) \geq \bar{P}. \end{cases}$$

This function is *continuous*. It is *not concave* because of the kink due to the “0-floor” of price. But it is (strictly) “quasi-concave” (kind of “bell shaped”). See the picture in BBoard folder.

- We will show that (1) w.l.o.g., *we can focus on deterministic conjectures*; hence, (2) *the risk attitudes of the firms do not matter*.
- Each i has B.R. function $r(q_{-i}) = \max \left\{ 0, \frac{\bar{P}-c}{2\beta} - \frac{1}{2} \sum_{j \neq i} q_j \right\}$.
Clearly, $r(0, \dots, 0) = \frac{\bar{P}-c}{2\beta} = q^M$. Given the assumptions, $r(\bar{q}, \dots, \bar{q}) = 0$. Iterating r we get different results according to $n = 2$, or $n > 2$ (see Rationalizability).

Quasi-Concavity

General definition

- Recall that the payoff function $u_i = v_i \circ g : A \rightarrow \mathbb{R}$ is i 's vNM utility of action profiles; hence, it depends on i 's *risk attitudes*.
- It is interesting to identify classes of games where risk attitudes do not matter for most purposes, i.e., where results are preserved under strictly increasing transformations of the payoff functions.

Definition

(*Q-Concavity*) Let X be a *convex* subset of a Euclidean space (or a real vector space). A function $f : X \rightarrow \mathbb{R}$ is **quasi-concave** if

$$\forall x', x'' \in X, \forall w \in [0, 1],$$

$$f(w x' + (1 - w) x'') \geq \min \{ f(x'), f(x'') \};$$

it is **strictly quasi-concave** if $\forall x', x'' \in X$ with $x' \neq x''$, $\forall w \in (0, 1)$,

$$f(w x' + (1 - w) x'') > \min \{ f(x'), f(x'') \}.$$

(Strict) Quasi-Concavity: Independence of risk attitudes

- **Remark (Invariance).** (Strict) Quasi-concavity is preserved by strictly increasing transformations: if f is (strictly) quasi-concave and $\varphi : f(X) \rightarrow \mathbb{R}$ is strictly increasing, then $\varphi \circ f$ is (strictly) quasi-concave.
- **Proof:** We consider the case of *strict* quasi-concavity.
 - **Preliminaries:** For all $y', y'' \in f(X)$, by def. $y', y'' \geq \min\{y', y''\}$; since φ is *increasing*,

$$\varphi(\min\{y', y''\}) = \min\{\varphi(y'), \varphi(y'')\}.$$

- Fix $x', x'' \in X$ with $x' \neq x''$ and $w \in (0, 1)$ arbitrarily. By *strict quasi-concavity* of f ,

$$f(wx' + (1 - w)x'') > \min\{f(x'), f(x'')\}.$$

By *strict* monotonicity of φ ,

$$\begin{aligned}\varphi(f(wx' + (1 - w)x'')) &> \varphi(\min\{f(x'), f(x'')\}) \\ &= \min\{\varphi(f(x')), \varphi(f(x''))\}. \blacksquare\end{aligned}$$

Quasi-Concavity: Characterization

- **Remark.** A real-valued function f is *quasi-concave* on its convex domain X IFF, for every $y \in f(X) \subseteq \mathbb{R}$, the set $\{x \in X : f(x) \geq y\}$ is *convex*.
- **Proof:**
 - **(If)** Let $\{x \in X : f(x) \geq y\}$ be *convex* for every $y \in f(X)$. Fix $x', x'' \in X$, $w \in [0, 1]$ arbitrarily. Let $y = \min \{f(x'), f(x'')\}$; with this, $f(x') \geq y$ and $f(x'') \geq y$; thus, $x', x'' \in \{x \in X : f(x) \geq y\}$. Since $\{x \in X : f(x) \geq y\}$ is convex, $(wx' + (1 - w)x'') \in \{x \in X : f(x) \geq y\}$, that is, $f(wx' + (1 - w)x'') \geq y = \min \{f(x'), f(x'')\}$.
 - **(Only if)** Let f be quasi-concave and fix $y \in f(X)$ arbitrarily. Fix $x', x'' \in \{x \in X : f(x) \geq y\}$ and $w \in [0, 1]$ arbitrarily. Then, $f(x') \geq y$ and $f(x'') \geq y$, that is, $\min \{f(x'), f(x'')\} \geq y$. Quasi-concavity of f implies $f(wx' + (1 - w)x'') \geq \min \{f(x'), f(x'')\} \geq y$. Thus, $(wx' + (1 - w)x'') \in \{x \in X : f(x) \geq y\}$. This shows that $\{x \in X : f(x) \geq y\}$ is convex. ■

Strict Quasi-Concavity: 1-Variable Characterization

Strict quasi-concavity of a (continuous) function of one variable $f : [a, b] \rightarrow \mathbb{R}$ means that f is “bell-shaped”, or strictly increasing, or strictly decreasing.

