Aggregation of Information and Beliefs: Asset Pricing Lessons from Prediction Markets*

Marco Ottaviani† Peter Norman Sørensen‡

February 2010

Abstract

This paper analyzes how asset prices in a binary market react to information when traders have heterogeneous prior beliefs. We show that the competitive equilibrium price underreacts to information when there is a bound to the amount of money traders are allowed to invest. Underreaction is more pronounced when prior beliefs are more heterogeneous. Even in the absence of exogenous bounds on the amount traders can invest, prices underreact to information provided traders become less risk averse as their wealth increases. In a dynamic setting, price changes are positively correlated over time. Thus the initial underreaction is followed by momentum.

Keywords: Aggregation of heterogeneous beliefs, Price reaction to information, Wealth effects.

JEL Classification: D82 (Asymmetric and Private Information), D83 (Search; Learning; Information and Knowledge), D84 (Expectations; Speculations).

*We thank Peter Bossaerts, Peter Ove Christensen, Tarek Coury, Morten Engberg, Erik Eyster, Christian Gollier, Piero Gottardi, Denis Gromb, Robin Hanson, Emeric Henry, Harrison Hong, Stephen Morris, Claudia Neri, Alessandro Pavan, Andrea Prat, Koleman Strumpf, Joel Watson, Justin Wolfers, Kathy Yuan, Anthony Ziegelmeyer, Eric Zitzewitz, and seminar participants at Aarhus, Bocconi, Brescia, Cambridge, Chicago, Copenhagen, Duke Law School, Erasmus University at Rotterdam, Fuqua School of Business, Gerzensee, Helsinki, Indiana, Institut Henri Poincaré (Paris), Keio (Tokyo), Kyoto, London Business School, London School of Economics, Lund, New York, Northwestern, Nottingham, Nottingham Trent, Oxford, Saïd Business School, Seoul, Toulouse, UCSD, Vienna, Warwick, WZB (Berlin), and Yale for helpful comments.

†Kellogg School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208-2013. Phone: +1–847–467–0684. Fax: +1–847–467–1777. E-mail: m-ottaviani@northwestern.edu. Web: http://www.kellogg.northwestern.edu/Faculty/ottaviani/homepage/.

‡Department of Economics, University of Copenhagen, Øster Farimagsgade 5, Building 26, DK–1353 Copenhagen K, Denmark. Phone: +45–3532–3056. Fax: +45–3532–3000. E-mail: peter.sorensen@econ.ku.dk. Web: http://www.econ.ku.dk/sorensen.

1 Introduction

Prediction markets are trading mechanisms that produce forecasts by aggregating the expectations of traders.[1] Given the simplicity of the trading environment and the availability of data on reasonably exogenous outcome realizations, prediction markets are ideal laboratories to test theories of market efficiency.[2] Inspired by some of the key features of prediction markets, this paper investigates how asset prices relate to the posterior beliefs of traders. Our analysis uncovers a novel theoretical mechanism through which prices underreact to information under the realistic assumption that traders with heterogeneous prior beliefs are subject to wealth effects. This mechanism can explain underreaction patterns that are widely documented in prediction markets (in the form of the favorite-longshot bias) as well as in more general financial markets (in the form of post-earning announcement drift and stock price momentum).

Consider a market for a binary event such as the outcome of a presidential election. Traders can take positions in two Arrow-Debreu contingent assets, each paying one dollar if the corresponding outcome occurs. Each trader's initial endowment is constant with respect to the outcome realization. Taking into account a typical institutional feature of prediction markets, in our baseline model we restrict each trader to investing a limited amount of money in the market.[3]

Given that traders have limited experience with the underlying events, we allow them to have heterogeneous *prior beliefs*.[4] These initial opinions are subjective and thus are uncorrelated with the realization of the outcome.[5] Having different prior beliefs, traders gain from trading actively. While trade may be motivated by the heterogeneity of prior beliefs, outside observers are typically interested in extracting information.[6] Therefore,

[1] It is often argued that forecasts generated by prediction markets are more accurate and less expensive than those obtained through more traditional methods, such as opinion polls or judgement by experts. See, for instance, Forsythe et al. (1992) and Berg et al. (2008) on the track record of the Iowa Electronic Markets since 1988. Partly thanks to their track record as forecasting tools, prediction markets are attracting growing interest as mechanisms to collect information and improving decision making in business and public policy contexts. See Hanson (1999), Wolfers and Zitzewitz (2004), and Hahn and Tetlock (2005).

