

Mixed Equilibrium

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Abstract

I start with a heuristic motivating example. The key result is the *Characterization Theorem* stating that a mixed equilibrium is a profile α^* of mixed actions such that, for each i , the support of α_i^* is included in the set of pure best replies to the co-players' mixed actions α_{-i}^* . The example clarifies the interpretation of this result: *a mixed equilibrium represents a stationary state of populations dynamics, with mixed actions describing statistical distributions of actions in populations rather than randomizations purposefully chosen by players*. It is shown that actions played with positive probability in some mixed equilibrium are rationalizable, hence, iteratively undominated. This and the Characterization Theorem yield an algorithm to compute mixed equilibria of finite 2-person games.

[These slides summarize Sections 61.1.-6.1.4 of Ch. 6 of “Game Theory: Analysis of Strategic Thinking” (GT-AST) devoted to the mixed (Nash) equilibrium concept. You should first read Section 4 of Ch. 5 on the interpretations of the Nash equilibrium concept.]

Heuristic example: owners and thieves

[See also alternative example and interpretation: “Hawk-Dove” game in Evolutionary Game Theory; e.g., Weibull (1995).]

- Interaction between n **owners** and n **thieves**, with n *large*. In each period, thieves are *matched at random* with owners: the probability that a particular thief (say, Mr. Lupin) finds himself near the home of a particular owner (say, Mr. Smith) is $1/n$.
- Each owner has objects that, upon burglary, can be stolen for a total value of V . He can activate his *Alarm* system at cost $c < V$, or *Not*, without knowing whether burglary will be attempted.
- Each thief can attempt *Burglary* or *Not*, without knowing if the alarm is on. If he does and the alarm is off, he steals the valuable objects and resells them for $V/2$; if the alarm is on, he cannot steal and incurs an expected penalty of $P/2$.

Owners and thieves: the game

- Assuming risk neutrality, each matched pair faces the following game (note, it turns out that it is not important to know the payoff function of the other):

$O \backslash T$	<i>Burglary</i> (β)	<i>No</i> ($1 - \beta$)
<i>Alarm</i> (α)	$V - c, -P/2$	$V - c, 0$
<i>No</i> ($1 - \alpha$)	$0, V/2$	$V, 0$

- Let
 - α =fraction of owners activating the *Alarm* ($0 \leq \alpha \leq 1$);
 - β =fraction of thieves attempting *Burglary* ($0 \leq \beta \leq 1$).
- Previous-period statistics $(\alpha_{t-1}, \beta_{t-1})$ are recorded and published.
- Viscosity*: In each period t , only a few **active** owners and thieves determined at random look at the statistics $(\alpha_{t-1}, \beta_{t-1})$ of the previous period and consider changing their choice; when they do and are indifferent, they keep the same choice. *Such statistics determine the current-period expectations of active agents.*

Owners and thieves: the dynamics

- An *active owner*

- compares the safe choice (*Alarm*), which is worth $V - c$ to the risky one (*No*), whose expected payoff is $(1 - \beta) V$.
- He chooses *Alarm* (resp. *No Alarm*) at t if $\beta_{t-1} > c/V$ (resp. $\beta_{t-1} < c/V$), and keeps the same choice of the previous period if $\beta_{t-1} = \beta^* := c/V$ (note, $0 < \beta^* < 1$).
- Thus, $\alpha_t > \alpha_{t-1}$ if $\beta_{t-1} > \beta^*$, $\alpha_t < \alpha_{t-1}$ if $\beta_{t-1} < \beta^*$, and $\alpha_t = \alpha_{t-1}$ if $\beta_{t-1} = \beta^*$.

- An *active thief*

- compares the safe choice (*No Burglary*), which is worth 0, with the risky one (*Burglary*), whose expected payoff is $(1 - \alpha) V/2 - \alpha P/2$.
- He attempts *Burglary* (resp. does *Not*) at t if $\alpha_{t-1} < V/(V + P)$ (resp. $\alpha_{t-1} > V/(V + P)$), and keeps the same choice of the previous period if $\alpha_{t-1} = \alpha^* := V/(V + P)$ (note, $0 < \alpha^* < 1$).
- Thus, $\beta_t > \beta_{t-1}$ if $\alpha_{t-1} < \alpha^*$, $\beta_t < \beta_{t-1}$ if $\alpha_{t-1} > \alpha^*$, and $\beta_t = \beta_{t-1}$ if $\alpha_{t-1} = \alpha^*$.

