

Multistage Games with Payoff Uncertainty

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Abstract

This lecture introduces the theory of multistage games with observed actions and *either incomplete information, or asymmetric information about an initial chance move*. It is focused on the description of these games and on the role of Bayes rule to model updated beliefs about co-players' types. The analytical representation blends the parameterized payoff functions (with differential information about parameters) used to study static games with incomplete information in Chapter 8 with the sequential game-tree representation of multistage games of Chapter 9. Graphically, the game is represented as a *forest* of trees, one for each possible profile of payoff functions, connected by information sets showing that (active) players cannot distinguish the different types of co-players. Conditional probability systems about the types and strategies of co-players yield “personal assessments” comprising conditional probabilities of actions and conditional probabilities of nodes in the information sets. Personal assessments must satisfy Bayes rule.

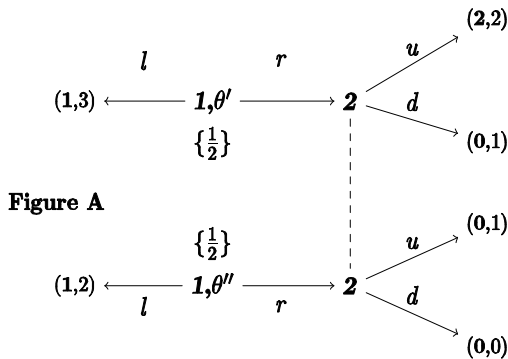
[These slides summarize and, in part, complement Sections 1, 2 and part of section 3 of Chapter 15 of GT-AST.]

Introduction

- We want to study multistage games with observed actions and incomplete information.
 - The same analytical tools can be applied to study games with complete, but imperfect and asymmetric information, e.g., Poker.
- To represent **incompleteness of information** in *static games*:
 - we used parameterized payoff functions $(u_i : \Theta \times A \rightarrow \mathbb{R})_{i \in I}$,
 - $\theta = (\theta_i)_{i \in I} \in \Theta$, where each $i \in I$ knows (only) θ_i (for simplicity, we neglect residual uncertainty).
- To represent the rules of interaction of a **multistage game with observed actions**:
 - we used a structure $\langle I, (A_i, \mathcal{A}_i(\cdot))_{i \in I} \rangle$ with $A_i = i$'s potentially feasible actions, $\mathcal{A}_i(\cdot) = i$'s constraint correspondence;
 - we derived a **tree** of histories $(\bar{H}, \preceq)$, with $\preceq = \text{"prefix of"}$, and $\bar{H} = H \cup Z$ (Z : terminal histories).
- To represent *incomplete information* (payoff uncertainty), posit $(u_i : \Theta \times Z \rightarrow \mathbb{R})_{i \in I}$.

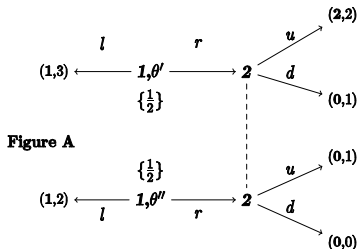
Graphical representation: a simple example

- We get a *forest of game trees* indexed by $\theta \in \Theta$, each one with payoff functions $(u_{i,\theta} : Z \rightarrow \mathbb{R})_{i \in I}$.
- Asymmetric information is graphically represented by **information sets**: for every $i \in I$, $\bar{\theta}_i \in \Theta_i$, and $h \in H$, the *indistinguishable* "nodes" $((\bar{\theta}_i, \theta_{-i}), h)$ ($\theta_{-i} \in \Theta_{-i}$) are connected by a dashed line. In Figure A, $i = 2$, $\Theta_2 = \{\bar{\theta}_2\}$, $\Theta_1 \cong \Theta = \{\theta', \theta''\}$, $h = (r)$.



Intuitive analysis of the example

- In Figure A we also represent the initial, exogenous belief of pl. 2 about θ . His (posterior) belief upon observing r may well be different: r is dominated for θ'' , but it is justifiable for θ' ; if 2 strongly believes in 1's rationality, then the posterior belief is $\mu_2(\theta''|r) = 1 - \mu_2(\theta'|r) = 0$.