[2] Indeed, the Iowa Electronic Markets were initially developed for educational purposes.

[3] For example, traders in the Iowa Electronic Markets are allowed to wager up to $500.

[4] Typically, prediction markets target unique events, such as the outcome of a presidential election or the identity of the winner in a sport contest.

[5] For the purpose of our analysis, traders' subjective prior beliefs play the role of exogenous parameters, akin to the role played by preferences. In the context of more general financial markets, Hong and Stein (2007) survey a large body of evidence that points to heterogeneity of opinions.

[6] This conceptual distinction between prior beliefs and information is standard. An individual revises

our model allows individual traders to have access to *information* about the outcome. Information has an objective nature because it is correlated with the outcome. To sharpen our result, we assume that all traders agree on their heterogeneous priors and interpret information in the same way, so that beliefs are "concordant" in Milgrom and Stokey's (1982) terminology.

How does the price react to information that becomes publicly available to all traders? How does the price relate to the traders' posterior beliefs? Given that traders agree on how to interpret information, differences in the posterior beliefs of traders are uniquely due to differences in the prior beliefs. However, we show that the market price does not behave like a posterior belief, because there is no "market" prior belief for which the equilibrium price is the Bayesian posterior update that incorporates the available information. This is striking in light of belief concordance—all traders agree on how to adjust their beliefs about the relative likelihood of the two states, but the relative asset price does not capture this agreement. Instead, the equilibrium price underreacts to information. This result amends the common interpretation that the price of an Arrow-Debreu asset represents the market's belief about the probability of the event.

To understand the mechanism driving the result, consider a hypothetical market based on which team, Italy or Denmark, will win a soccer game. Suppose that the risk-neutral traders are subjectively more optimistic about Italy winning, the further south they live. In equilibrium, traders living south of a certain threshold latitude bet all they can on the asset that pays if Italy wins. Likewise, traders north of the threshold latitude invest in the Denmark asset the maximum amount of money allowed.

Now, what happens when traders observe information more in favor of Italy winning? This information causes the price of the Italy asset to be higher, while contemporaneously reducing the price of the Denmark asset, compared to the case with less favorable information. As a result, the southern traders (who are optimistic about Italy) are able to buy fewer Italy assets, which are now more expensive. Similarly, the northern traders can afford, and thus demand, more Denmark assets, now cheaper. Hence, the market would have an excess supply of the Italy asset and excess demand for the Denmark asset. For the market to equilibrate, some northern traders must turn to the Italian side. In summary,

her belief when learning someone else's information rather than prior belief. As Aumann (1976) notes, "reconciling subjective probabilities makes sense if it is a question of implicitly exchanging information, but not if we are talking about 'innate' differences in priors."

when information more favorable to an outcome is available, the marginal trader who determines the price has a prior belief that is *less* favorable to that outcome. Through this countervailing adjustment, the heterogeneity in priors dampens the effect of information on the price.

Note that underreaction also results if the information that is revealed was initially privately held by the traders. For such distribution of private information across traders, we show that there exists a rational expectations equilibrium (REE) in which all relevant information is fully revealed. Nevertheless, the price underreacts to the information jointly possessed by the traders. This result is consistent with the widespread observation of the favorite-longshot bias in betting markets, whereby the price of favorites underestimates the corresponding empirical probability, while the price of longshots overestimates it.

Our explanation for underreaction is crucially different from the one proposed in the fledgling literature on prediction markets, pioneered by Ali (1977) and recently revived by Manski (2006), Gjerstad (2005), and Wolfers and Zitzewitz (2005). They analyze the relation between the equilibrium price and the median or average belief of traders, depending on the traders' preferences for risk. Underreaction is explained with the auxiliary assumption that the median or average belief corresponds to the empirical probability. But this assumption is contentious. If the traders' beliefs really have information content, their positions should depend on the information about these beliefs that is contained in the market price. This tension underlies the rational expectations critique of the Walrasian approach to price formation with heterogeneous beliefs (see the discussion in Chapter 1 of Grossman, 1989). Our analysis shows that underreaction results without this auxiliary assumption, and thus is immune to the critique.

