

Bargaining with Alternating Offers

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Game Theory: Analysis of Strategic Thinking

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Abstract

We present three models of sequential bargaining: the **Ultimatum Game**, **Two-Period Bargaining with Alternating Offers**, and **Infinite-Horizon Bargaining with Alternating Offers**. The first two models can be analyzed with backward induction (of a kind). The subgame perfect equilibrium (SPE) of the infinite-horizon model is obtained *heuristically* relying on the analysis of the two-period model, and then using the OD principle to show that the solution so obtained is indeed an SPE. It turns out that this is the *unique SPE*.

[These slides summarize Chapter 14 of GT-AST]

Introduction

- *Bargaining is ubiquitous in economic life:*
- Economic interaction (e.g., in production and/or exchange) generates a surplus, and often *agents can commit to surplus maximizing behavior and on how to split the surplus.*
- **Bargaining** is the process of offers, replies, and—possibly—counteroffers; this process may terminate, or not, with such an agreement.
- Focusing on the case of a fixed potential surplus, we study bargaining between *two agents* according to a precise **alternating-offers protocol** (agents discount future consumption):
 - An **offer** of how to split is made.
 - If the offer is *accepted* the split is implemented and agents consume immediately their shares. If it is *rejected*, the process *moves on to the next period*.
 - In the next period, *either* agents consume *default* shares (exogenous end), *or* the *previous-period responder makes the next offer*.

Ultimatum Game: A Discrete Example

Example: size-1

surplus to split.

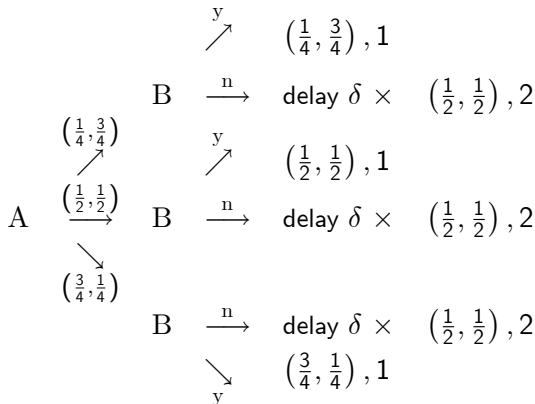
Game

form
of the

Ultimatum Game

with

0.25-increments



$t = 1$ $t = 2$, default split

- Ann offers (x_A, x_B) , Bob: y (immediate consumption), or n (delayed cons. of default shares). The BI-solution depends on δ .

Ultimatum Game with a Continuum of Offers: Model

- Set of **offers** $X = \left\{ (x_A, x_B) \in \mathbb{R}_+^{\{A,B\}} : x_A + x_B = 1 \right\}$.
- **Default split** $(\bar{x}_A, \bar{x}_B) \in X$ in case of *disagreement*.
- Set of **outcomes** $Y = X \times \{1, 2\}$, (x, t) = split x in period t .
- **Period** $t = 1$:
 - Ann proposes $(x_A, x_B) \in X$;
 - Bob replies yes (y), or no (n);
 - if y , immediate consumption of (x_A, x_B) [outcome $((x_A, x_B), 1)$];
 - if n , go to period $t = 2$.
- **Period** $t = 2$: consumption of default split $(\bar{x}_A, \bar{x}_B)$ [outcome $((\bar{x}_A, \bar{x}_B), 2)$].
- **Intertemporal preferences**: $((x_i, 1 - x_i), 2) \sim_i ((\delta x_i, 1 - \delta x_i), 1)$, with $\delta \in (0, 1)$ discount factor (common, just for simplicity).

Ultimatum Game with a Continuum of Offers: BI-Solution

- Bob:

- accepts (x_A, x_B) if $x_B > \delta \bar{x}_B$,
- rejects (x_A, x_B) if $x_B < \delta \bar{x}_B$,
- indifferent if $x_B = \delta \bar{x}_B$, 2 cases: y and n.

- Ann:

- **case-y**: Ann chooses (x_A, x_B) to maximize the value function

$$V_A^*(x_A, x_B) = \begin{cases} x_A & \text{if } x_A \leq 1 - \delta \bar{x}_B \\ \delta \bar{x}_A & \text{if } x_A > 1 - \delta \bar{x}_B \end{cases}$$

since $\delta \bar{x}_A = \delta(1 - \bar{x}_B) < 1 - \delta \bar{x}_B$, Ann offers
 $(x_A, x_B) = (1 - \delta \bar{x}_B, \delta \bar{x}_B)$.