- Also, d is conditionally dominated by u . A rational pl. 2 would choose u after r . Anticipating this, a rational player 1 of type θ' chooses r .
- Intuitive solution: (r', ℓ'', u) [where $r'.\ell'' = (r \text{ if } \theta', \ell \text{ if } \theta'')$]

Multistage games with payoff uncertainty

A **multistage game with payoff uncertainty** (and observed actions) is a structure

$$\hat{\Gamma} = \langle I, (\Theta_i, A_i, \mathcal{A}_i(\cdot), u_i)_{i \in I} \rangle$$

where

- $\langle I, (A_i, \mathcal{A}_i(\cdot))_{i \in I} \rangle$ is a **multistage game tree**;
- $\theta_i \in \Theta_i$ are information **types** (traits, tastes, private information);
- $\Theta := \times_{i \in I} \Theta_i$ (more generally, $\Theta \subseteq \times_{i \in I} \Theta_i$);
- $u_i : \Theta \times Z \rightarrow \mathbb{R}$ is i 's parameterized **payoff function**;
 - u_i is derivable from $g : \Theta \times Z \rightarrow Y$ and $v_i : \Theta_i \times Y \rightarrow \mathbb{R}$;
 - generalization: $\mathcal{A}_i(\cdot, \cdot) : \Theta_i \times H \rightrightarrows A_i$, $a_i \in \mathcal{A}_i(\theta_i, h)$ feasible for θ_i given h ;
 - $\Theta = \{\bar{\theta}\}$ singleton (or θ -indep. payoff) $\Rightarrow$ *complete information*.
- To ease notation, we neglect residual uncertainty (Θ_0 is a singleton, not shown).

An example

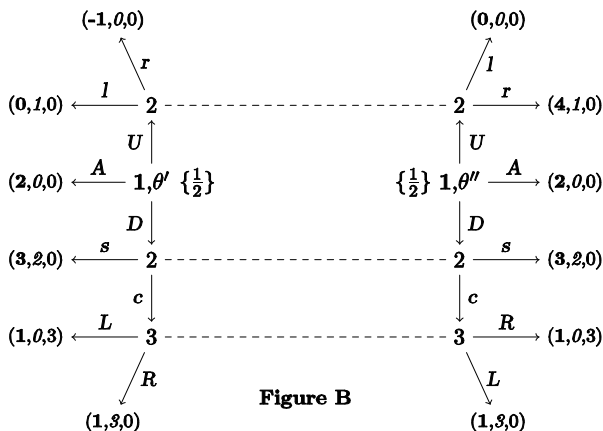
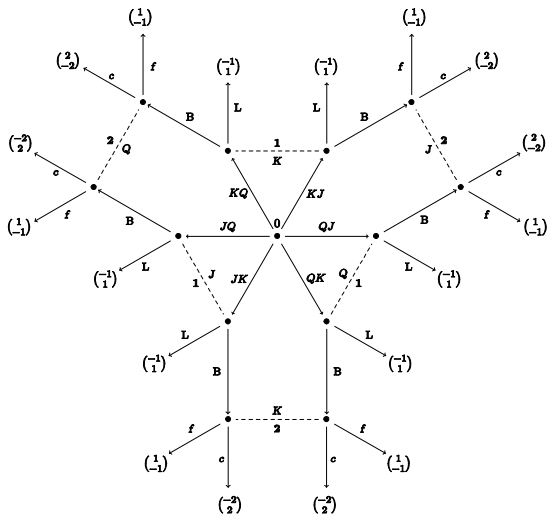


Figure B

- $\Theta = \{\theta', \theta''\} \cong \Theta_1$, only pl. 1 is informed.
 - Dashed lines join “nodes” (θ, h) in the same information set of the active player (pl. 2 and 3 have the same information).
 - (Numbers in $\{\cdot\}$: exogenous common prior on Θ , not part of $\hat{\Gamma}_i$)



- **Mini-Poker:** θ_i is acquired information (not personal trait).
 - $\Theta_i = \{J, Q, K\}$, $\Theta = \{JK, JQ, KQ, KJ, QJ, QK\} \subset \Theta_1 \times \Theta_2$.
 - Initial node of pl. 0=chance move (random order of cards).

