Rationalizability in Multistage games

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Game Theory: Analysis of Strategic Thinking

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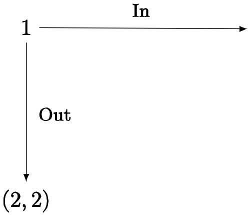
Abstract

The analysis of strategic thinking can be extended from static to multistage games in multiple ways. The reason is that in multistage games we have to address a new issue. Consider step 2 of a rationalizability procedure in a two-person game. It is clear how belief in rationality shapes the initial conjecture of player i: the co-player's $(-i)$ unjustifiable strategies must have zero probability. But i may assign zero subjective probability also to some justifiable strategies of $-i$. What if $-i$ moves according to one of those justifiable strategies that i deems impossible, thus surprising her? Would i still believe that $-i$ is rational after a surprising action? We mostly focus on two ways of addressing this problem. (1) An "everything-goes" approach, in which strategic reasoning only restricts initial conjectures, allowing player i to revise her conjecture arbitrarily if she is surprised by some actions. (2) A much more demanding approach, in which strategic reasoning also shapes—according to a "best rationalization principle"—the revised conjectures of players when they are surprised. Approach (1) yields initial rationalizability. Approach (2) yields strong rationalizability. We present these concepts and characterizations by means of iterated dominance. [These slides summarize and complement Chapter 11.1-3 of GT-AST.]



$1\backslash 2$	B_2	S_2
B_1	$3,1$	$0,0$
S_1	$0,0$	$1,3$

- How should pl. 1 and 2 reason strategically in the BoSOO? Here is an anticipation of what is to come:
  - In.S_1 is dominated by Out, hence unjustifiable. Thus pl. 2 is certain that In.S_1 will not be executed.
  - If pl. 2, for whatever reason, initially believes that pl. 1 chooses Out, he is surprised by In. Should pl. 2 still believe that pl. 1 is rational?
  - If Yes (*strong belief in rationality*), upon observing In, pl. 2 should believe In.B_1 and choose $B_{2.}$ Anticipating this, pl. 1 plays In.B_1.
  - If Not (only *initial belief in rationality*), no further inference.

Updated, or Revised Conjectures

- In the analysis of rational planning, we found it convenient to express the conjecture of i about the co-player $-i$ as a behavior strategy $\beta^i \in \times_{h \in H} \Delta(\mathcal{A}_{-i}(h))$ (a "correlated" behavior strategy with 2 or more co-players).
- But in order to analyze strategic reasoning, that is, where conjectures come from, it is better to use probability measures over S_{-i}. Indeed, if i believes that $-i$ is rational, i must assign probability zero to the unjustifiable "ways of behaving" s_{-i}.
- This is fine for the initial conjecture that i holds at the root, viz. $\mu^i_\varnothing$. But what if i is surprised by the actions taken by $-i$ in observed history h, that is, what if i observes h with $\mu^i_\varnothing(S_{-i}(h)) = 0$? In this case we cannot obtain updated probabilities from $\mu^i_\varnothing$ by conditioning.
- Thus, we assume that for every "conditioning event" $S_{-i}(h)$ about $-i$'s behavior, i would hold an **updated, or revised conjecture** $\mu^i(\cdot|S_{-i}(h)) \in \Delta(S_{-i}(h))$, and these updated/revised *conjectures* should be *mutually consistent*.

Conditional Probability Systems

- Consider the collection of observable events about $-i'$s behavior $\mathcal{H}_{-i} = \{C_{-i} \subseteq S_{-i} : \exists h \in H, C_{-i} = S_{-i}(h)\}$.

Definition

A **Conditional Probability System (CPS)** is an array of probability measures $\mu^i = (\mu^i(\cdot|C_{-i}))_{C_{-i} \in \mathcal{H}_{-i}} \in (\Delta(S_{-i}))^{\mathcal{H}_{-i}}$ such that:

1. (Believe what you observe) $\forall C_{-i} \in \mathcal{H}_{-i}$, $\mu^i(C_{-i}|C_{-i}) = 1$;
2. (Chain rule) $\forall C_{-i}, D_{-i} \in \mathcal{H}_{-i}$, $\forall E_{-i} \subseteq S_{-i}$, if $E_{-i} \subseteq D_{-i} \subseteq C_{-i}$ then $\mu^i(E_{-i}|C_{-i}) = \mu^i(E_{-i}|D_{-i}) \cdot \mu^i(D_{-i}|C_{-i})$.