- **case-n**: the value function is as above with $\leq$ (resp. $>$) replaced by $<$ (resp. $\geq$); hence, no maximum! *No SPE in this case!*

- **Comments:**

- *Non-existence* in **case-n** is just a technicality due to the continuum, it *disappears with discrete offers*.
- *To solve by BI, break responder's indifference (relevant tie) with yes.*

Alternating Offers with 2 Periods of Bargaining

- Set of **Outcomes** $Y = X \times \{1, 2, 3\}$,
 $((x_i, 1 - x_i), t + 1) \sim_i ((\delta x_i, 1 - \delta x_i), t)$.
- **Period** $t = 1$:
 - Ann proposes $x^1 = (x_A^1, x_B^1) \in X$;
 - Bob replies yes (y), or no (n);
 - if y, immediate consumption of x^1 [outcome $(x^1, 1)$];
 - if n, go to period $t = 2$.
- **Period** $t = 2$:
 - Bob proposes $x^2 = (x_A^2, x_B^2) \in X$;
 - Ann replies yes (y), or no (n);
 - if y, immediate consumption of x^2 [outcome $(x^2, 2)$];
 - if n, go to period $t = 3$.
- **Period** $t = 3$: consumption of default split $\bar{x} = (\bar{x}_A, \bar{x}_B)$ [outcome $(\bar{x}, 3)$].

Alternating Offers with 2 Periods: BI-Solution

• Period 2:

- Ann accepts iff $x_A^2 \geq \delta \bar{x}_A$;
- Bob chooses (x_A^2, x_B^2) to maximize

$$V_B^*((x_A^1, x_B^1), n, (x_A^2, x_B^2)) = \begin{cases} x_B^2 & \text{if } x_B^2 \leq 1 - \delta \bar{x}_A \\ \delta \bar{x}_B & \text{if } x_B^2 > 1 - \delta \bar{x}_A \end{cases}$$

and, thus, offers $(\hat{x}_A^2, \hat{x}_B^2) = (\delta \bar{x}_A, 1 - \delta \bar{x}_A)$.

• Period 1:

- Bob accepts iff $x_B^1 \geq \delta \hat{x}_B^2 = \delta (1 - \delta \bar{x}_A)$, which is the present value of going to $t = 2$;
- Ann chooses (x_A^1, x_B^1) to maximize $V_A^*(x_A^1, x_B^1)$; thus, she offers $(\hat{x}_A^1, \hat{x}_B^1) = (1 - \delta (1 - \delta \bar{x}_A), \delta (1 - \delta \bar{x}_A))$.

• Comments

- In the unique SPE, *agreement is reached immediately* due to impatience and complete information (knowledge of the stingiest acceptable offer).
- *Impatience* (low δ) *yields a first-mover advantage*.

Comparison Between SPE and Rationalizability

- We defined strong & backward rationalizability for *finite* multistage games: consider a *finite*, but *fine-grained grid of offers*:

$$X_k = \left\{ x \in X : x_A \in \left\{ 0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1 \right\} \right\} \quad (\text{e.g., } k = 100).$$

There may be *multiple SPEs* if responders may be indifferent (*relevant ties*), which depends of δ and k ; but they are close to each other: *offers of different SPEs can differ by $1/k$ at most*.

- For every SPE path the *first offer is accepted* and the *responder accepts if indifferent*.
- If there are *no relevant ties* (NRT), there is a unique SPE obtained by BR/BI with path $\hat{z} = ((\hat{x}_A^1, \hat{x}_B^1), y)$, and *this is also the unique strongly rationalizable path* (because $\text{PI} \ \& \ \text{NRT} \Rightarrow \zeta(S^\infty) = \{\hat{z}\}$).
- If there are *relevant ties* and there is an SPE path $((\hat{x}_A^1, \hat{x}_B^1), y)$ so that Bob is *indifferent*, then *rationalizability allows for delay*, i.e., also $((\hat{x}_A^1, \hat{x}_B^1), n, \dots)$, because *rationalizability does not require an indifferent responder to accept*.