- For simplicity, focus on two-person games.
 - With complete information, we could represent a conjecture of i as $\beta^i = \left(\beta^i(\cdot|h) \right)_{h \in H} = \beta_{-i} \in \times_{h \in H} \Delta(\mathcal{A}_{-i}(h))$, which is like a behavior strategy of $-i$, and we obtained a subjective decision tree for i .
 - With incomplete info, we need a **personal assessment** (β^i, μ_i) : **conjecture** β^i must have the form

$$\beta^i = \left(\beta^i(\cdot|\theta_{-i}, h) \right)_{\theta_{-i} \in \Theta_{-i}, h \in H} \in (\times_{h \in H} \Delta(\mathcal{A}_{-i}(h)))^{\Theta_{-i}},$$

because pl. i of type θ_i expects that $-i$'s behavior depends of his type θ_{-i} ; also, we have to specify updated beliefs about θ_{-i} given each $h \in H$: a **system of beliefs**

$$\mu_i = (\mu_i(\cdot|h))_{h \in H} \in (\Delta(\Theta_{-i}))^H$$

(we also write $\mu_i(\cdot|\theta_i, h)$ to emphasize that θ_i is known/given).

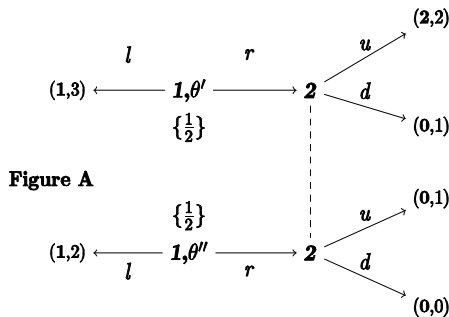
- **Note:** μ_i should be derived from β^i (if possible) using *Bayes rule*.
- From (β^i, μ_i) we will derive conditional values for type θ_i .

Conditional Probability Systems (CPSs)

- In the (rationalizability) analysis of *static* games with *incomplete* information, we considered conjectures $\mu^i \in \Delta(\Theta_{-i} \times A_{-i})$.
- In the (rationalizability) analysis of *multistage* games with *complete* information, we considered CPSs $\mu^i \in \Delta^H(S_{-i})$.
- In the (rationalizability) analysis of *multistage* games with *incomplete* information, we can use CPSs $\bar{\mu}^i \in \Delta^H(\Theta_{-i} \times S_{-i})$, where—as before— $S_{-i} = \times_{h \in H} \mathcal{A}_{-i}(h)$ are the co-player's pure strategies (we write $\bar{\mu}^i$ to distinguish from systems of beliefs $\mu_i \in (\Delta(\Theta_{-i}))^H$).
- We can *derive a personal assessment* (β^i, μ_i) from a CPS $\bar{\mu}^i$:
 $\forall (\theta_{-i}, h) \in \Theta_{-i} \times H, \forall a_{-i} \in \mathcal{A}_{-i}(h),$
 $\mu_i(\theta_{-i}|h) = \bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h)|h),$ if $\bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h)|h) > 0$
then

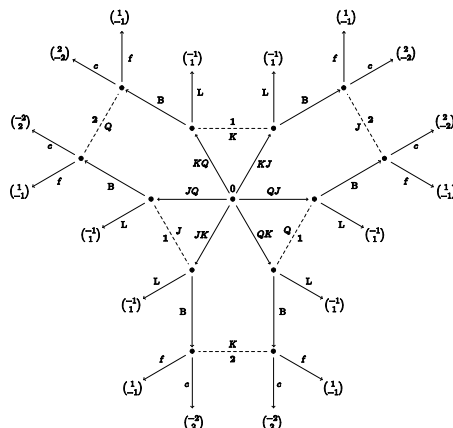
$$\beta^i(a_{-i}|\theta_{-i}, h) = \frac{\bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h, a_{-i})|h)}{\bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h)|h)}.$$

Example of Personal Assessment: Signaling



- $\mu_2(\theta'|\emptyset) = p(\theta') = 1/2,$
- $\beta^2(r|\theta') > 0, \beta^2(r|\theta'') = 0,$
- $\mu_2(\theta'|r) = \frac{\beta^2(r|\theta')p(\theta')}{\beta^2(r|\theta')p(\theta') + \beta^2(r|\theta'')p(\theta'')} = 1.$

Example of Personal Assessment: Poker



- $p(\theta) = 1/6$ for each $\theta \in \{JK, JQ, KQ, KJ, QJ, QK\}$.
- $\beta^2(B|K_1) = \beta^2(B|Q_1) = 1$, $\beta^2(B|J_1) = q \in (0, 1)$.
- $\mu_2(J_1|Q_2, B) = \frac{1}{6}q / (\frac{1}{6}q + \frac{1}{6}) = q / (q + 1)$.