- **Remark.** Fix any *compact interval* $[a, b] \subseteq \mathbb{R}$ and let the function $f : [a, b] \rightarrow \mathbb{R}$ be *continuous*. Then, f is *strictly quasi-concave* IFF it has a *unique maximizer* x^* , it is *strictly increasing* on $[a, x^*]$ (if $a < x^*$), and it is *strictly decreasing* on $[x^*, b]$ (if $x^* < b$).
- Note, a (continuous) strictly quasi-concave function $f : [a, b] \rightarrow \mathbb{R}$ may attain its maximum at the boundary (endpoints), it is strictly *increasing* if $x^* = b$ and strictly *decreasing* if $x^* = a$.

Strict Quasi-Concavity: Proof of the 1-V Characterization

- **Proof:** By compactness and continuity there is at least one maximizer $x^* \in \arg \max_{x \in [a,b]} f(x)$ (Weierstrass).
 - Suppose that f is strictly quasi-concave and that $a < x^*$. Let $a \leq x' < x'' < x^*$. We prove that $f(x'') > f(x')$. This implies that f is strictly increasing on $[a, x^*]$. Indeed, $x'' = (wx' + (1-w)x^*)$, where $w = \frac{x^* - x''}{x^* - x'} \in (0, 1)$. Thus, $f(x'') = f(wx' + (1-w)x^*) > \min\{f(x'), f(x^*)\} = f(x')$ where the inequality holds by strict quasi-concavity of f , and the second equality holds because x^* is a maximizer. The proof for the case $x^* < x' < x'' \leq b$ is similar.
 - Suppose that f has a unique maximizer x^* , it is strictly increasing on $[a, x^*]$ (if $a < x^*$), and it is strictly decreasing on $[x^*, b]$ (if $x^* < b$). Fix $x', x'' \in [a, b]$ with $x' < x''$, and $w \in (0, 1)$. Let $x(w) := wx' + (1-w)x''$, then, $x' < x(w) < x''$. If $x(w) \leq x^*$, then $f(x(w)) > f(x') \geq \min\{f(x'), f(x'')\}$ because f is strictly increasing on $[a, x^*]$. If $x(w) \geq x^*$, then $f(x(w)) > f(x'') \geq \min\{f(x'), f(x'')\}$ because f is strictly decreasing on $[x^*, b]$. Thus, f is strictly quasi-concave. ■

Nice Games

Definition

Definition

(Nice Games) A compact-continuous game $\langle I, (A_i, u_i)_{i \in I} \rangle$ is **nice** if, for every player $i \in I$, $A_i = [\underline{a}_i, \bar{a}_i] \subseteq \mathbb{R}$ is an *interval* and each section $u_i(\cdot, a_{-i}) : A_i \rightarrow \mathbb{R}$ of the payoff function ($a_{-i} \in A_{-i}$) is *strictly quasi-concave*.

- **Examples.** Many economic models, such as oligopolies (e.g., the aforementioned Cournot model), beauty contests, and network interactions are nice games.
- Many models satisfying all the nice-game properties except compactness can be “compactified”. For example, in the above-mentioned oligopoly model, the set of justifiable outputs is $[0, q^M]$ (with q^M = monopoly output) with any capacity $\bar{q} > q^M$ such that $P((|I| - 1)\bar{q}) < c$, and *also without capacity constraints*.

Nice Games

Best reply functions

- **Terminology.** A correspondence $\varphi : X \rightrightarrows Y$ is called **function** if $\varphi(x)$ is a *singleton* for every $x \in X$.

Lemma

(Continuous BR) In every nice game, for each player $i \in I$, the best reply correspondence r_i restricted to A_{-i} is a continuous function.

- **Proof.** Each section $u_i(\cdot, a_{-i})$ has a unique maximizer $r_i(a_{-i})$ by strict quasi-concavity.
 - Suppose that $a_{-i}^k \rightarrow a_{-i}^*$. By compactness, we may assume w.l.o.g. that $\lim_{k \rightarrow \infty} r_i(a_{-i}^k) = a_i^*$ for some $a_i^* \in A_i$.
 - We prove $a_i^* = r_i(a_{-i}^*)$. Since $r_i(a_{-i}^k)$ is the maximizer for each k and by continuity of u_i (as $k \rightarrow \infty$),

$$\forall a_i \in A_i, \forall k \in \mathbb{N}, u_i(r_i(a_{-i}^k), a_{-i}^k) \geq u_i(a_i, a_{-i}^k)$$

$$\forall a_i \in A_i, u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \text{ thus } a_i^* = r_i(a_{-i}^*). \blacksquare$$

Nice Games

Dominance by pure actions and best replies

- Let

$ND_{i,p} := \{\bar{a}_i \in A_i : \forall a_i \in A_i, \exists a_{-i} \in A_{-i}, u_i(\bar{a}_i, a_{-i}) \geq u_i(a_i, a_{-i})\}$
denote the set of pure actions not dominated by *pure* actions.