- Mutual consistency:
  - if $h' \neq h''$ reveal different behavior of i, but the *same behavior of* $-i$, $S_{-i}(h') = C_{-i} = S_{-i}(h'')$, then *same* $\mu^i(\cdot|C_{-i})$;
  - the **chain rule** can be expressed as $\mu^i(D_{-i}|C_{-i}) > 0 \Longrightarrow \mu^i(E_{-i}|D_{-i}) = \frac{\mu^i(E_{-i}|C_{-i})}{\mu^i(D_{-i}|C_{-i})}$.

CPSs and Weak Sequential Optimality

- To ease notation, we write $\mu^i(\cdot|S_{-i}(h)) = \mu^i(\cdot|h) \in \Delta(S_{-i}(h))$ and we let $\Delta^H(S_{-i})$ denote the set of CPSs on $-i$.
- Natural map (CPS) $\mu^i \mapsto \beta^i$ (conj.): for all $h \in H$ and $a_{-i} \in \mathcal{A}_{-i}(h)$,

$$\beta^i(a_{-i}|h) = \mu^i(S_{-i}(h, a_{-i})|h).$$

- Note, by the chain rule, given $\bar{h} \prec h$ [which implies $S_{-i}(\bar{h}) \supseteq S_{-i}(h)$]

$$\mu^i(S_{-i}(h)|\bar{h}) > 0 \Longrightarrow \beta^i(a_{-i}|h) = \frac{\mu^i(S_{-i}(h, a_{-i})|\bar{h})}{\mu^i(S_{-i}(h)|\bar{h})}.$$

- With this, we can define the set of weakly sequentially optimal strategies given μ^i as $r_i\left(\beta^i\right)$ with β^i *derived from* μ^i.
- Equivalent direct definition:

$$r_i(\mu^i) := \left\{\bar{s}_i \in S_i : \forall h \in H_i(\bar{s}_i), \bar{s}_i \in \arg\max_{s_i \in S_i(h)} U_i(s_i, \mu^i(\cdot|h))\right\}.$$

Initial Common Belief in Rationality

- Recall: s_i is **justifiable** if $s_i \in r_i\left(\mu^i\right)$ for some CPS μ^i (equiv.: if $s_i \in r_i\left(\beta^i\right)$ for some conj. β^i).
- Similarly to static games, the *behavioral implication of rationality is justifiability.* Let s_i describe i's behavior and t_i what i "thinks" about others. We may conceive events about i as sets of pairs (s_i, t_i). $R_i =$[i is rational]. Then $(s_i, t_i) \in R_i$ implies that s_i is justifiable.
- With this, we first consider assumptions about strategic reasoning very *similar to* those analyzed for *static* games: *players are rational and there is common initial belief in rationality.*
- Write $\mathrm{B}_{i,\varnothing}(E_{-i})$ for "i initially believes E_{-i}" and $\mathrm{B}_{\varnothing}(E)$ ($E = \times_{j \in I} E_j$) for "every $i \in I$ initially believes E_{-i}." We study the behavioral implications of R, $R \cap \mathrm{B}_{\varnothing}(R)$, $R \cap \mathrm{B}_{\varnothing}(R) \cap \mathrm{B}^2_{\varnothing}(R)$, ..., $R \cap \bigcap_{m=1}^{\infty} \mathrm{B}^m_{\varnothing}(R)$ by extending the rationalization operator of static games to an "initial" rationalization operator for multistage games.

Initial Rationalization Operator

- Say that CPS $\mu^i \in \Delta^H(S_{-i})$ **initially believes** $C_{-i} \subseteq S_{-i}$ if $\mu^i(C_{-i}|\varnothing) = 1$. Let $\Delta^H_\varnothing(C_{-i}) = \{\mu^i \in \Delta^H(S_{-i}) : \mu^i(C_{-i}|\varnothing) = 1\}$ denote the set of CPSs of i that initially believe C_{-i}.
- **Note:** Let C_{-i} be the set of justifiable strategies of $-i$ (those consistent with rationality); even if $\mu^i \in \Delta^H_\varnothing(C_{-i})$, we may have $C_{-i} \cap S_{-i}(h)$ nonempty and yet $\mu^i(C_{-i}|h) = 0$ if $\mu^i(S_{-i}(h)|\varnothing) = 0$ (if i is surprised by $-i$'s actions in h).
- Similarly to static games, let $\mathcal{C}$ denote the collection of Cartesian subsets $C = \times_{i \in I} C_i \subseteq S$. Define the ("initial") rationalization operator $\rho : \mathcal{C} \to \mathcal{C}$ as follows: for all $C \in \mathcal{C}$,

$$\rho(C) = \times_{i \in I} \left\{ s_i \in S_i : \exists \mu^i \in \Delta^H_\varnothing(C_{-i}), s_i \in r_i\left(\mu^i\right) \right\}.$$