Our baseline model with bounded wealth is already applicable to financial markets where typically traders have a finite wealth and/or can borrow a finite amount of money due to imperfections in the credit market. Next, we turn to our analysis of a classic financial market, where traders are risk averse but may not be exogenously constrained by their wealth or by the amount they can borrow. Building on Milgrom and Stokey (1982), we allow traders to have arbitrary risk preferences, heterogeneous priors, and concordant information.[7] To the well-known characterization of equilibrium, we add a comparative

[7] To encompass the prediction market model as a special case, the full model further allows for wealth constraints, even though these are not necessary for the results.

statics analysis of the first-round equilibrium price with respect to information. We show that underreaction holds under the empirically plausible assumption that traders have decreasing absolute risk aversion, *even* when no exogenous bound is imposed on the traders' wealth. As in the baseline model, when favorable information is revealed, traders who take long positions on the asset that now becomes more expensive suffer a negative wealth effect. Hence these traders become more risk averse and cut back their positions.

This analysis integrates results from the classic literature on belief aggregation (Wilson, 1968, Lintner, 1969, and Rubinstein, 1974) with work on the reaction of prices to information (Grossman, 1976, and Radner, 1979). As we show, wealth effects generate underreaction to information because the price allocates an increased weight to traders with beliefs that are contrary to the realized information. Underreaction results once we relax simultaneously two of Grossman's (1976) assumptions, no wealth effects and common prior:

- Wealth effects are essential to the result, because otherwise heterogeneous beliefs can be aggregated, as shown by Wilson (1968). As compared to Varian's (1989) generalization of Grossman (1976) to heterogeneous priors, we further introduce wealth effects by imposing a limit on the amount that traders can invest (in the baseline prediction market model) *or*, more generally, by relaxing the assumption of constant absolute risk aversion.

- Heterogeneity in priors cannot be replaced by heterogeneity in endowments across traders. Extending a classic result by Rubinstein (1974), we show that there is no systematic underreaction when traders have common prior but heterogeneous endowments for a prominent class of preferences (hyperbolic absolute risk aversion with common cautiousness parameter) for which wealth effects are allowed. With heterogeneous endowments, more favorable information induces all traders, buyers and sellers, to take more extreme positions. With heterogeneous priors, instead, more favorable information induces optimists (who buy) to buy less, but induces pessimists (who sell) to sell more, so that the price must equilibrate in a direction contrary to information.

In a dynamic extension of the model, we characterize the correlation of the price changes resulting in the periods that follow the first round of trade. We find that the first-round underreaction is followed by price momentum. Intuitively, the arrival of additional information over time partly undoes the initial underreaction. As such, our analysis uncovers a novel explanation for why prices in financial markets underreact to information and exhibit momentum—a long-standing puzzle documented by a large empirical literature in finance (for example see Jagadeesh and Titman, 1993).

There is a substantial theoretical literature about price reaction to information. In one strand of the literature, market prices deviate from fundamental value because of patterns in noise trading—recent contributions are Cespa and Vives (2009) and Serrano-Padial (2010). Another strand relaxes the assumption of concordant beliefs by allowing traders to interpret information differently or incorrectly. Thus Barberis, Shleifer, and Vishny (1998) derive momentum by assuming that traders are mistaken about the correct information model, while Hong and Stein (1999) posit that information diffuses gradually and is not fully understood by all traders.[8] Allen, Morris, and Shin (2006) assume that the information model is not common knowledge and find that traders overweight the common public information. In contrast to this literature, we obtain the striking result of underreaction and momentum even when all traders agree about the correct interpretation of information.

Our work also relates to an extensive finance literature in which heterogeneous beliefs across traders give rise to a speculative premium in asset prices—see Miller (1977), Harrison and Kreps (1978), and Morris (1996), and the survey by Scheinkman and Xiong (2004). In this literature, traders are typically subject to short-sale constraints, so that in equilibrium the entire net supply of an asset is held by the most optimistic trader. Once this trader's belief is taken as the market belief, there is no systematic underreaction to information. To this literature we contribute an analysis of a more balanced market, constraining both optimists and pessimists. Our key observation is that the identity of the marginal trader and, more generally, the weight accorded to the beliefs of different traders are endogenously determined and depend in a predictable way on the realized information.

In sequence, Section 2 illustrates our underreaction result in the context of a prediction market with risk-neutral traders. Section 3 turns to the full model of a financial market with risk-averse traders. Section 4 shows that momentum arises in a natural dynamic extension of the model. Section 5 concludes. The appendix collects the proofs.

[8] Recent contributions are Fostel and Geanakoplos (2008) and Palfrey and Wang (2009).

