

# Multistage Games with Payoff Uncertainty: Rational Planning

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## Abstract

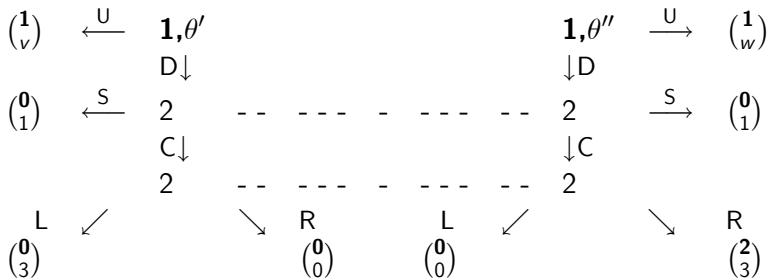
This lecture extends the analysis of rational planning to multistage games with (observed actions and) payoff uncertainty.

[These slides summarize and, in part, complement Section 3 of Chapter 15 of GT-AST.]

# Introduction

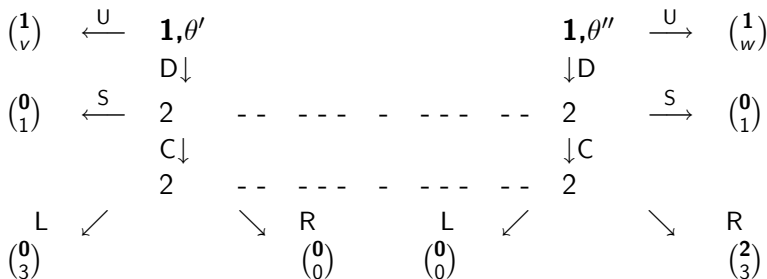
- We want to study rational planning in multistage games with observed actions and payoff uncertainty.
- With this aim, we extend our representation of  $i$ 's beliefs about  $-i$ :
  - We start with conditional probability systems (CPSs)  $\bar{\mu}^i \in \Delta^H(\Theta_{-i} \times S_{-i})$  over others' information types  $\theta_{-i}$  and strategies (ways of behaving)  $s_{-i}$ , thus extending the analysis of beliefs used to study rationalizability in multistage games with complete information.
  - Next we derive pairs  $(\beta^i, \mu_i)$  assigning conditional probabilities  $\beta^i(a_{-i}|\theta_{-i}, h)$  to actions and conditional probabilities  $\mu_i(\theta_{-i}|h)$  to types. [ $\beta^i(\cdot|\theta_{-i}, h)$  is arbitrary if  $\mu_i(\theta_{-i}|h) = 0$ , but this is going to be innocuous.]
  - If  $(\beta^i, \mu_i)$  is derived from a CPS, it must satisfy Bayes rule whenever possible and is called “**Bayes consistent** personal assessment”.
- With this, we obtain results about *rational planning*.

# Running Example: (Conditional) Beliefs, 1/2



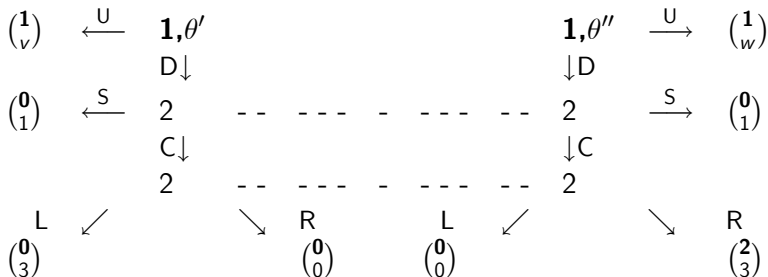
- Only player **1** (denoted in **bold**) is informed:  $\Theta_1 \cong \Theta = \{\theta', \theta''\}$ .
  - Payoffs  $v$  and  $w$  of player 2 do not matter.  $H = \{\emptyset, (D), (D, C)\}$ .
  - Consider a CPS  $\bar{\mu}^2 \in \Delta^H(\Theta \times S_1)$ , with conditioning events  $\Theta \times S_1(h)$  ( $h \in H$ ), where  $S_1(\emptyset) = S_1 = \{U, D\}$ ,  $S_1(D) = S_1(D, C) = \{D\}$  (C does not reveal anything about pl. 1).
  - Abbreviations:** We often write  $\bar{\mu}^2(\{(\theta, s_1)\} | \Theta \times S_1(h)) =: \bar{\mu}^2(\theta, s_1 | h)$ , omitting  $h = \emptyset$ .

