

Static Games with Incomplete Information: Payoff Uncertainty

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Abstract

Some justifications of solution concepts make sense under the assumption that the rules of the game and players' personal preferences are common knowledge. A situation of strategic interaction features **incomplete information** when this is not the case. We represent this with **games with payoff uncertainty**, whereby the payoff functions depend on a vector of parameters about which players have partial and asymmetric knowledge. The game features **private values** if there is common knowledge of the outcome function, and **interdependent values** otherwise. It is relatively straightforward to extend rationalizability and pure self-confirming equilibrium to allow for payoff uncertainty.

[These slides summarize and complement parts of Sections 8.1-3 and 8.7 of Ch. 8 of GT-AST]

Introduction

- Whether a solution or equilibrium concept is consistent with incomplete information is a matter of *interpretation*. We must look at the conceptual *motivations*:
- Standard *rationalizability* (iterated deletion of strictly dominated actions) is explicitly motivated as representing the behavioral implications of rationality and common belief in rationality *under complete information* (common knowledge of the payoff functions).
- *Nash equilibrium* can be motivated as an “*obvious way to play the game*”: See deductive interpretation, and self-enforcing agreement interpretation. Also this makes sense *under the complete information assumption*.
- *Deductive interpretation of NE*: it makes sense when there is a unique rationalizable outcome, see above.

Incomplete Information and Self-Enforcing Agreements

- *Self-enforcing agreement interpretation of NE*: Again we need complete information (or maybe something “close” to it) in order to make sense of this interpretation.
- Consider the following game and the Pareto dominant agreement (t, ℓ) . The *agreement is self-enforcing if there is common belief* (or “almost common belief”) *that there is no incentive to deviate from (t, ℓ)* .

	ℓ	r
t	100,100	0,99
b	99,0	99,99

- Would Rowena (row player) play t if she is not sure of the payoff function of Colin (column player)? What if she is not sure that Colin is sure of her payoff function? What if...?
- [But **note**: as long as each player knows her/his payoff function, there is *no need to assume complete information to make sense of Nash equilibrium as description of rest points of adaptive processes*.]

Environments with Incomplete Information

- Rules of the game $\Rightarrow$ *outcome/consequence function* $g : A \rightarrow Y$.
- Each player $i \in I$ ranks (lotteries over) outcomes according to (the expectation of) a vNM utility function $v_i : Y \rightarrow \mathbb{R}$.
- In environments with **incomplete information** there is *lack of common knowledge of g* (outcome function) and/or $(v_i)_{i \in I}$ (personal preferences).
- Such situation can be described with *parameterized payoff functions*

$$u_i : \Theta \times A \rightarrow \mathbb{R},$$

with

- $\theta \in \Theta$ parameter affecting payoffs,

$$\theta = (\theta_0, (\theta_i)_{i \in I}) \in \Theta = \Theta_0 \times (\times_{i \in I} \Theta_i)$$

- $i \in I$ knows only θ_i = private information of i about payoffs.

Interpretation, Distributed Knowledge

- *Intuition*: it is common knowledge that $\theta \in \Theta$, Θ_i represents what is commonly believed possible about i 's traits known to him (e.g., tastes, abilities), the “larger” Θ_i the more uncertain are the other players about such traits.
- If Θ_i is a *singleton* ($i \in I$), that is, $\Theta_i = \{\bar{\theta}_i\}$, it means that *what i knows is common knowledge* (it is common knowledge that $\theta_i = \bar{\theta}_i$) and Θ_i can be neglected: indeed, $\Theta_0 \times \left(\times_{j \in I \setminus \{i\}} \Theta_j \right)$ and Θ have the same cardinality; hence, they are (intuitively) isomorphic.
- Θ_0 represents the *residual uncertainty* that would remain if the players could pool their private information.
- We often *focus* on the case where Θ_0 is a singleton: there is *no residual uncertainty* after pooling private information (in this case it is said that there is “**distributed knowledge**” of θ). Thus, we will often neglect Θ_0 .

We distinguish between the case of **private values**, where u_i depends only on θ_i , and **interdependent values**, where u_i may depend on the whole θ .

- **Private values:** *Common knowledge of outcome function g , but lack of common knowledge of preferences $(v_i)_{i \in I}$:*
 - (it is common knowledge that) each i knows his vNM utility function $v_i \Rightarrow$ parameterized representation $v_i : \Theta_i \times Y \rightarrow \mathbb{R}$.
 - Note: $\{w_i \in \mathbb{R}^Y : \exists \theta_i \in \Theta_i, w_i = v_{i, \theta_i}\}$ is the set of utility functions that each $j \neq i$ thinks i might have $\Rightarrow$ get

$$u_i(\theta_i, a) = v_i(\theta_i, g(a))$$

- Note: *under private values* we may assume w.l.o.g. that there is *distributed knowledge of θ* (Θ_0 singleton).