Lemma

(Equivalence to Certainty) Fix any player $i \in I$ in a nice game. The set of best replies to deterministic conjectures is a compact interval and it coincides with the set of pure actions undominated by pure actions; hence, it also coincides with the set of justifiable actions and undominated actions: $r_i(A_{-i}) = r_i(\Delta(A_{-i})) = ND_i = ND_{i,p}$.

- This result greatly simplifies the computation of justifiable and rationalizable actions in nice games, allowing us to neglect probabilistic conjectures.

Nice Games

Dominance by pure actions and BR: proof of Lemma of Equivalence to Certainty

- **Proof.** It is trivially true that $r_i(A_{-i}) \subseteq r_i(\Delta(A_{-i})) \subseteq ND_i \subseteq ND_{i,p}$ (actually, $r_i(\Delta(A_{-i})) = ND_i$ by the WP Lemma). We prove $ND_{i,p} \subseteq r_i(A_{-i})$, or—equivalently—that every $a_i \in A_i \setminus r_i(A_{-i})$ is dominated by some $a'_i \in A_i$.
 - The image of a compact and connected set through a real-valued continuous function is a *compact interval*.
 - Thus, continuity of r_i (see previous lemma) yields $r_i(A_{-i}) = [\underline{a}_i^*, \bar{a}_i^*]$, where $\underline{a}_i^* := \min r_i(A_{-i})$, $\bar{a}_i^* := \max r_i(A_{-i})$.
 - If $\underline{a}_i \leq a_i < \underline{a}_i^*$, for every $a_{-i} \in A_{-i}$, $a_i < \underline{a}_i^* \leq r_i(a_{-i})$ and strict q-concavity yields $u_i(a_i, a_{-i}) < u_i(\underline{a}_i^*, a_{-i})$, because $u_i(\cdot, a_{-i})$ is strictly increasing on $[\underline{a}_i, r_i(a_{-i})]$; hence, a_i is dominated by $\underline{a}_i^*$.
 - A similar argument shows that if $\bar{a}_i^* < a_i \leq \bar{a}_i$ then a_i is dominated by $\bar{a}_i^*$. ■

Nice Games

Point rationalizability

- Looking at the iterated elimination of actions that are not justified by *deterministic* conjectures, we obtain the “**point rationalizability**” concept (conjectures are concentrated on one point). This is *just an auxiliary analytical tool* with no conceptual justification. It is typically easy to calculate and it yields a subset of the rationalizable set.
- Recall that $\mathcal{C}$ is the collection of **closed** (hence, **compact**) Cartesian subsets of $A := \times_{i \in I} A_i$. For each $C \in \mathcal{C}$, let

$$r(C) := \times_{i \in I} r_i(C_{-i}).$$

Since r_i is continuous and C_{-i} is closed, $r_i(C_{-i})$ is closed, for each i ; hence, $r(C)$ is closed and $r(C) \in \mathcal{C}$. Thus, $C \mapsto r(C)$ is a *self-map* on $\mathcal{C}$ and we can define r^k in the usual way (r^0 is the identity on $\mathcal{C}$, $r^k = r \circ r^{k-1}$ for $k \in \mathbb{N}$).

- Remark.** $r : \mathcal{C} \rightarrow \mathcal{C}$ is *monotone*.

Nice Games

Iterated dominance by pure actions

- Similarly, we can look at the iterated elimination of pure actions *dominated by pure actions* by iterating the operator ND_p defined (for each $C \in \mathcal{C}$) by $\text{ND}_{i,p}(C) :=$

$$:= \{ \bar{a}_i \in C_i : \forall a_i \in C_i, \exists a_{-i} \in C_{-i}, u_i(\bar{a}_i, a_{-i}) \geq u_i(a_i, a_{-i}) \},$$

$$\text{ND}_p(C) := \times_{i \in I} \text{ND}_{i,p}(C).$$

- It can be shown that each $\text{ND}_{i,p}(C)$ is a closed set. Hence, $\text{ND}_p(C) \in \mathcal{C}$, and $C \mapsto \text{ND}_p(C)$ is a self-map on $\mathcal{C}$. Thus, we can define $(\text{ND}_p)^k = \text{ND}_p \circ (\text{ND}_p)^{k-1}$ for every $k \in \mathbb{N}$ (where $(\text{ND}_p)^0$ is the identity on $\mathcal{C}$).
- Like ND also ND_p is a “restriction operator”: $\text{ND}_p(C) \subseteq C$ (by definition), but it is *not* a *monotone* operator (show it by example).