2 Prediction Market Model

Our baseline model is inspired by the rules of the Iowa Electronic Markets for a binary prediction market, in which traders can take positions on whether an event, E, is realized (e.g., the Democratic candidate wins the 2008 presidential election) or not. There are two Arrow-Debreu assets corresponding to the two possible realizations: one asset pays out 1 currency unit if event E is realized and 0 otherwise, while the other asset pays out 1 currency unit if the complementary event E^c is realized and 0 otherwise.[9]

Traders enter the market by first obtaining an equal number of both assets. Essentially, the designer of the prediction market initially endows each trader i with w_{i0} units of each of the two assets. One important feature of the market is that there is a limit on how much money each trader can invest.[10] After entering the market, traders can exchange their assets with other traders. A second key feature of the market is that traders are not allowed to hold a negative quantity of either asset. As explained below in more detail, these two restrictions (on the amount of money invested and on the number of assets a trader can sell) impose a bound on the number of asset units that each individual trader can purchase and eventually hold.[11]

Markets clear when the aggregate demand for asset 1 precisely equals the aggregate demand for asset 2. We normalize the sum of the two asset prices to one, and focus on the price p of the asset paying in event E.

We assume that there is a continuum I of risk-neutral traders who aim to maximize their subjective expected wealth.[12] Trader i maximizes $\pi_i w_i(E) + (1 - \pi_i) w_i(E^c)$, where π_i denotes the trader's subjective belief. We now turn to the process that determines the trader's subjective belief, π_i.

[9] The state of the world is given exogenously and cannot be affected by the traders. This assumption is realistic in the case of prediction markets on economic statistics, such as non-farm payroll employment. When applied to corporate decision making, prediction market traders might have incentives to manipulate the outcome. Ottaviani and Sørensen (2007) analyze outcome manipulation, disregarding the wealth effect on which we concentrate in this paper. Lieli and Nieto-Barthaburu (2009) extend the analysis to allow for the possibility of feedback, whereby a decision maker acts on the basis of the information revealed by the market.

[10] For example, in the Iowa Electronic Markets each trader cannot invest more than \$500. Exemption from anti-gambling legislation is granted to small stake markets created for educational purposes.

[11] Our main result (Proposition 2) hinges on the property that this bound is endogenous to the model, because the number of assets each trader eventually holds depends on the market-clearing prices.

[12] The results derived in this section immediately extend to the case of risk-loving traders, whose behavior is also to adopt an extreme asset position. We turn to risk-averse traders in Section 3.

Initially, trader i has subjective prior belief q_i. Before trading, all traders can observe a public signal s. Conditional on state $\omega \in \{E, E^c\}$, we let $f(s|\omega)$ denote the probability density of the signal. The likelihood ratio for signal realization s is defined as $L(s) = f(s|E)/f(s|E^c)$. The only constraint imposed on the signal distribution is that there is zero probability of fully state-revealing signals, so $L(s) \in (0, \infty)$ with probability one. If trader i observes the realized signal s, then by Bayes' rule the subjective posterior belief π_i satisfies

$$\frac{\pi_i}{1-\pi_i} = \frac{q_i}{1-q_i} L(s). \tag{1}$$

Hence, $L(s)$ is a sufficient statistic for the signal s.

For convenience, we normalize the aggregate endowment of each asset to 1. The initial distribution of assets over individuals is described by the cumulative distribution function G. Thus $G(q) \in [0,1]$ denotes the share of all assets initially held by individuals with subjective prior belief less than or equal to q. We assume that G is continuous, and that G is strictly increasing on the interval where $G \notin \{0,1\}$.[13]

We assume that the model (i.e., preferences, prior beliefs, and signal distributions) and the rationality of all traders are common knowledge. This means that all traders agree on the conditional distributions $f(s|\omega)$, even though they have heterogeneous prior beliefs—thus posterior beliefs are concordant, the leading case considered by Milgrom and Stokey (1982).

Before proceeding, we discuss the roles played by heterogeneous priors and concordant beliefs in our model. Our results crucially depend on the heterogeneity of posterior beliefs across traders. The assumption that traders have concordant beliefs about a publicly observed signal serves to make our main result particularly striking. Even though each and every individual trader's belief is updated in a Bayes rational way in response to the same information, the market price moves by less than Bayes' rule would predict. On the other hand, we depart from the parsimonious assumption that traders share a common prior mostly on grounds of realism.[14] In reality, traders are unlikely to have experienced similar events in the past.[15]

[13] The assumption that the priors are continuously distributed is made to simplify the analysis, but is not essential for our underreaction result. See the discussion in footnote 16.