Interdependent Values

- **Interdependent values:** *lack of common knowledge of outcome function g , which may depend on θ_0 or on personal traits such as some players' "ability").*
 - *common knowledge of preferences $(v_i)_{i \in I}$ (simplest case) $\Rightarrow$ parameterized representation $g : \Theta \times A \rightarrow Y$; note: $\{\gamma \in Y^A : \exists \theta \in \Theta, \gamma = g_\theta\}$ is the set of possible outcome functions $\Rightarrow$ get*

$$u_i(\theta, a) = v_i(g(\theta, a)).$$

- *More generally, if neither the outcome function nor preferences are common knowledge, each v_i is parameterized by θ_i and*

$$u_i(\theta, a) = v_i(\theta_i, g(\theta, a)).$$

- *Interdependence:* The value for i depends on what j knows, e.g., a personal trait of j .

Example

Cournot oligopoly *model (quantity setting)*: firm $i = 1, \dots, n$ produces $q_i \geq 0$ units of homogeneous good

- ▶ Inverse demand $P(Q) = [\bar{p} + \theta_0 - Q]_+$ (with $[x]_+ := \max\{0, x\}$, $Q = \sum_{i=1}^n q_i$)
- ▶ Cost function of firm i : $C_i(q_i, \theta_i) = \theta_i q_i$, $0 \leq q_i \leq \bar{q}$ ($\bar{q}$ =common capacity)
- ▶ Common knowledge of risk neutrality *and of sets* $\Theta_0, \Theta_1, \dots, \Theta_n$
- ▶ Payoff of i : $u_i(\theta_0, \theta_i, q_1, \dots, q_n) = \left([\bar{p} + \theta_0 - \sum_{j=1}^n q_j]_+ - \theta_i \right) q_i$
- ▶ *There are private values and distributed knowledge of θ if there is common knowledge of market demand (Θ_0 singleton)*

Example

Team production: Team agents $i = 1, \dots, n$, i exerts effort $e_i \geq 0$

- ▶ *Cost of effort (in units of output) $C_i(e_i, k_i) = k_i e_i^2$, $k_i \in K_i \subseteq \mathbb{R}_+$*
- ▶ *Production function: $y = \prod_{i=1}^n e_i^{p_i}$, $p_i \in P_i \subseteq \mathbb{R}_+$*
- ▶ *$\theta_i = (k_i, p_i) \in K_i \times P_i = \Theta_i$*
- ▶ *Common knowledge of (output-)risk neutrality and of sets $\Theta_i = K_i \times P_i$*
- ▶ *Payoff function of i : $u_i(k_1, p_1, \dots, k_n, p_n, e_1, \dots, e_n) = \frac{1}{n} \prod_{j=1}^n e_j^{p_j} - k_i e_i^2$*
- ▶ *Private values iff sets $P_1, \dots, P_n$ are singletons (productivities are common knowledge), otherwise interdependent values*

Games with Payoff Uncertainty

- We can represent (simultaneous) strategic interaction under *incomplete information* with the mathematical structure

$$\hat{G} = \langle I, \Theta_0, (\Theta_i, A_i, u_i : \Theta \times A \rightarrow \mathbb{R})_{i \in I} \rangle;$$

it is assumed that the interactive situation represented by $\hat{G}$ is common knowledge. This is called **game with payoff uncertainty**; θ_i is called the **information-type** of i [special: payoff-type].

- *Interpretation:* θ_0 affects the payoffs of somebody (if $\theta'_0 \neq \theta''_0$, then $\exists i \in I, u_i(\theta'_0, \cdot) \neq u_i(\theta''_0, \cdot)$). But part, or all, of i 's private information θ_i *may be payoff irrelevant*. Yet even payoff-irrelevant information may be strategically relevant (e.g., θ_i may be the report to i by an art expert about the authenticity of a painting for sale).
- Take the obvious extension to payoff uncertainty of the definition of “**compact-continuous game**.” To extend “**nice game**,” add to the obvious properties the *convexity* (or connectedness) of each Θ_i .