- **Remark** As in static games, ρ is *monotone*: $\forall D, E \in \mathcal{C}$, $E \subseteq D \Rightarrow \rho(E) \subseteq \rho(D)$. Thus, $(\rho^m(S))_{m \in \mathbb{N}}$ is (weakly) *decreasing*.

Initial Rationalizability

- The behavioral implications of each assumption $R \cap \bigcap_{k=1}^{m} \mathrm{B}_{\varnothing}^{k}(R)$ are given by the following table:

Assumptions	Behavioral implications
R	$\rho(S)$ (justifiable strategy profiles)
$R \cap \mathrm{B}_{\varnothing}(R)$	$\rho(\rho(S)) = \rho^2(S)$
...	...
$R \cap \bigcap_{k=1}^{m} \mathrm{B}_{\varnothing}^{k}(R)$	$\rho^{m+1}(S)$
...	...
$R \cap \bigcap_{m=1}^{\infty} \mathrm{B}_{\varnothing}^{m}(R)$	$\rho^{\infty}(S)$

- $\rho^{\infty}(S)$ is the set of **initially rationalizable** strategy profiles.

Conditional Dominance

- Recall: $U_i(s) = u_i(\zeta(s))$. With this, the EU of σ_i given s_{-i} is: $U_i(\sigma_i, s_{-i}) = \sum_{s_i \in S_i} U_i(s_i, s_{-i}) \sigma_i(s_i)$.

Definition

Strategy $\bar{s}_i$ is **conditionally dominated** if there are a history $h \in H_i(\bar{s}_i)$ and a mixed strategy $\sigma_i \in \Delta(S_i(h))$ s.t.

$$\forall s_{-i} \in S_{-i}(h),\ U_i(\sigma_i, s_{-i}) > U_i(\bar{s}_i, s_{-i}).$$

- **Remark** *If a strategy $\bar{s}_i$ is dominated, then $\bar{s}_i$ is also conditionally dominated, but the converse does not hold* (see the Entry Game).

Lemma

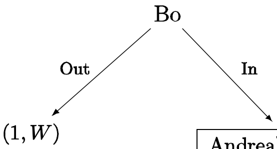
A strategy is justifiable **if and only if** *it is not conditionally dominated.*

Characterization via (Iterated) Dominance

1. A strategy is *justifiable* IFF it is *not conditionally dominated.* Let $NCD \subseteq S$ denote the set of profiles of non-conditionally dominated strategies. Then $\rho(S) = NCD$.
   1. *What about the following steps?*
   2. Let $\mathrm{ND}(\cdot)$ denote the "*un-dominance*" operator of the *strategic form* $\mathcal{N}(\Gamma) = \langle I, (S_i, U_i)_{i \in I} \rangle$. Then:
2. $\rho^2(S) = \mathrm{ND}(NCD)$.
3. $\rho^3(S) = \mathrm{ND}^2(NCD)$. (...)

- $\rho^{m+1}(S) = \mathrm{ND}^m(NCD)$.
- **Intuition**: Let $\rho^m_{-i}(S) = \mathrm{proj}_{S_{-i}} \rho^m(S)$. If $\mu^i(\rho^m_{-i}(S) | \varnothing) = 1$ and i is not "surprised" by h ($\mu^i(S_{-i}(h) | \varnothing) > 0$), then the chain rule forces i to keep believing in $\rho^m_{-i}(S)$: $\mu^i(\rho^m_{-i}(S) | h) = \mu^i(\rho^m_{-i}(S) \cap S_{-i}(h) | h) = 1$. But if i is "surprised" by h ($\mu^i(S_{-i}(h) | \varnothing) = 0$) it may well be the case that $\mu^i(\rho^m_{-i}(S) | h) = 0$ even if $\rho^m_{-i}(S) \cap S_{-i}(h)$ is not empty. Indeed, at step $m + 1$, we only assume that i *initially* believes $\rho^m_{-i}(S)$. ♡