[14] As in most work on heterogeneous priors, prior beliefs are given exogenously in our model. We refer to Brunnermeier and Parker (2005) for a model in which heterogeneous prior beliefs arise endogenously.

[15] The common prior assumption is sensible when traders are dealing with objective uncertainty and with commonly experienced events, but it is not an implication of rational decision making.

2.1 Competitive Equilibrium

This section characterizes the equilibrium when traders are allowed to exchange assets with other traders in a competitive market. By normalization, the prices of the two assets sum to one, and we focus on the equilibrium determination of the relative price p for the asset that pays out in event E.

For every L, trader i's demand solves this trader's maximization problem, given belief $\pi_i(L)$ satisfying (1), and given market price $p(L)$. Market clearing requires the price to be such that aggregate net demand is zero, or that the aggregate holding of each asset equals aggregate wealth (normalized to 1).

Solving the choice problem of the risk-neutral traders is straightforward. Suppose trader i has information with likelihood ratio L resulting in a posterior belief equal to π_i, and suppose that the market price is p. The subjective expected return on the asset that pays out in event E is $\pi_i - p$, while the other asset's expected return is $(1-\pi_1)-(1-p) = p - \pi_i$. With the designer's constraint on asset portfolios, individual demand thus satisfies the following: if $\pi_i > p$, trader i exchanges the entire endowment of the E^c asset into $(1-p)\, w_{i0}/p$ units of the E asset. The final portfolio is then w_{i0}/p units of the E asset and 0 of the E^c asset. Conversely, when $\pi_i < p$, the trader's final portfolio is 0 of the E asset and $w_{i0}/(1-p)$ of the E^c asset. Finally, when $\pi_i = p$, the trader is indifferent between any trade.

Proposition 1 *The competitive equilibrium price, p, is the unique solution to the equation*

$$p = 1 - G\left(\frac{p}{(1-p)\, L + p}\right) \tag{2}$$

and is a strictly increasing function of the information realization L.

2.2 Underreaction to Information

Using Bayes' rule (1), we can always interpret the price as the posterior belief of a hypothetical individual with initial belief $p/[(1-p)\, L + p]$, where the price p is a function of L, according to (2). This implied ex ante belief then may be interpreted as an aggregate of the heterogeneous subjective prior beliefs of the individual traders. Our main result is that this belief depends in a systematic fashion on the information that is initially available to traders. This means that the aggregation of beliefs cannot be separated from the realization of information.

Proposition 2 *If beliefs are truly heterogeneous, i.e. $q_i \neq q_j$ for some pair of traders, then the implied ex ante belief for the market*

$$\frac{p}{(1-p)\,L + p}$$

is strictly decreasing in L.

The arrival of more favorable pre-trade information yields a higher market price that nevertheless underreacts to the information. Consider the inference of an observer, such as the market designer, any of the traders, or a truly outside observer. Given common knowledge of the model and the observation of a price that reveals information L, this observer's posterior probability, $\pi(L)$, for the event E derived from any fixed prior belief q satisfies (1), or

$$\log\left(\frac{\pi(L)}{1-\pi(L)}\right) = \log\left(\frac{q}{1-q}\right) + \log L. \tag{3}$$

The expression on the left-hand side is the posterior log-likelihood ratio for event E, which clearly moves one-to-one with changes in $\log L$. Proposition 2 implies that this observer's belief reacts more than the price:

Proposition 3 *If beliefs are truly heterogeneous, then for any two different information realizations, L and $L' > L$, we have*

$$\log\left(\frac{\pi(L')}{1-\pi(L')}\right) - \log\left(\frac{\pi(L)}{1-\pi(L)}\right) > \log\left(\frac{p(L')}{1-p(L')}\right) - \log\left(\frac{p(L)}{1-p(L)}\right) > 0. \tag{4}$$