Rationality and Common Belief in Rationality

- Games with payoff uncertainty are sufficient to describe certain aspects of strategic thinking, specifically, *rationality and common belief in rationality*.
- Write $B_i(E)$ for “ i believes E ” (with prob. 1), and $B(E) = \bigcap_{i \in I} B_i(E)$ for “everybody believes E ,” R_i for “ i is rational,” $R = \bigcap R_i$ for “everybody is rational.”
- What actions of i are consistent with R (rationality), $B(R)$ (mutual belief in rationality), $B(B(R))$, $B(B(B(R)))$... $R \cap CB(R)$?

Example: Incomplete Information on One Side

Example

Possible payoff functions given by the following tables. Player 1 (Rowena) knows θ while player 2 (Colin) does not ($\Theta \cong \Theta_1$)

$\hat{G}^1 :$	θ'	ℓ	r		θ''	ℓ	r
	t	4,0	2,1		t	2,0	0,1
	b	3,1	1,0		b	0,1	1,2

► $R_1 \Rightarrow [t \text{ if } \theta']$, because t dominates b given $\theta = \theta'$ (recall, Row. knows θ) $\Rightarrow (\theta', b)$ is inconsistent with rationality (delete).
► $R_2 \cap B_2(R_1) \Rightarrow r$, because $u_2(\theta, x, \ell) < u_2(\theta, x, r)$ for all $(\theta, x) \neq (\theta', b)$ (those consistent with R_1).
► $R_1 \cap B_1(R_2) \cap B_1(B_2(R_1)) \Rightarrow$ Row. picks best reply to r given θ
 $\Rightarrow [b \text{ if } \theta = \theta'']$.

Envelope Game

Example

Players 1 and 2 receive an envelope. Envelope of i contains θ_i Euros, with $\theta_i = 1, \dots, K$. Each player can offer to exchange (OE) by paying transaction cost $\varepsilon > 0$ (small). Exchange executed IFF both offer:

$\hat{G}^2$:

$a_i \backslash a_j$	OE	No
OE	$\theta_j - \varepsilon$	$\theta_i - \varepsilon$
No	θ_i	θ_i

Note: A rational player i offers to exchange only if she assigns positive probability to event $[\theta_j > \theta_i] \cap [a_j = \text{OE}]$. ▶ $R_i \Rightarrow [a_i = \text{No if } \theta_i = K]$ because OE is dominated in this case. ▶ $R_i \cap B_i(R_j) \Rightarrow [a_i = \text{No if } \theta_i = K - 1]$ because ... ▶ $R_i \cap B_i(R_j) \cap B_i(B_j(R_i)) \Rightarrow [a_i = \text{No if } \theta_i = K - 2]$ because ... ▶ It can be shown that:
 $R \cap \text{CB}(R) \Rightarrow (\forall \theta_i, a_i = \text{No given } \theta_i)$ (no-trade!).

Rationalizability in Games with Payoff Uncertainty

- To ease notation, assume *distributed knowledge*: $\Theta \cong \times_{i \in I} \Theta_i$.
- Given conjecture $\mu^i \in \Delta(\Theta_{-i} \times A_{-i})$ and private information $\theta_i \in \Theta_i$, let

$$r_i(\mu^i, \theta_i) := \arg \max_{a_i \in A_i} \mathbb{E}_{\mu^i}(u_{i, \theta_i, a_i})$$

where $u_{i, \theta_i, a_i} : \Theta_{-i} \times A_{-i} \rightarrow \mathbb{R}$ is the section of u_i at (θ_i, a_i) ; in the *finite support case*

$$\mathbb{E}_{\mu^i}(u_{i, \theta_i, a_i}) = \sum_{(\theta_{-i}, a_{-i}) \in \text{supp} \mu^i} u(\theta_i, \theta_{-i}, a_i, a_{-i}) \mu^i(\theta_{-i}, a_{-i})$$

- Let $C_i \subseteq \Theta_i \times A_i$ (with $\text{proj}_{\Theta_i} C_i = \Theta_i$); interpretation: set of “surviving” pairs (see previous examples); $C_{-i} = \times_{j \neq i} C_j$, $\mathcal{C}$ collection of (closed) Cartesian products.
- Define the (monotone) **rationalization operator** $\rho : \mathcal{C} \rightarrow \mathcal{C}$.

$$\begin{aligned} \rho_i(C_{-i}) &= \{(\theta_i, a_i) \in \Theta_i \times A_i : \exists \mu^i \in \Delta(C_{-i}), a_i \in r_i(\mu^i, \theta_i)\} . \\ \rho(C) &= \times_{i \in I} \rho_i(C_{-i}). \end{aligned}$$