Owners and thieves: steady state

- From the previous intuitive description of the dynamics of (α_t, β_t) , the state of the system, we see that the **steady state** (rest point), or **mixed equilibrium**, $(\alpha^*, \beta^*) = (V / (V + P), c / V)$ has the following features:
 - α^* is the (expected) fraction of owners activating the Alarm that makes (active) *thieves indifferent between the actions played by a positive fraction of them* (in this elementary example, all actions);
 - β^* is the fraction of thieves attempting Burglary that makes (active) *owners indifferent between the actions played by a positive fraction of them* (in this elementary example, all actions).
- **Take home message:**
 - *A mixed equilibrium describes steady-state statistics*, or frequency distributions, $(\alpha_i^*, \alpha_{-i}^*) \in \Delta(A_i) \times \Delta(A_{-i})$.
 - In 2-person games, if each action of each player/role is played by a positive fraction of agents in that role, then α_i^* (resp. α_{-i}^*) solves the indifference condition of $-i$ (resp. i). Thus, α_i^* (resp. α_{-i}^*) is *not chosen by any individual: agents do not randomize*.

Mixed extension of a game

Definition

The **mixed extension** of a *finite* game $G = \langle I, (A_i, u_i)_{i \in I} \rangle$ is the game $\bar{G} = \langle I, (\Delta(A_i), \bar{u}_i)_{i \in I} \rangle$ where, for all $i \in I$ and $\alpha \in \times_{j \in I} \Delta(A_j)$,
$$\bar{u}_i(\alpha) = \sum_{a \in A} u_i(a) \underbrace{\prod_{j \in I} \alpha_j(a)}_{\mathbb{P}_\alpha(a)}.$$

• Note:

- Expected payoffs are computed under the assumption that the actions of different players are *statistically independent*.
- Each function $\bar{u}_i : \times_{j \in I} \Delta(A_j) \rightarrow \mathbb{R}$ is *continuous* and *multi-affine*, that is, affine (concave and convex) in each variable α_j .
- The definition can be extended to “non pathological” infinite games, e.g., compact-continuous games, letting $\bar{u}_i(\alpha) = \mathbb{E}_{\times_{j \in I} \alpha_j}(u_i)$, where $\times_{j \in I} \alpha_j$ is the product measure on A [$\times_{j \in I} \alpha_j$ is such that $\forall C \in \mathcal{C}$, $(\times_{j \in I} \alpha_j)(C) = \prod_{j \in I} \alpha_j(C_j)$].

Mixed equilibrium: definition and existence

Definition

A mixed action profile $\alpha = (\alpha_i)_{i \in I}$ is a **mixed** (Nash) **equilibrium** of a game G if it is a Nash equilibrium (NE) of the mixed extension $\bar{G}$.

- **Note:** Pure equilibria are a special, degenerate kind of mixed equilibria (check that you understand why).
- Many games have no pure equilibria (e.g., Matching Pennies, Rock-Scissor-Paper, ...), but all finite games have mixed equilibria.

Theorem

(Existence) *Every finite game G has at least one mixed equilibrium.*

- **Proof:**

- We must show that the mixed extension $\bar{G}$ has an NE.
- For each i , (1) $\Delta(A_i) \subseteq \mathbb{R}^{A_i}$ is (nonempty) compact and convex;
(2) $\bar{u}_i(\alpha)$ is continuous in α and affine (hence concave) in α_i . Thus, $\bar{G}$ satisfies the sufficient conditions for existence of an NE.