# Running Example: (Conditional) Beliefs, 2/2



- Derive from CPS  $\bar{\mu}^2$  a corresponding personal assessment  $(\beta^2, \mu_2)$  to obtain a *subjective decision tree* for pl. 2:
  - $\mu_2(\theta) = \bar{\mu}^2(\{\theta\} \times S_1)$  (prior exogenous belief of pl. 2). Assume  $0 < \mu_2(\theta') < 1$ .
  - $\beta^2(D|\theta) = \bar{\mu}^2(\theta, D) / \bar{\mu}^2(\{\theta\} \times S_1) = \bar{\mu}^2(\theta, D) / \mu_2(\theta)$ .
  - $\mu_2(\theta|D) = \bar{\mu}^2(\{(\theta, D)\} | \Theta \times S_1(D)) = \bar{\mu}^2(\{(\theta, D)\} | \Theta \times S_1(D, C)) = \mu_2(\theta|(D, C))$ .

# Running Example: Rational Planning by Folding Back



- Here, only part  $\mu_2$  of 2's personal assessment  $(\beta^2, \mu_2)$  matters.
  - Let  $q := \mu_2(\theta' | D) = \mu_2(\theta' | (D, C))$ ; with this,  $q < \frac{1}{2} \Rightarrow \hat{s}_2(D, C) = R$ ,  $q > \frac{1}{2} \Rightarrow \hat{s}_2(D, C) = L$ ,  $q = \frac{1}{2} \Rightarrow \text{indiff}$ .
  - $\hat{V}_2^q((D, C)) = \max\{3q, 3(1 - q)\} \geq \frac{3}{2} > 1$ ; thus,  $\hat{s}_2(D) = C$  for every  $q$ , i.e., for every  $\bar{\mu}^2 \in \Delta^H(\Theta \times S_1)$ .
  - **Key:**  $\mu_2(\theta' | D) = \mu_2(\theta' | (D, C))$ , otherwise there may be no sequentially optimal strategy!

# Beliefs in Multistage Games with Payoff Uncertainty

- Fix a (finite) **multistage game with payoff uncertainty** and observed actions  $\hat{\Gamma} = \langle I, (\Theta_i, A_i, \mathcal{A}_i(\cdot), u_i)_{i \in I} \rangle$ .
- To represent *strategic thinking* as rationalizability:
  - We will merge elements of Ch. 8 (static games with incomplete information) and Ch. 11 (rationalizability in multistage games with complete information).
  - With this goal, beliefs are conveniently represented as CPSs  $\bar{\mu}^i = (\bar{\mu}^i(\cdot|h))_{h \in H} \in \Delta^H(\Theta_{-i} \times S_{-i})$ , recalling that, for all  $h', h'' \in H$ ,

$$S_{-i}(h') = S_{-i}(h'') \Rightarrow$$

$$\bar{\mu}^i(\cdot|h') = \bar{\mu}^i(\cdot|\Theta_{-i} \times S_{-i}(h')) = \bar{\mu}^i(\cdot|\Theta_{-i} \times S_{-i}(h'')) = \bar{\mu}^i(\cdot|h'')$$

- To represent *rational planning* (and later, for equilibrium analysis):
  - it is convenient to work with personal assessments  $(\beta^i, \mu_i)$  satisfying *Bayes consistency*,
  - which—essentially—follows if  $(\beta^i, \mu_i)$  is derived from a CPS  $\bar{\mu}^i$ .