Rehearsing Bayes Rule: Preliminaries

- For the first time, we are *truly* using Bayes rule!
- Abstract *finite* setting: **uncertainty space**

$$\Omega = \Theta \times X,$$

e.g., $\theta \in \Theta$ is the composition of a k -color n -balls urn, $x \in X$ is a sequence of colors of balls drawn from the urn.

- The **likelihood function** $\theta \mapsto P(\cdot|\theta) \in \Delta(X)$ gives the probability law for $x \in X$ for each parameter value $\theta \in \Theta$.
- As in *Bayesian statistics*, we posit $P(\cdot) \in \Delta(\Theta \times X)$ and interpret $P(x|\theta) = P(\{(\theta, x)\} | \{\theta\} \times X)$, where

$$P(\{(\theta, x)\} | \{\theta\} \times X) = \frac{P(\{(\theta, x)\})}{P(\{\theta\} \times X)} =: \frac{P(\theta, x)}{P(\theta)}$$

(recall: $P(E|F) = P(E \cap F|F) = P(E \cap F)/P(F)$ if $P(F) > 0$).

Rehearsing Bayes Theorem (or Rule)

- The joint probability of θ and x can be expressed as

$$P(x|\theta) P(\theta) = P(\theta, x) = P(\theta|x) P(x).$$

- The marginal, or **predictive** probability of x is

$$P(x) := P(\Theta \times \{x\}) = \sum_{\theta' \in \Theta} P(\theta', x) = \sum_{\theta' \in \Theta} P(x|\theta') P(\theta').$$

- Therefore (**Bayes theorem**) the “posterior” probability of any $\theta \in \Theta$ given any evidence $x \in X$ is

$$P(\theta|x) = \frac{P(\theta, x)}{P(x)} = \frac{P(x|\theta) P(\theta)}{\sum_{\theta' \in \Theta} P(x|\theta') P(\theta')}.$$

- (1) Bayes rule only uses the likelihood function $\theta \mapsto P(\cdot|\theta)$ and the “prior” $\text{marg}_{\Theta} P \in \Delta(\Theta)$.
- (2) In statistics, we may assume $P(x) > 0$ for all x , *not in GT* where x is endogenous (actions).

Bayes Rule: Connection to Games

- Take the perspective of an *external observer* of a multistage game with incomplete information.
- The “probability law” is $\beta = (\beta(\cdot|\theta, h))_{\theta \in \Theta, h \in H} \in (\times_{h \in H} \Delta(\mathcal{A}(h)))^\Theta$ (obtained from a profile of type-dependent behavior strategies).
- There is an exogenous prior $p \in \Delta(\Theta)$; write $\beta(a|\theta, \emptyset) = \beta(a|\theta)$; for each $a \in \mathcal{A}(\emptyset)$ such that $\sum_{\theta' \in \Theta} \beta(a|\theta') p(\theta') > 0$ and each $\theta \in \Theta$,

$$\mu(\theta|a) = \frac{\beta(a|\theta) p(\theta)}{\sum_{\theta' \in \Theta} \beta(a|\theta') p(\theta')}.$$

- Let $\mu(\cdot|h) \in \Delta(\Theta)$ be given; for each $a \in \mathcal{A}(h)$ such that $\sum_{\theta' \in \Theta} \beta(a|\theta', h) \mu(\theta'|h) > 0$ and each $\theta \in \Theta$,

$$\mu(\theta|h, a) = \frac{\beta(a|\theta, h) \mu(\theta|h)}{\sum_{\theta' \in \Theta} \beta(a|\theta', h) \mu(\theta'|h)}.$$

-  BATTIGALLI, P., E. CATONINI, AND N. DE VITO (2025): *Game Theory: Analysis of Strategic Thinking*. Typescript, Bocconi University.
-  BATTIGALLI, P. (2025): *Mathematical Language and Game Theory*. Typescript, Bocconi University.