Behavioral Implications of RCBR

Assumptions about behavior and beliefs	Implications for $(\theta_i, a_i)_{i \in I}$
R	$\rho(\Theta \times A)$
$R \cap B(R)$	$\rho^2(\Theta \times A)$
$R \cap B(R) \cap B^2(R)$	$\rho^3(\Theta \times A)$
...	...
$R \cap (\bigcap_{k=1}^m B^k(R))$	$\rho^{m+1}(\Theta \times A)$
...	...
$R \cap (\bigcap_{k=1}^{\infty} B^k(R)) = R \cap CB(R)$	$\rho^{\infty}(\Theta \times A)$

Theorem

If $\hat{G}$ is finite or compact-continuous, then

$$\rho^{\infty}(\Theta \times A) = \rho(\rho^{\infty}(\Theta \times A)) \quad \text{and} \quad \text{proj}_{\Theta} \rho^{\infty}(\Theta \times A) = \Theta.$$

Furthermore, for each $C \in \mathcal{C}$, $C \subseteq \rho(C)$ implies $C \subseteq \rho^{\infty}(\Theta \times A)$.

Comments on RCBR and Rationalizability

- The previous theorem extends Theorems 2 and 3 of GT-AST from games with complete information to games with incomplete information (payoff uncertainty):
- $\rho^\infty(\Theta \times A) = \rho(\rho^\infty(\Theta \times A))$ is the “fixed set property” of the rationalizable set: after countably many iterations there is no need to re-start the iterated deletion procedure.
- $\text{proj}_\Theta \rho^\infty(\Theta \times A) = \Theta$ means that, for every $(\theta_i)_{i \in I} \in \Theta$, the *set of rationalizable actions* for information-type θ_i is *not empty*.
- $C \subseteq \rho(C) \Rightarrow C \subseteq \rho^\infty(\Theta \times A)$ means that every (Cartesian) subset of $\Theta \times A$ with the Best Reply Property is included in the rationalizable set.

Justifiability and Dominance

- Fix Cartesian subset $C = \times_{i \in I} C_i$ with $C_i \subseteq \Theta_i \times A_i$. Let $C_{i,\theta_i} := \{a_i \in A_i : (\theta_i, a_i) \in C_i\}$ (section of set C_i at θ_i).

Definition

Mixed action α_i **dominates** a_i **given** θ_i **within** C , written $\alpha_i \gg_{(\theta_i, C)} a_i$, if $\text{supp} \alpha_i \subseteq C_{i,\theta_i}$ and

$$\forall (\theta_{-i}, a_{-i}) \in C_{-i}, \quad u_i(\theta_i, \theta_{-i}, \alpha_i, a_{-i}) > u_i(\theta_i, \theta_{-i}, a_i, a_{-i}).$$

Lemma

Fix a finite or compact-continuous $\hat{G}$; let $C = \times_{i \in I} C_i$ be non-empty and compact. For all $i \in I$ and $(\theta_i, a_i^*) \in C_i$ the following are equivalent:

- (1) $\nexists \alpha_i$ s.t. $\alpha_i \gg_{(\theta_i, C)} a_i^*$ (a_i^* undominated given θ_i within C)
- (2) $\exists \mu^i \in \Delta(C_{-i})$ s.t. $a_i^* \in \arg \max_{a_i \in C_{i,\theta_i}} \mathbb{E}_{\mu^i}(u_{i,\theta_i,a_i})$.

Iterated Dominance

- The previous result (with $C = \Theta \times A$) extends the Wald-Pearce Lemma on justifiability and dominance to simultaneous-moves games with incomplete information (payoff uncertainty).
- For each $C \in \mathcal{C}$ ($\forall i \in I, C_i \subseteq \Theta_i \times A_i$), define $\text{ND}(C)$ as follows:

$$\begin{aligned}\text{ND}_i(C) &= C_i \setminus \left\{ (\theta_i, a_i) \in C_i : \exists \alpha_i \in \Delta(C_{i,\theta_i}), \alpha_i \gg_{(\theta_i, C)} a_i \right\}, \\ \text{ND}(C) &= \times_{i \in I} \text{ND}_i(C).\end{aligned}$$

- Similarly, dominance by pure actions (very relevant for nice games) gives

$$\begin{aligned}\text{ND}_{p,i}(C) &= C_i \setminus \left\{ (\theta_i, a_i) \in C_i : \exists a'_i \in C_{i,\theta_i}, a'_i \gg_{(\theta_i, C)} a_i \right\}, \\ \text{ND}_p(C) &= \times_{i \in I} \text{ND}_{p,i}(C).\end{aligned}$$

Rationalizability and Iterated Dominance

Extension of the equivalence "*rationalizable IFF iteratively undominated*" to games with payoff uncertainty (r is the point-rationalization operator obtained with deterministic conjectures):