Alternating Offers with Infinite Horizon: Model

- Set of **outcomes** $Y = \{d\} \cup (X \times \mathbb{N})$, $d = \text{permanent disagreement}$, $((x_A, x_B), t) = ((0, 1), t) \sim_A d$ (similarly for B).
- **Period** t odd ($t = 1, 3, \dots$):
 - Ann proposes $x^t = (x_A^t, x_B^t) \in X$;
 - Bob replies yes (y), or no (n);
 - if y, immediate consumption of x^t [outcome (x^t, t)];
 - if n, go to period $t + 1$ (even).
- **Period** $t + 1$ even ($t + 1 = 2, 4, \dots$):
 - Bob proposes $x^{t+1} = (x_A^{t+1}, x_B^{t+1}) \in X$;
 - Ann replies yes (y), or no (n);
 - if y, immediate consumption of x^{t+1} [outcome $(x^{t+1}, t + 1)$];
 - if n, go to period $t + 2$ (odd).
- **Period** $t + 2$ (odd): the bargaining protocol re-starts with Ann proposing (no default split in case of disagreement, keep on bargaining if n).

Alternating Offers with Infinite Horizon: Heuristic Solution

- The model is **stationary**: from every history h at which i has to propose, the “same” infinite-horizon bargaining game starts (similarly, from every history (h, x) at which i has to respond to x , the “same” infinite-horizon subgame starts).
- With this, it is reasonable to *look for a stationary* solution $(\hat{s}_A, \hat{s}_B)$ such that, if i has to propose at h , s/he demands $\hat{x}_i$ for her/himself and the offer is accepted. (In this case, the value for the responder $-i$ of saying no is $\delta \hat{x}_{-i}$, since s/he is the next period proposer and expects to get $\hat{x}_{-i}$.)
- Thus, replace the default split $(\bar{x}_A, 1 - \bar{x}_A)$ of the 2-period model with the yet *unknown* (to us) solution split $(\hat{x}_A, 1 - \hat{x}_A)$.
- Obtain as a 2-period solution for A: $\xi_A(\hat{x}_A) := 1 - \delta(1 - \delta \hat{x}_A)$ and solve the fixed-point equation $\hat{x}_A = \xi_A(\hat{x}_A)$ to obtain $\hat{x}_A = 1 - \delta(1 - \delta \hat{x}_A) = 1 - \delta + \delta^2 \hat{x}_A$.
- **Candidate stationary offer:** $\hat{x}_A = \frac{1-\delta}{1-\delta^2} = \frac{1-\delta}{(1-\delta)(1+\delta)} = \frac{1}{1+\delta}$.

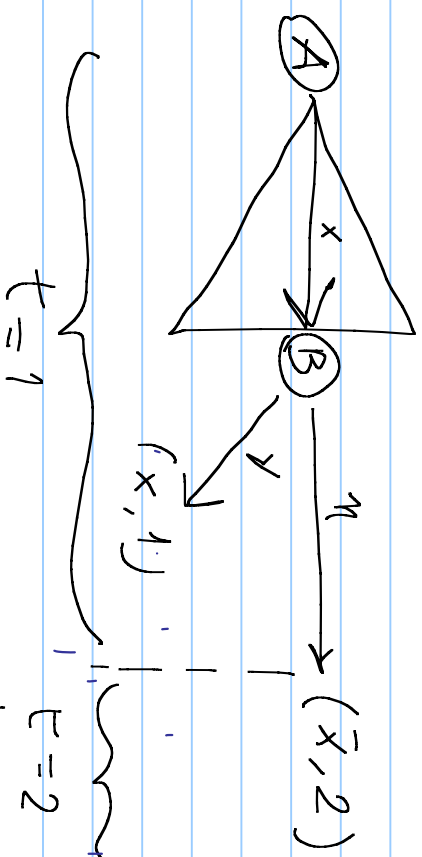
Infinite-Horizon Model: Verification, Uniqueness

- The **candidate stationary solution** $(\hat{s}_A, \hat{s}_B)$ is:
 - for each h at which A [respectively, B] *proposes*,
 $\hat{s}_A(h) = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ [respectively, $\hat{s}_B(h) = \left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\right)$];
 - for each $h' = (h, (x_A, x_B))$ at which A [respectively, B] *responds*,
 $\hat{s}_A(h, (x_A, x_B)) = y$ if $x_A \geq \frac{\delta}{1+\delta}$, and $\hat{s}_A(h, (x_A, x_B)) = n$ otherwise
[respectively, $\hat{s}_B(h, (x_A, x_B)) = y$ if $x_B \geq \frac{\delta}{1+\delta}$, and
 $\hat{s}_B(h, (x_A, x_B)) = n$ otherwise].
- Is this an SPE? *Yes!* The *OD principle applies* to this game (it is compact-continuous). To prove that $(\hat{s}_A, \hat{s}_B)$ is a SPE it is *enough to verify that each $\hat{s}_i$ is One-Step Optimal given $\hat{s}_{-i}$* , which is quite easy :-) [Do it!]
- **Theorem:** $(\hat{s}_A, \hat{s}_B)$ is the unique SPE. [We skip the proof.]
- **Comment:** Differently from repeated games, the long-but-finite-horizon SPE-solution approximates the infinite-horizon SPE-solution.