The "Clash of Musicians" with Outside Option (CoMOO)



Andrea\Bo	C	M
home	1, 0	1, 0
c	2, 4	0, 0
m	0, 0	4, 2

- Strategy **home** of Andrea is conditionally dominated. If $W = 1$ then
  - $\rho(S) = NCD = \{\mathbf{c}, \mathbf{m}\} \times \mathbf{S}_b^r$ ($\mathbf{S}_b^r = \{\text{Out}, \text{In.C}, \text{In.M}\}$ reduced strat)
  - $\rho^2(S) = \rho(NCD) = \{\mathbf{c}, \mathbf{m}\} \times \{\text{In.C}, \text{In.M}\}$ END.
- If $W = 3$, then In.M is dominated:
  - $\rho(S) = NCD = \{\mathbf{c}, \mathbf{m}\} \times \{\text{Out}, \text{In.C}\}$,
  - $\rho^2(S) = \rho(NCD) = \{\mathbf{c}, \mathbf{m}\} \times \{\text{Out}, \text{In.C}\}$ END.

Strong Belief

- In the BoSOO and CoMOO(W=3) initial rationalizability allows a "surprised" second mover to give up her belief in the co-player's rationality *even if reaching the subgame* [history $h = (\mathrm{In})$] *is consistent with the co-player's rationality.*
- Now we want to assume that each player believes in the co-player's rationality *whenever possible*, that is, conditional on each $S_{-i}(h)$ that contains some justifiable strategy (profile). We call this "**strong belief** in rationality."
- We also want to assume strong belief in rationality and in some level of "strategic sophistication."
- Say that CPS $\mu^i \in \Delta^H(S_{-i})$ **strongly believes** C_{-i} if

$$\forall h \in H,\ S_{-i}(h) \cap C_{-i} \neq \emptyset \Rightarrow \mu^i(C_{-i}|h) = 1.$$

Let $\Delta^H_{\mathrm{sb}}(C_{-i})$ denote the **set of CPSs that strongly believe** C_{-i}. Thus, if i strongly believes in $-i$'s rationality, it must be the case that i's CPS μ^i about $-i$'s behavior strongly believes the set $C_{-i} = NCD_{-i}$ of justifiable strategies of $-i$.

Best Rationalization Principle

- Let $\mathrm{SB}_i\left(E_{-i}\right)$ denote the event that i **strongly believes** E_{-i}, where E_{-i} is an event concerning $-i$; e.g., $\mathrm{SB}_i\left(R_{-i}\right)$ is the event "i strongly believes in $-i$'s rationality."
- We consider the following assumptions about each player i (2 players):
  1. $R_i^1 = R_i$
  2. $R_i^2 = R_i \cap \mathrm{SB}_i\left(R_{-i}\right) = R_i^1 \cap \mathrm{SB}_i\left(R_{-i}^1\right)$
  3. $R_i^3 = R_i \cap \mathrm{SB}_i\left(R_{-i}\right) \cap \mathrm{SB}_i\left(R_{-i} \cap \mathrm{SB}_{-i}\left(R_i\right)\right) = R_i^2 \cap \mathrm{SB}_i\left(R_{-i}^2\right)$

  (...)
- $R_i^{n+1} = R_i^n \cap \mathrm{SB}_i\left(R_{-i}^n\right)$, $n \in \mathbb{N}$, $R_i^\infty = \bigcap_{m\in\mathbb{N}} R_i^m$.
- **Best rationalization principle:** In words, i starts ascribing to $-i$ the "highest degree of strategic sophistication" $\left(R_{-i}^\infty\right)$ and, if she observes h that contradicts it, she falls back on the largest m (smallest R_{-i}^m) s.t. R_{-i}^m is consistent with h.
- This is also called "forward-induction **reasoning**" (as opposed to "backward-induction reasoning", to be studied later).

Strong Rationalizability

- It turns out that the **behavioral implications of** assumptions R_i^n ($n \in \mathbb{N} \cup \{\infty\}$) are given by the sets of strategies S_i^n defined below:

Definition

Consider the following elimination procedure.

(Step $n = 0$) For each $i \in I$, let $S_i^0 = S_i$. Also, let $S_{-i}^0 = \times_{j \neq i} S_j$ and $S^0 = S$.