Theorem

If $\hat{G}$ is finite or compact-continuous, for all $m = 1, 2, \dots, \infty$,
$$\rho^m(\Theta \times A) = \text{ND}^m(\Theta \times A).$$

Theorem

If $\hat{G}$ is nice, point-rationalizability, rationalizability, and pure iterated dominance coincide, that is, for all $m = 1, 2, \dots, \infty$,
$$r^m(\Theta \times A) = \rho^m(\Theta \times A) = \text{ND}_p^m(\Theta \times A);$$

furthermore, the projections of these sets onto A ($\text{proj}_A \rho^m(\Theta \times A)$, $m \in \mathbb{N} \cup \{\infty\}$) are closed order-intervals (products of closed intervals).

Rationalizability in the Envelope Game

Example

Recall: The game is symmetric, the envelope of i contains $\notin \theta_i$, with $\theta_i = 1, \dots, K$. Each player can offer to exchange (OE) by paying a small transaction cost $\varepsilon > 0$. Exchange executed IFF both offer:

$\hat{G}^2$:

$a_i \backslash a_j$	OE	No
OE	$\theta_j - \varepsilon$	$\theta_i - \varepsilon$
No	θ_i	θ_i

Recall: A rational player i offers to exchange only if she assigns positive probability to $[\theta_j > \theta_i] \cap [a_j = OE]$. With this, for each i :
► OE is dominated for $\theta_i = K$; delete $(\theta_i, a_i) = (K, OE)$.
► Given that (K, OE) is deleted for j , OE is dominated for $\theta_i = K - 1$; delete $(K - 1, OE)$.
► ... Given that (θ_j, OE) is deleted for each $\theta_j \in \{K - k + 1, \dots, K\}$ ($1 \leq k < K$), OE is dominated for $\theta_i = K - k$; delete $(K - k, OE)$...
► $\rho^\infty(\Theta \times A) = \rho^K(\Theta \times A) = (\{1, \dots, K\} \times \{No\})^2$ (no type trades).

Rationalizability in a Duopoly with Incomplete Inform., I

Example

The Cournot model presented above (with $n = 2$) is a *nice game with payoff uncertainty*. Thus, look at best replies (B.R.) to *deterministic* conjectures.

Assume: $\theta_0 = 0$ commonly known, marg. cost $\theta_i \in [0, 1]$, $\bar{p} > 2$ (highest average cost much lower than $\bar{p}$), $\bar{q}$ large ($\bar{q} > \bar{p} - 1$).

The model has *private values* and is *symmetric*: $A_i = [0, \bar{q}]$, $\Theta_i = [0, 1]$, and each firm i 's payoff depends on θ_i and q_{-i} in the same way.

Hence, common B.R. function

$$r(\theta_i, q_{-i}) = \left[\frac{\bar{p} - \theta_i}{2} - \frac{1}{2}q_{-i} \right]_+$$

where $\frac{\bar{p} - \theta_i}{2} = r(\theta_i, 0)$ = monopolistic output for cost-type θ_i .

Example

(Cont.) Let $\underline{r}(q_{-i}) = \left[\frac{\bar{p}-1-q_{-i}}{2} \right]_+$, $\bar{r}(q_{-i}) = \left[\frac{\bar{p}-q_{-i}}{2} \right]_+$ be the B.R. functions of, respectively, the *least efficient* ($\theta_i = 1$) and *most efficient* ($\theta_i = 0$) cost-type (see picture at the end).

Look at min and max output at each rationalizability step $k \in \mathbb{N}_0$:

$\underline{q}(k) = \underline{r}(\bar{q}(k-1))$ and $\bar{q}(k) = \bar{r}(\underline{q}(k-1))$, with

$\underline{q}(0) = 0$, $\bar{q}(0) = \bar{q}$. Then, $\text{proj}_A \rho^k(\Theta \times A) = [\underline{q}(k), \bar{q}(k)]^2$ with

$$\underline{q}(1) = \underline{r}(\bar{q}) = \left[\frac{\bar{p}-1-\bar{q}}{2} \right]_+ = 0, \quad \bar{q}(1) = \bar{r}(0) = \frac{\bar{p}}{2},$$

$$\underline{q}(2) = \underline{r}\left(\frac{\bar{p}}{2}\right) = \frac{\bar{p}-2}{4}, \quad \bar{q}(2) = \bar{r}\left(\frac{\bar{p}-2}{4}\right) = \frac{\bar{p}}{2},$$

$$\underline{q}(3) = \underline{r}\left(\frac{\bar{p}}{2}\right) = \frac{\bar{p}-2}{4}, \quad \bar{q}(3) = \bar{r}\left(\frac{\bar{p}-2}{4}\right) = \frac{3\bar{p}+2}{8},$$

$$\underline{q}(4) = \underline{r}\left(\frac{3\bar{p}+2}{8}\right) = \frac{5\bar{p}-10}{16}, \quad \bar{q}(4) = \bar{r}\left(\frac{5\bar{p}-10}{16}\right) = \frac{3\bar{p}+2}{8},$$