# Conditional Probability Systems (CPSs)

- In the (rationalizability) analysis of static games with *incomplete* information, we considered conjectures  $\mu^i \in \Delta(\Theta_{-i} \times A_{-i})$ .
- In the (rationalizability) analysis of multistage games with *complete* information, we considered CPSs  $\mu^i \in \Delta^H(S_{-i})$ .
- In the (rationalizability) analysis of multistage games with *incomplete* information, we can use CPSs  $\bar{\mu}^i \in \Delta^H(\Theta_{-i} \times S_{-i})$ , where (as before)  $S_{-i} = \times_{h \in H} \mathcal{A}_{-i}(h)$  are the co-players' pure strategies (we write  $\bar{\mu}^i$  to distinguish from systems of beliefs  $\mu_i \in (\Delta(\Theta_{-i}))^H$ ).
- We can *derive a personal assessment*  $(\beta^i, \mu_i)$  from a CPS  $\bar{\mu}^i$ : for all  $(\theta_{-i}, h) \in \Theta_{-i} \times H$  and  $a_{-i} \in \mathcal{A}_{-i}(h)$ ,  
 $\mu_i(\theta_{-i}|h) = \bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h)|h)$  and, if  
 $\bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h)|h) > 0$ , then

$$\beta^i(a_{-i}|\theta_{-i}, h) = \frac{\bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h, a_{-i})|h)}{\bar{\mu}^i(\{\theta_{-i}\} \times S_{-i}(h)|h)}.$$



# Bayes Consistency of Personal Assessments

- If  $(\beta^i, \mu_i)$  is derived from a CPS  $\bar{\mu}^i$ , then it has to be **Bayes consistent**: For all  $h \in H$ ,  $a_{-i} \in \mathcal{A}_{-i}(h)$ ,  $\theta_{-i}$ , write
  - $\mathbb{P}^{\beta^i}(a_{-i}|\theta_{-i}, h) := \beta^i(a_{-i}|\theta_{-i}, h)$ ,  $\mathbb{P}^{\mu_i}(\theta_{-i}|h) := \mu_i(\theta_{-i}|h)$ ,
  - $\mathbb{P}^{\beta^i, \mu_i}(\theta_{-i}, a_{-i}|h) := \beta^i(a_{-i}|\theta_{-i}, h) \mu_i(\theta_{-i}|h)$ ,
  - $\mathbb{P}^{\beta^i, \mu_i}(a_{-i}|h) = \sum_{\theta'_{-i}} \mathbb{P}^{\beta^i, \mu_i}(\theta'_{-i}, a_{-i}|h)$   
 $= \sum_{\theta'_{-i}} \beta^i(a_{-i}|\theta'_{-i}, h) \mu_i(\theta'_{-i}|h)$ .
  - If  $\mathbb{P}^{\beta^i, \mu_i}(a_{-i}|h) > 0$ , write  $\mu_i(\theta_{-i}|h, a_{-i}) := \frac{\mathbb{P}^{\beta^i, \mu_i}(\theta_{-i}, a_{-i}|h)}{\mathbb{P}^{\beta^i, \mu_i}(a_{-i}|h)}$   
 $= \frac{\beta^i(a_{-i}|\theta_{-i}, h) \mu_i(\theta_{-i}|h)}{\sum_{\theta'_{-i}} \beta^i(a_{-i}|\theta'_{-i}, h) \mu_i(\theta'_{-i}|h)}$  (BR).
  - **Bayes consistency**: for all  $h \in H$  s.t.  $L(\hat{\Gamma}(h)) > 1$ ,  $a_i \in \mathcal{A}_i(h)$ ,  $a_{-i} \in \mathcal{A}_{-i}(h)$ , and  $\theta_{-i}$

$$\mu_i(\theta_{-i}|h, (a_i, a_{-i})) = \mu_i(\theta_{-i}|h, a_{-i}),$$

where  $\mu_i(\theta_{-i}|h, a_{-i})$  satisfies (BR) whenever possible. (Hence,  $\mu_i(\cdot|h, (a_i, a_{-i}))$  is independent of own-action  $a_i$ .)