(Step $n > 0$) For each $i \in I$, let

$$\begin{aligned} \Delta_i^n &= \bigcap_{m=0}^{n-1} \Delta_{\text{sb}}^H (S_{-i}^m); \\ S_i^n &= \left\{ s_i \in S_i : \exists \mu^i \in \Delta_i^n, s_i \in r_i(\mu^i) \right\}. \end{aligned}$$

Also, let $S_{-i}^n = \times_{j \neq i} S_j^n$ and $S^n = \times_{i \in I} S_i^n$.

Finally, for each $i \in I$, let $S_i^\infty := \bigcap_{n>0} S_i^n$, and $S^\infty := \times_{i \in I} S_i^\infty$. For each $i \in I$, the strategies in S_i^∞ are called **strongly rationalizable**.

Characterization via Dominance, 1/2

- **Note:** $\times_{i\in I} S_i^1 = \rho(S) = NCD$. We want to go on and define a reduction procedure using conditional dominance that characterizes strong rationalizability $(S_i^n)_{i\in I, n\in\mathbb{N}}$.
- Fix $i \in I$, a nonempty $C = \times_{j\in I} C_j \subseteq S$, and $\bar{s}_i \in C_i$. Let $H(C) := \{h \in H : S(h) \cap C \neq \emptyset\}$.

Definition

Strategy $\bar{s}_i \in C_i$ is **conditionally dominated in** C if there are $h \in H_i(\bar{s}_i) \cap H(C)$ and $\sigma_i \in \Delta(C_i \cap S_i(h))$, such that

$$\forall s_{-i} \in C_{-i} \cap S_{-i}(h), \quad \sum_{s_i \in C_i \cap S_i(h)} \sigma_i(s_i) U_i(s_i, s_{-i}) > U_i(\bar{s}_i, s_{-i}).$$

We say that $\bar{s}_i \in C_i$ is **conditionally undominated in** C if it is *not* conditionally dominated in C. The set of strategies of player i that are *conditionally undominated in* C is denoted by $\mathrm{NCD}_i(C)$ and $\mathrm{NCD}(C) = \times_{i\in I}\mathrm{NCD}_i(C)$.

Characterization via Dominance, 2/2

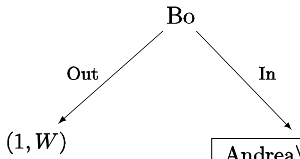
- **Iterated Conditional Dominance:** The reduction procedure of eliminating at each step n *all* the strategies conditionally dominated within the set of "survivors" of steps 1, ..., $n-1$ yields the weakly decreasing sequence $(\mathrm{NCD}^n(S))_{n\in\mathbb{N}}$.

Theorem

Strong rationalizability is characterized by iterated conditional dominance, that is, for all $n \in \mathbb{N} \cup \{\infty\}$, $S^n := \times_{i\in I} S_i^n = \mathrm{NCD}^n(S)$.

- In the *Entry Game* (and in every leader-follower game), initial and strong rationalizability coincide, and stop after 2 steps.
- In general, $S^1 = \mathrm{NCD}(S) = \rho(S)$, and $S^n = \mathrm{NCD}^n(S) \subseteq \rho^n(S)$ for all $n \geq 2$.
- E.g., analyze the *BoSOO* and *BoS with a dissipative action* (Burn)!

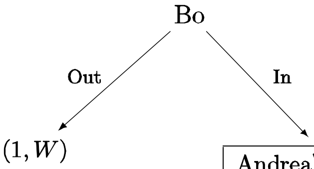
CoMOO (W=3): Initial vs Strong Rationalizability, 1/2



Andrea\Bo	C	M
home	1, 0	1, 0
c	2, 4	0, 0
m	0, 0	4, 2

- **home** is cond. dominated. If $W = 3$, In.M is dominated.
  - *Initial rationalizability* yields $\rho^{\infty}(S) = \rho(S) = NCD = \{\mathbf{c}, \mathbf{m}\} \times \{\text{Out}, \text{In.C}\}$.
  - **Problem:** Andrea may be initially certain of Out; if In, she would be surprised and could give up her belief in Bo's rationality.
  - Under *strong belief in rationality*, Andrea would be certain of In.C given In, and reply with *c*. Anticipating this, Bo plays In.C.