Show: $\text{proj}_A \rho^\infty(\Theta \times A) = \left[\lim_{\ell \rightarrow \infty} (\underline{r} \circ \bar{r})^\ell(0), \bar{r}\left(\lim_{\ell \rightarrow \infty} (\underline{r} \circ \bar{r})^\ell(0)\right) \right]^2$

and compute $\underline{q}(\infty) = \underline{r}(\bar{q}(\infty))$, $\bar{q}(\infty) = \bar{r}(\underline{q}(\infty))$.

Contextual Restrictions on Beliefs

- In many applications it is plausible to assume that the context makes some restrictions on players' beliefs about θ **transparent**, i.e., true and commonly believed (see examples in the book).
- It may also make sense to assume that restrictions on beliefs about both θ *and* behavior (i.e., on conjectures) are transparent.
- Such restrictions may depend on the information-type θ_i . We represent them with a restricted set of conjectures $\Delta_{i,\theta_i} \subseteq \Delta(\Theta_{-i} \times A_{-i})$ (keep assuming distributed knowledge of θ) for each i and $\theta_i \Rightarrow$ profile of restricted sets $\Delta = (\Delta_{i,\theta_i})_{i \in I, \theta_i \in \Theta_i}$.
- Modified rationalization operator (monotone!): for each $C \in \mathcal{C}$,

$$\rho_{i,\Delta}(C_{-i}) = \{(\theta_i, a_i) : \exists \mu^i \in \Delta_{i,\theta_i} \cap \Delta(C_{-i}), a_i \in r_i(\mu^i, \theta_i)\},$$

$$\rho_{\Delta}(C) = \times_{i \in I} \rho_{i,\Delta}(C_{-i}).$$

Directed Rationalizability

- The transparent restrictions (represented by) Δ “direct” the rationalizability procedure toward some results. Hence, the approach is called “**directed rationalizability**.”
- Say that Δ represents **restrictions on exogenous beliefs** if, for every i and θ_i there is some $\bar{\Delta}_{i,\theta_i} \subseteq \Delta(\Theta_{-i})$ s.t.
$$\Delta_{i,\theta_i} = \left\{ \mu^i \in \Delta(\Theta_{-i} \times A_{-i}) : \text{marg}_{\Theta_{-i}} \mu^i \in \bar{\Delta}_{i,\theta_i} \right\}$$
only beliefs about exogenous θ_{-i} are restricted. Then, for every information-type, the set of rationalizable actions is nonempty (*finiteness* for simplicity).

Theorem

Fix a finite game with payoff uncertainty $\hat{G}$ and a profile Δ of restrictions about exogenous beliefs. Then

$$\rho_{\Delta}^{\infty}(\Theta \times A) = \rho_{\Delta}(\rho_{\Delta}^{\infty}(\Theta \times A)), \text{proj}_{\Theta} \rho_{\Delta}^{\infty}(\Theta \times A) = \Theta.$$

Furthermore, for each $C \in \mathcal{C}$, $C \subseteq \rho_{\Delta}(C)$ implies $C \subseteq \rho_{\Delta}^{\infty}(\Theta \times A)$.

Self-Confirming Equilibrium with Incomplete Info., I

- Intuitively, the *SCE* concept does not capture sophisticated strategic reasoning. Players' conjectures are only disciplined by long-run evidence. Again, to ease notation, assume $\Theta \cong \times_{i \in I} \Theta_i$ (distributed knowledge).
- As long as each player knows her payoff function (private values), we should get the same concept introduced in the previous lecture. We will make this formal.
- Feedback is modeled by functions $f_i : \Theta \times A \rightarrow M_i$ ($i \in I$). Recall that $f_{i,\theta_i,a_i} : \Theta_{-i} \times A_{-i} \rightarrow M_i$ is the **section** of f_i **at** (θ_i, a_i) . If i observes m_i given (θ_i, a_i) , she infers that the unknown profile (θ_{-i}, a_{-i}) must belong to the subset

$$f_{i,\theta_i,a_i}^{-1}(m_i) = \{(\theta'_{-i}, a'_{-i}) : f_i(\theta_i, a_i, \theta'_{-i}, a'_{-i}) = m_i\}.$$