Mixed equilibrium: characterization

- **Recall:** $A_i \subseteq \Delta(A_i)$, $r_i(\mu^i) := A_i \cap \arg \max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i)$, and $\alpha_i^* \in \arg \max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i)$ IFF $\text{supp} \alpha_i^* \subseteq r_i(\mu^i)$ (**Lemma 1** in GT-AST).
- In two-person games, conjectures and mixed actions of the co-players are represented by the same mathematical objects, the elements of $\Delta(A_{-i})$.
- It follows that, in two-person games, α^* is a mixed equilibrium IFF $\text{supp} \alpha_i^* \subseteq r_i(\alpha_{-i}^*)$ for each i .
- More generally, write $r_i(\alpha_{-i}) = r_i(\times_{j \neq i} \alpha_j)$, where $\times_{j \neq i} \alpha_j \in \Delta(A_{-i})$ is the product measure on A_{-i} obtained from α_{-i} under statistical independence. *Lemma 1* (GT-AST) yields:

Theorem

(Characterization) In every finite (or compact-continuous) game G , for every mixed action profile $\alpha^* = (\alpha_i^*)_{i \in I}$, α^* is a mixed equilibrium of G IFF $\text{supp} \alpha_i^* \subseteq r_i(\alpha_{-i}^*)$ for each $i \in I$.

Mixed equilibrium and rationalizability

- The **Characterization Theorem** and **Theorem 3** of GT-AST imply that *the actions profiles played with positive probability in a mixed Nash equilibrium are rationalizable, hence iteratively undominated*:

Theorem

In every finite (or compact-continuous) game G , for every mixed equilibrium α^ of G , $\times_{i \in I} \text{supp} \alpha_i^* \subseteq \rho^\infty(A) = \text{ND}^\infty(A)$.*

- **Proof:**

- Recall, by Theorem 3 (GT-AST), for every $C \in \mathcal{C}$, $C \subseteq \rho(C) \Rightarrow C \subseteq \rho^\infty(A)$.
- Fix a mixed equilibrium α^* and let $C = \times_{i \in I} \text{supp} \alpha_i^*$, we prove $C \subseteq \rho(C)$. Indeed, by Lemma 1 (GT-AST) $C_i = \text{supp} \alpha_i^* \subseteq r_i(\alpha_{-i}^*)$. Also, $\text{supp}(\times_{j \neq i} \alpha_j^*) = \times_{j \neq i} \text{supp}(\alpha_j^*) = C_{-i}$. Thus, $r_i(\alpha_{-i}^*) \subseteq r_i(\Delta(C_{-i}))$ for each $i \in I$, and $C \subseteq \times_{i \in I} r_i(\Delta(C_{-i})) = \rho(C)$.

Computation of the set of mixed equilibria: 2 players

- The set of mixed equilibria of *finite 2-person games* can be computed by solving a sequence of linear programming (LP) problems.
- **First.** Iteratively delete (in any order) the dominated actions (checking whether an action in a finite game is dominated is an LP problem) and obtain $A^* := \text{ND}^\infty(A) = \rho^\infty(A)$. By Theorem 3 (GT-AST), the search for mixed equilibria of G can be limited to the restricted game $G^* = \langle I, (A_i^*, u_i^*)_{i \in I} \rangle$, where $u_i^* = u_i|_{A^*}$ is the restriction of u_i to A^* .
- **Second.** By the Characterization Theorem, for every nonempty Cartesian subset $C \subseteq A^*$, α^C is a mixed equilibrium of G^* (and G) with $\text{supp} \alpha_i^C = C_i$ for each $i \in I$ IFF for some $(y_i^C)_{i \in I} \in \mathbb{R}^I$ and each i , α_{-i}^C solves the following *system of linear equalities and inequalities*:
 - $\forall a_i \in C_i, \sum_{a_{-i} \in C_{-i}} \alpha_{-i}(a_{-i}) u_i(a_i, a_{-i}) = y_i^C$ (**indifference cond.**)
 - $\forall a'_i \in A_i^* \setminus C_i, \sum_{a_{-i} \in C_{-i}} \alpha_{-i}(a_{-i}) u_i(a'_i, a_{-i}) \leq y_i^C$ (**incentive cond.**)

Computation of the set of mixed equilibria: comments

- **Note:**

- **(The co-player does not care!)** The indifference and incentive conditions of i are solved w.r.t. the mixed action α_{-i} of the *co-player* $-i$. But the co-player could not care less about choosing a mixed action satisfying such conditions! Go back to the heuristic example to interpret this.
- **(Number of equilibria)** Let $I = \{1, 2\}$. There are at most $(2^{|A_1^*|} - 1) \times (2^{|A_2^*|} - 1)$ Cartesian sets C supporting mixed equilibria. Except for “non-generic” games, this is an upper bound on the number of mixed equilibria (an indifference between payoffs, or linear combinations of payoffs, may yield a continuum of equilibria; think of examples).
- **(Complexity)** The search for mixed equilibria is exponentially complex in the number of (rationalizable) actions.