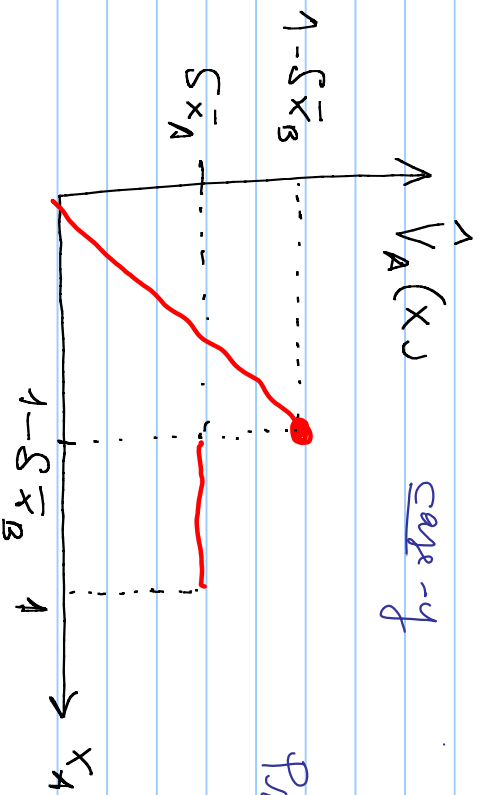
-  BATTIGALLI, P., E. CATONINI, AND N. DE VITO (2025): *Game Theory: Analysis of Strategic Thinking*. Typescript, Bocconi University.
-  BATTIGALLI, P. (2025): *Mathematical Language and Game Theory*. Typescript, Bocconi University.

PICTURES BARGAINING

1-Period model: Ultimatum game (with "fan of offers")



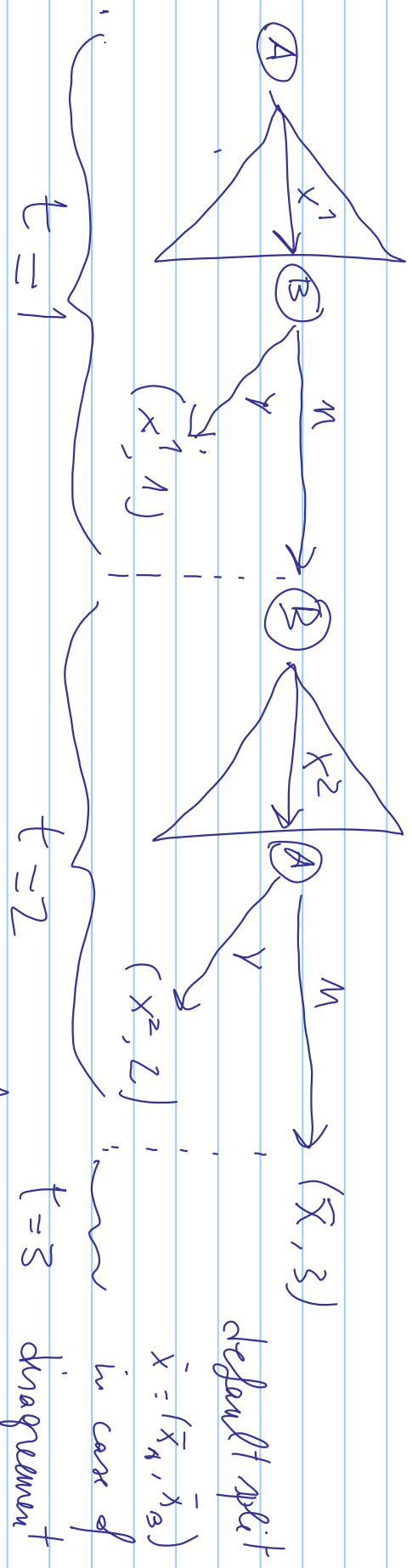
Picture of the US game form



default split $\bar{x} = (x_A, x_B)$
if Ann's offer is rejected

Problem of Ann if Bob is expected to accept when indifferent: "immense" discontinuity at indifference point
"case-n", "bad discontinuity"

2-period model (stylized picture with "form of offers")



Solution of period 2:

$$\hat{x}^2 = (Sx_A, 1 - Sx_A)$$

BI-problem of

Ann in period 1

$S\hat{x}_B^2$ indifference for B

$$\hat{x}_B^2 = 1 - S\hat{x}_A$$