Nice Games

Rationalizability and iterated dominance

- The following noteworthy result simplifies the computation of the rationalizable set, showing that it is independent of risk attitudes (if the outcome function g is deterministic). In particular, *rationalizability coincides with point rationalizability and with the iterated elimination of pure actions dominated by pure actions.*

Theorem

(Nice Rationalizability) In every nice game, for every $k \in \mathbb{N}_0 \cup \{\infty\}$, $r^k(A)$ is a product of compact intervals and

$$r^k(A) = \rho^k(A) = \text{ND}^k(A) = (\text{ND}_\rho)^k(A).$$

Proof of Nice Rationalizability Theorem

- **Proof.** For each $C \in \mathcal{C}$, define the “deterministic-conjecture analog” of $\bar{\rho}(C)$: $\bar{r}(C) := \times_{i \in I} \bigcup_{a_{-i} \in C_{-i}} \arg \max_{a_i \in C_i} u_i(a_i, a_{-i})$. As

we did for ρ and $\bar{\rho}$, one can show that $\bar{r}^k(A) = r^k(A)$ for every k . Next, we prove the theorem by induction.

- *Basis.* Since $\gamma^0(A) = A$ for each $\gamma \in \{r, \rho, \text{ND}, \text{ND}_\rho\}$, the result holds trivially for $k = 0$.
- *Inductive step.* Suppose that $r^k(A)$ is a product of compact intervals and $r^k(A) = \rho^k(A) = \text{ND}^k(A) = (\text{ND}_\rho)^k(A)$ (IH). Then, the restricted game with constrained set of action profiles $C = \bar{r}^k(A) = r^k(A)$ is a *nice* game; thus, the previous Lemma and (IH) yield

$$\bar{r}(r^k(A)) = \bar{\rho}(\rho^k(A)) = \text{ND}(\text{ND}^k(A)) = \text{ND}_\rho((\text{ND}_\rho)^k(A)),$$

where $\bar{r}(r^k(A))$ is a product of compact intervals. Since $\bar{r}^{k+1}(A) = r^{k+1}(A)$, $\bar{\rho}^{k+1}(A) = \rho^{k+1}(A)$, and $\gamma^{k+1} = \gamma \circ \gamma^k$ for each $\gamma \in \{r, \bar{r}, \rho, \text{ND}, \text{ND}_\rho\}$, we obtain

$$r^{k+1}(A) = \rho^{k+1}(A) = \text{ND}^{k+1}(A) = (\text{ND}_\rho)^{k+1}(A). \blacksquare$$

Rationalizability in the Cournot oligopoly

- The game is *nice* and we can *apply the previous results*. By symmetry, each firm has the same BR function, $r : [0, \bar{q}]^{n-1} \rightarrow [0, \bar{q}]$ ($n = |I|$ is the number of firms). Also, $r(0, \dots, 0) = q^M$, and $r(q, \dots, q) = 0$ for each q s.t. $P((n-1)q) \leq c$. With this, $r(C) = \times_{i \in I} r(C_{-i})$ for every $C \in \mathcal{C}$.
- For $n = 2$, $r : [0, \bar{q}] \rightarrow [0, \bar{q}]$ is a *self-map*, and it can be iterated, with $r^0(0) = 0$, $r^1(0) = r(0) = q^M$, $r^2(0) = r(r^1(0)) = r(q^M)$, etc. With this

$$\rho^{2\ell+1}(A) = \left[r^{2\ell}(0), r^{2\ell+1}(0) \right]^2, \ell = 0, 1, \dots \text{ (odd iterations)}$$

$$\rho^{2\ell}(A) = \left[r^{2\ell}(0), r^{2\ell-1}(0) \right]^2, \ell = 1, 2, \dots \text{ (even iterations)}$$

$$\lim_{\ell \rightarrow \infty} r^{2\ell}(0) \nearrow q^* \nwarrow \lim_{\ell \rightarrow \infty} r^{2\ell+1}(0), \text{ with } q^* = r(q^*) \text{ symm. NE}$$

- For $n \geq 3$, $\forall k \in \mathbb{N} \cup \{\infty\}$, $\rho^k(A) = [0, q^M]^n$.

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