Self-Confirming Equilibrium with Incomplete Info., II

Definition

Fix a game with payoff uncertainty $\hat{G}$ and a profile of feedback functions $f = (f_i : \Theta \times A \rightarrow M_i)_{i \in I}$. A profile of actions and conjectures $(a_i^*, \mu^i)_{i \in I}$ is a (pure) **self-confirming equilibrium of $(\hat{G}, f)$ at θ** if, for each $i \in I$, (1, B.R.) $a_i^* \in r_i(\mu^i, \theta_i)$ and (2, CONF) $\mu^i \left(f_{i, \theta_i, a_i^*}^{-1}(f_i(a^*, \theta)) \right) = 1$.

- For any fixed $\theta \in \Theta$, let $(\hat{G}_\theta, f_\theta) := \langle I, (A_i, u_{i, \theta} : A \rightarrow \mathbb{R}, f_{i, \theta} : A \rightarrow M_i)_{i \in I} \rangle$ (a game with feedback as in the previous lecture).
- **Remark** Fix $a^* \in A$ and $\theta^* \in \Theta$ arbitrarily; if $\hat{G}$ has private values, then a^* is part of an SCE of $(\hat{G}, f)$ at θ^* IF AND ONLY IF a^* is part of an SCE of $(\hat{G}_{\theta^*}, f_{\theta^*})$.
- **Proof** Let $(a_i^*, \mu^i)_{i \in I}$ be an SCE of $(\hat{G}, f)$ at θ^* . For each $i \in I$, let $\bar{\mu}^i = \text{marg}_{A_{-i}} \mu^i \in \Delta(A_{-i})$ (marginal of μ^i onto A_{-i}). Since each u_i is independent of θ_{-i} , $(a_i^*, \bar{\mu}^i)_{i \in I}$ must be an SCE of $(\hat{G}_{\theta^*}, f_{\theta^*})$. ■

Incomplete Information and Properties of Feedback

- Recall, $f_{i,\theta_i,a_i} : \Theta_{-i} \times A_{-i} \rightarrow M_i$ and $u_{i,\theta_i,a_i} : \Theta_{-i} \times A_{-i} \rightarrow \mathbb{R}$ are the **sections** of f_i and u_i at (θ_i, a_i) .
- Let $\mathcal{F}_{-i}(\theta_i, a_i)$ denote the “ex post information partition” of $\Theta_{-i} \times A_{-i}$ given (θ_i, a_i) :

$$\mathcal{F}_{-i}(\theta_i, a_i) = \left\{ C_{-i} \in 2^{\Theta_{-i} \times A_{-i}} : \exists m_i \in M_i, C_{-i} = f_{i,\theta_i,a_i}^{-1}(m_i) \right\}.$$

- $f_i : \Theta \times A \rightarrow M_i$ satisfies
 - own-action independence** of feedback about others (**OAI**) if $\mathcal{F}_{-i}(\theta_i, a'_i) = \mathcal{F}_{-i}(\theta_i, a''_i)$ for all θ_i and all a'_i, a''_i *justifiable* (undominated) for θ_i ;
 - observed payoffs (OP)** if $f_{i,\theta_i,a_i}(\theta'_{-i}, a'_{-i}) = f_{i,\theta_i,a_i}(\theta''_{-i}, a''_{-i}) \Rightarrow u_{i,\theta_i,a_i}(\theta'_{-i}, a'_{-i}) = u_{i,\theta_i,a_i}(\theta''_{-i}, a''_{-i})$ for all $\theta_i, a_i, \theta'_{-i}, a'_{-i}$ and θ''_{-i}, a''_{-i} (u_{i,θ_i,a_i} is constant on each cell of $\mathcal{F}_{-i}(\theta_i, a_i)$, for all θ_i, a_i).

- $(\hat{G}, f)$ satisfies OAI and OP if each f_i does ($i \in I$).
- **Examples:**
 - The Envelope Game and Cournot Game with $f_i = u_i$ for each $i \in I$ satisfy OP (obviously), but do *not* satisfy OAI.
 - The Cournot Game with known inverse demand function $P(\cdot)$ and $f_i(\theta, \cdot) = P(\cdot)$ for all i and θ satisfies OP and OAI.