- If  $i$  is the only active player at  $h$ ,  $\mu_i(\theta_{-i}|h, a_i) = \mu_i(\theta_{-i}|h)$ .

# One-Step and Sequential Optimality

- Fix  $(\beta^i, \mu_i)$ ,  $\theta_i$  and  $\beta_i \in \times_{h \in H} \Delta(\mathcal{A}_i(h))$ .
  - For all  $h \in H$ ,  $z \in Z(h)$ ,  $a_i \in \mathcal{A}_i(h)$ ,  $a_{-i} \in \mathcal{A}_{-i}(h)$ ,  $\theta_{-i}$  let
  - $\mathbb{P}^{\beta_i, \beta^i}(z | \theta_{-i}, h) = \text{prob. of } z \text{ conditional on } h \text{ given } \theta_{-i}$ ,
  - $V_{\theta_i}^{\beta_i, \beta^i}(\theta_{-i}, h) = \sum_{z \in Z(h)} u_i(\theta_i, \theta_{-i}, z) \mathbb{P}^{\beta_i, \beta^i}(z | \theta_{-i}, h)$ ,
  - $V_{\theta_i}^{\beta_i, \beta^i, \mu_i}(h) = \sum_{\theta'_{-i}} V_{\theta_i}^{\beta_i, \beta^i}(\theta'_{-i}, h) \mu_i(\theta'_{-i} | h)$ ,
  - $V_{\theta_i}^{\beta_i, \beta^i, \mu_i}(h, a_i) = \sum_{\theta'_{-i}, a'_{-i}} V_{\theta_i}^{\beta_i, \beta^i}(\theta'_{-i}, (h, (a_i, a'_{-i}))) \beta^i(a'_{-i} | \theta'_{-i}, h) \mu_i(\theta'_{-i} | h)$ .

## Definition

Behavior strategy  $\beta_i$  is **one-step optimal** given  $(\beta^i, \mu_i)$  if, for all

$h \in H$ ,  $\text{supp} \beta_i(\cdot | h) \subseteq \arg \max_{a_i \in \mathcal{A}_i(h)} V_{\theta_i}^{\beta_i, \beta^i, \mu_i}(h, a_i)$ ;

$\beta_i$  is **sequentially optimal** given  $(\beta^i, \mu_i)$  if, for all  $h \in H$ ,

$V_{\theta_i}^{\beta_i, \beta^i, \mu_i}(h) = \max_{s_i \in S_i(h)} V_{\theta_i}^{s_i, \beta^i, \mu_i}(h)$ .

# The One-Deviation Principle

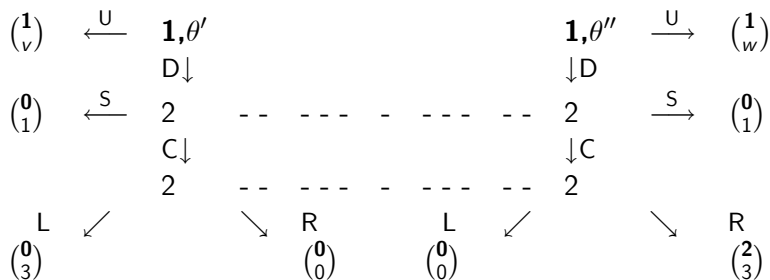
- The results about rational planning can be extended to allow for incomplete information (payoff uncertainty). In particular, one can prove a version of the OD Principle:

## Theorem

*For all [behavior] strategies  $s_i$   $[\beta_i]$  and **Bayes consistent** personal assessments  $(\beta^i, \mu_i)$ ,  $s_i$   $[\beta_i]$  is one-step optimal given  $(\beta^i, \mu_i)$  IFF it is sequentially optimal given  $(\beta^i, \mu_i)$ .*

- The proof is similar to the complete-information case. The novelty is that we also need a system of beliefs  $\mu_i \in (\Delta(\Theta_{-i}))^H$  and that the personal assessment  $(\beta^i, \mu_i)$  has to be Bayes consistent.