CoMOO (W=3): Initial vs Strong Rationalizability, 2/2



Andrea\Bo	C	M
home	1, 0	1, 0
c	2, 4	0, 0
m	0, 0	4, 2

- **home** is cond. dominated. If $W = 3$, In.M is dominated, thus:
  - $S_a^1 \times \mathbf{S}_b^1 = \mathrm{NCD}^1(S_a \times \mathbf{S}_b) = \{\mathbf{c}, \mathbf{m}\} \times \{\mathrm{Out}, \mathrm{In.C}\}$ $[= \rho^\infty(S)]$.
  - $S_a^2 \times \mathbf{S}_b^2 = \mathrm{NCD}^2(S_a \times \mathbf{S}_b) = \{\mathbf{c}\} \times \{\mathrm{Out}, \mathrm{In.C}\}$ (**m** conditionally dominated in $S_a^1 \times \mathbf{S}_b^1$).
  - $S_a^3 \times \mathbf{S}_b^3 = \mathrm{NCD}^3(S_a \times \mathbf{S}_b) = \{\mathbf{c}\} \times \{\mathrm{In.C}\}$ $[\subset \rho^\infty(S)]$ END.

Justifiability and Admissibility

- We can relate the previous concepts to weak dominance in the (reduced) strategic form.

Definition

Strategy $\bar{s}_i$ is **weakly dominated** (or inadmissible) **in** $C \in \mathcal{C}$ if there is $\sigma_i \in \Delta(C_i)$ s.t.

$$\begin{aligned} \forall s_{-i} &\in C_{-i},\ U_i(\sigma_i, s_{-i}) \geq U_i(\bar{s}_i, s_{-i}), \\ \exists \bar{s}_{-i} &\in C_{-i},\ U_i(\sigma_i, \bar{s}_{-i}) > U_i(\bar{s}_i, \bar{s}_{-i}); \end{aligned}$$

otherwise, $\bar{s}_i$ is **admissible** (non weakly dominated) **in** C.

Definition

A subset $N \subseteq \mathbb{R}^n$ is **negligible** if its closure has zero measure. [Examples: a finite set of points in $\mathbb{R}$, (subsets of) unions of finitely many lines in $\mathbb{R}^2$, or planes in $\mathbb{R}^3$.]

Rationalizability and Iterated Admissibility

Lemma

Fix a finite game tree and a nonempty $C \in \mathcal{C}$. For all payoff function profiles $u = (u_i : Z \to \mathbb{R})_{i \in I} \in \mathbb{R}^{Z \times I}$ except at most a negligible set, conditionally undominated and admissible strategies in C coincide; in particular (with $C = S$), $\rho(S) = S^1 = NCD = NWD$.

- Given $C \in \mathcal{C}$, $\mathrm{NWD}(C)$ denotes the set of strategy profiles not weakly dominated in C (restricted strategic form $\langle I, (C_i, U_i|C)_{i \in I} \rangle$).

Theorem

Fix a finite game tree. For all profiles $u \in \mathbb{R}^{Z \times I}$ except at most a negligible set, for all $n \in \mathbb{N}$, $S^n = \mathrm{NWD}^n(S)$.

- Verify the result in all the foregoing examples.

Backward Rationalizability

- Initial and strong rationalizability rely on the assumption that co-players' observed behavior—even when "surprising"—is interpreted as evidence about co-players' plans.
- **Backward rationalizability** instead *allows for the possibility that unexpected co-players' behavior is due to "mistakes"* in carriying out their plans, and yet no further "mistakes" will happen in the continuation-game, so that co-players' continuation strategies will be consisten with rationality and "strategic reasoning" in the continuation game. [Chapter 11.4 of GT-AST 11.4 gives the details.]
- Backward rationalizability characterizes the behavioral implications of *Rationality and "Common Future Belief in Rationality".* [Battigalli & De Vito (2021) offer an in-depth analysis and foundation.]

Two-Stage Games

- In *two-stage games*, find the backward rationalizable strategies with the following **backward procedure**:
  - 1. First, find the *rationalizable actions* in each *last-stage* (second-stage) subgame.
  - 2. Next *iteratively delete* strategies of the two-stage game that are not best replies to conjectures assigninig probability 0 to already deleted strategies, *taking into account Step 1*, i.e., co-players' strategies that select non-rationalizable actions in the second (last) stage must have probability 0.
- **Note**: The three versions of rationalizability for multistage games coincide in the even more special case of leader-follower games (two-stage games with perfect information).

References

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