Numerical example: two pure equilibria, one mixed

Payoffs of the **row pl.** (*column pl.*) in **bold** (*Italics*). First delete *b*, then delete *r*, then compute equilibria.

	<i>ℓ</i>	<i>c</i>	<i>r</i>
<i>t</i>	4, 2	1, 1	4, 1
<i>m</i>	1, 1	2, 4	1, 2
<i>b</i>	2, 0	1, 0	0, 1

→

	<i>ℓ</i>	<i>c</i>	<i>r</i>
<i>t</i>	4, 2	1, 1	4, 1
<i>m</i>	1, 1	2, 4	1, 2

→

	<i>ℓ</i> (λ)	<i>c</i>
<i>t</i> (τ)	4, 2	1, 1
<i>m</i>	1, 1	2, 4

→

mixed	Nash	eq.
$\{(\tau^*, \lambda^*) = (\frac{1}{4}, \frac{3}{4}),$	$(t, \ell),$	$(m, c)\}$

indifference of row: $4\lambda + (1 - \lambda) = \lambda + 2(1 - \lambda) \Rightarrow \lambda^* = 1/4$.

indifference of col.: $2\tau + (1 - \tau) = \tau + 4(1 - \tau) \Rightarrow \tau^* = 3/4$.

Numerical example: one mixed equilibrium, no pure NE

Payoffs of the **row pl.** (*column pl.*) in **bold** (*Italics*). First delete *b*, then delete *r*, then compute equilibria.

	<i>ℓ</i>	<i>c</i>	<i>r</i>
<i>t</i>	4, 1	1, 2	4, 1
<i>m</i>	1, 4	2, 1	1, 2
<i>b</i>	2, 0	1, 0	0, 1

→

	<i>ℓ</i>	<i>c</i>	<i>r</i>
<i>t</i>	4, 1	1, 2	4, 1
<i>m</i>	1, 4	2, 1	1, 2

→

	<i>ℓ</i> (λ)	<i>c</i>
<i>t</i> (τ)	4, 1	1, 2
<i>m</i>	1, 4	2, 1

→ mixed eq.
 $(\tau^*, \lambda^*) = (\frac{1}{4}, \frac{3}{4})$

No pure NE.

indifference of row: $4\lambda + (1 - \lambda) = \lambda + 2(1 - \lambda) \Rightarrow \lambda^* = 1/4$.

indifference of col.: $\tau + 4(1 - \tau) = 2\tau + (1 - \tau) \Rightarrow \tau^* = 3/4$.

Mixed eq. and Maxmin in 2-person 0-sum games, I

- Consider a finite, 2-person, *constant-sum* game; w.l.o.g. we can assume it is 0-sum: $I = \{1, 2\}$, $u_1 = -u_2$.
- To ease notation, let $X = \Delta(A_1)$, $Y = \Delta(A_2)$, respectively denote the mixed action sets of pl. 1, pl. 2.
- With this, let and $V : X \times Y \rightarrow \mathbb{R}$ denote the expected payoff function of pl. 1: for all (x, y) ,

$$V(x, y) := \sum_{a_1 \in A_1} \sum_{a_2 \in A_2} x(a_1) u_1(a_1, a_2) y(a_2).$$

Mixed eq. and Maxmin in 2-person 0-sum games, II

- The proof of the following is left as an exercise:

Theorem

(Maxmin) *In a finite 2-person, 0-sum game, each player has a unique mixed equilibrium payoff, the equilibrium payoff of pl. 1, called “value” of the game V^* , which satisfies*

$$\max_{x \in X} \min_{y \in Y} V(x, y) = V^* = \min_{y \in Y} \max_{x \in X} V(x, y).$$

(Exchangeability) *for any two equilibria (x', y') and (x'', y'') , also (x', y'') and (x'', y') are equilibria.*

- It is the first important result in GT (independently proved). It still plays an important role in GT.
- In 0-sum games, it is *as if* the equilibrium were unique, and if each player optimizes *as if* the other player were “spying” on him, then a mixed equilibrium with a saddle-point property obtains.

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