# The Need for Bayes Consistency



- If  $(\beta^2, \mu_2)$  is derived from a CPS, then it is Bayes consistent,  $\mu_2(\theta'|D) = \mu_2(\theta'|(D, C))$ , one-step optimality is equivalent to sequential optimality, and the optimal strategies select C if D.
- Suppose  $(\beta^2, \mu_2)$  is *not* derived from a CPS and

$$\mu_2(\theta'|D) < \frac{1}{3}, \mu_2(\theta'|(D, C)) > \frac{1}{2}.$$

Then, one-step optimality yields L if (D, C) and S if D.

# Conditional Dominance

- We can extend the definition of conditional dominance to this incomplete-information environment.
- Write:  $U_i(\theta, s) := u_i(\theta, \zeta(s))$ , and  $U_i(\theta, \sigma_i, s_{-i}) = \mathbb{E}_{\sigma_i}(U_i(\theta, \cdot, s_{-i}))$  for  $\sigma_i \in \Delta(S_i)$ .

## Definition

Strategy  $s_i$  is **conditionally dominated for type**  $\theta_i$  if there are  $h \in H_i(s_i)$  and  $\sigma_i \in \Delta(S_i(h))$  such that

$$\forall \theta_{-i}, \forall s_{-i} \in S_{-i}(h), U_i(\theta_i, \theta_{-i}, s_i, s_{-i}) < U_i(\theta_i, \theta_{-i}, \sigma_i, s_{-i}).$$

- **Exercise:** Show that (reduced) strategy S of the running example is conditionally dominated.

# Justifiability and Conditional Dominance

- As for the complete-information case, we use notions of optimality and justifiability that are invariant w.r.t. behavioral equivalence:

## Definition

A strategy  $\bar{s}_i$  is **weakly sequentially optimal for type**  $\theta_i$  given  $(\beta^i, \mu_i)$ , written  $\bar{s}_i \in r_i(\theta_i, \beta^i, \mu_i)$ , if

$V_{\theta_i}^{\bar{s}_i, \beta^i, \mu_i}(h) = \max_{s_i \in S_i(h)} V_{\theta_i}^{s_i, \beta^i, \mu_i}(h)$  for all  $h \in H_i(\bar{s}_i)$ ;  $\bar{s}_i$  is **justifiable for type**  $\theta_i$  if  $\bar{s}_i \in r_i(\theta_i, \beta^i, \mu_i)$  for some **Bayes consistent**  $(\beta^i, \mu_i)$ .

- Remark** If  $\bar{s}_i \in r_i(\theta_i, \beta^i, \mu_i)$  and  $s_i$  is behaviorally equivalent to  $\bar{s}_i$  then  $s_i \in r_i(\theta_i, \beta^i, \mu_i)$ . Hence,  $\bar{s}_i$  is justifiable for  $\theta_i$  IFF every behaviorally equivalent  $s_i$  is justifiable for  $\theta_i$ .

## Lemma

For every  $s_i \in S_i$  and  $\theta_i \in \Theta_i$ ,  $s_i$  is justifiable for  $\theta_i$  IFF it is not conditionally dominated for  $\theta_i$ .

-  BATTIGALLI, P., E. CATONINI, AND N. DE VITO (2025): *Game Theory: Analysis of Strategic Thinking*. Typescript, Bocconi University.
-  BATTIGALLI, P. (2025): *Mathematical Language and Game Theory*. Typescript, Bocconi University.