Theorem

Suppose that $(\hat{G}, f)$ satisfies OAI and OP and fix $a^ \in A$ and $\theta^* \in \Theta$ arbitrarily; then a^* is part of an SCE at θ^* of $(\hat{G}, f)$ IF AND ONLY IF a^* is a Nash equilibrium of $\hat{G}_{\theta^*} = \langle I, (A_i, u_{i,\theta^*} : A \rightarrow \mathbb{R})_{i \in I} \rangle$.*

References

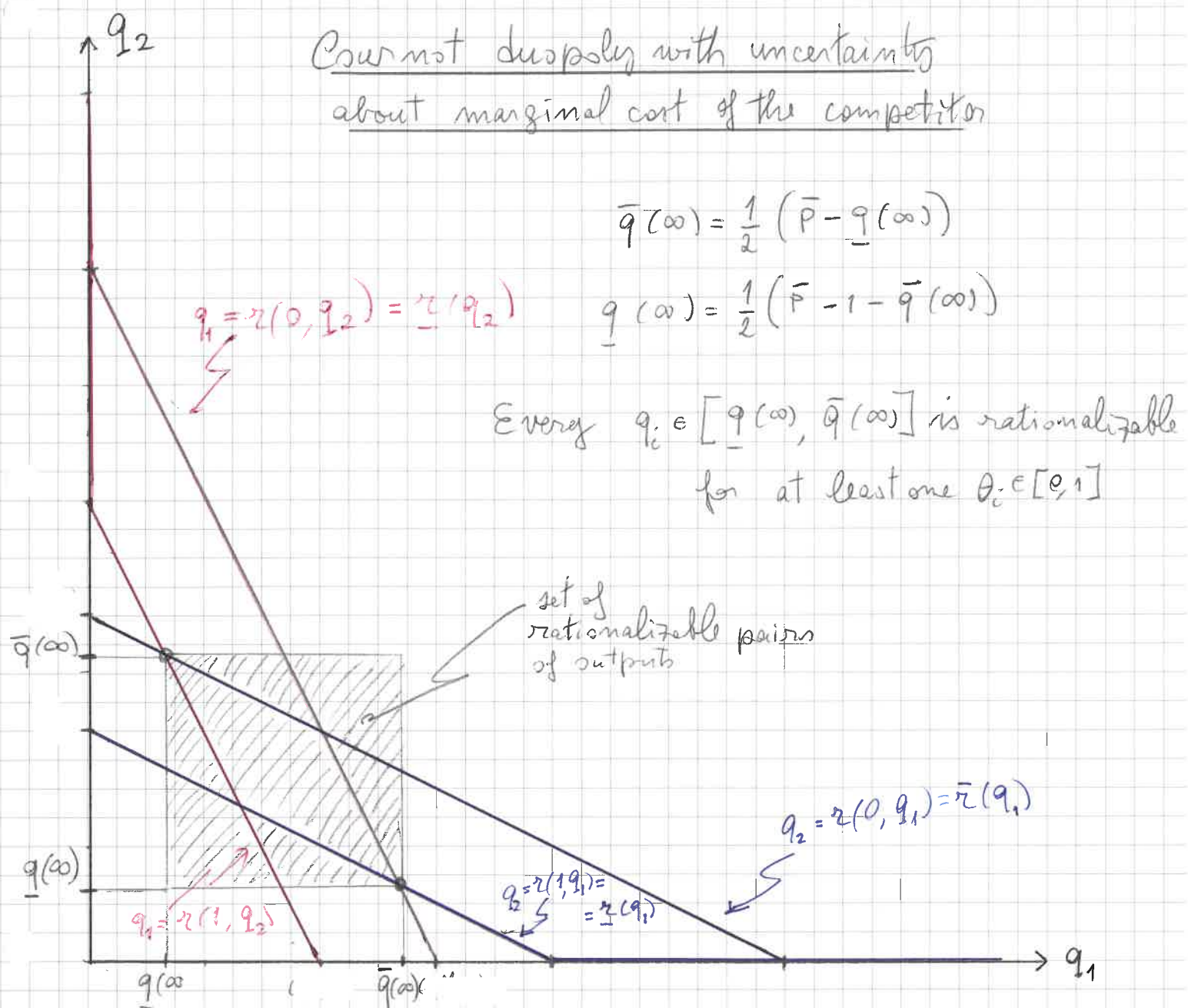


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Cournot duopoly with uncertainty about marginal cost of the competitor



θ_i = marg. cost of firm i (private information)

$\theta_i \in [0, 1]$ $\theta_i = 0$ most efficient type $\theta_i = 1$ least efficient type

Inverse demand $P(q_1 + q_2) = \max\{0, \bar{P} - (q_1 + q_2)\}$

Ex ante symmetric nice game with payoff uncertainty