To understand the intuition for this underreaction result, consider what happens when traders have information more favorable to event E (corresponding, say, to the Democratic candidate winning the election), i.e., when L is higher. According to (2), the price of the E asset, p, is clearly higher when L is higher. Now, this means that traders who are optimistic about a Democratic victory can buy fewer units of asset E, because the bound w_{i0}/p is decreasing in p. In addition, traders who are pessimistic about a Democratic victory can buy more units of asset E^c, which they want to buy. If all the traders who were purchasing E before the increase in L were still purchasing E at the higher price that results with higher L, there would be insufficient demand for E. Similarly, there would also be excess demand for E^c. To balance the market it is necessary that some traders who were betting on the Republican candidate before now change sides and put their money on the Democratic candidate. In the new equilibrium, the price must change to move traders from the pessimistic to the optimistic side. Thus the indifferent trader who determines the equilibrium price at the margin holds a more pessimistic prior belief about Democratic victory, the more favorable to Democratic candidate (i.e., the higher) the information, L, is. Hence, although the price, p, rises with the information, L, it rises more slowly than a posterior belief, because of this negative effect on the prior belief of the marginal trader.[16]

The underreaction result derived in this section is driven by the restriction on the amount of money invested (see footnote 10) and, therefore, on the number of assets a trader can sell. In turn, this restriction imposes a bound on the number of assets that each individual can purchase and eventually hold. The result hinges on the fact that this bound (equal to w_{i0}/p) is inversely related to the equilibrium price.[17]

The result says that outcomes favored by the market occur more often than if the market price is interpreted as a probability—and, conversely, longshots win less frequently than suggested by the market price. To see how this effect arises in our context, compare the market price, $p(L)$, with the posterior belief, $\pi(L)$, held by an outside observer with (fixed) prior belief q:

Proposition 4 *If beliefs are truly heterogeneous, there exists a market price* $p^* \in [0,1]$ *with the property that* $p(L) > p^*$ *implies* $\pi(L) > p(L)$*, and* $p(L) < p^*$ *implies* $\pi(L) < p(L)$*.*

Thus there is a threshold level, p^*, for the realized market price, such that a market price below p^* classifies event E as a longshot and a market price above p^* makes E a favorite. A market observer expects longshots to occur less often than indicated by the market price, and favorites to occur more often. Proposition 3 offers a new informational explanation of the favorite-longshot bias, a widely-documented fact in the empirical literature on betting markets (see Thaler and Ziemba, 1988, and Snowberg and Wolfers, 2005).[18]

[16] The assumption that prior beliefs are continuously distributed is not essential. If the population is finite, or there are gaps in the distribution, then there can be ranges of information, L, over which the equilibrium price is constant. In that case, the rational expectations equilibrium cannot fully reveal L, but can still reveal the information needed for the proper allocation of the assets. Underreaction still occurs with respect to the revealed information.

[17] Imagine that the market designer imposes a direct cap on the number of assets that each trader can buy, rather than on the budget each trader can invest. Then, over a large range of information realizations, a constant set of optimists (or pessimists) buys the full allowance of the E (or E^c) asset. Since the marginal trader is constant, there is no underreaction. However, prediction markets typically bound the traders' budget, rather than the asset position.

[18] Wolfers and Zitzewitz (2004) report that the favorite-longshot bias has been observed in a prediction market for Standard and Poor's 500 index.

Ali (1977, Theorem 2) provides an antecedent to our underreaction result. In a model of equilibrium betting with heterogeneous prior beliefs, Ali notes that if the median bettor thinks that one outcome (defined to be the favorite) is more likely than the other, then the equilibrium fraction of parimutuel bets on this favorite outcome is lower than the belief of the median bettor. Under the key assumption that the belief of the median bettor is the correct benchmark for the empirical probability, Ali concludes that favorites are underbet as compared to longshots. Instead, we make no such assumption, and remain agnostic about the relationship between (the distribution of prior) beliefs and the empirical chance of the outcome. Rather, we derive underreaction as a comparative statics result with respect to the revelation of information. Our analysis is consistent with Ali's, but our approach is more appropriate for empirical testing, because the different opinions underlying the heterogeneous priors of traders have no information content and, thus, should have no bearing on the empirical probabilities.

2.3 Comparative Statics

We are now ready to show that underreaction is more pronounced if traders' heterogeneous beliefs are more dispersed. In analogy with Rothschild and Stiglitz's (1970) definition of mean preserving spread, define distribution G' to be a *median-preserving spread* of distribution G if G and G' have the same median m and satisfy $G'(q) \geq G(q)$ for all $q \leq m$ and $G'(q) \leq G(q)$ for all $q \geq m$.

Proposition 5 *Suppose that G' is a median-preserving spread of G, denoting the common median by m. Then, more underreaction results under G' than under G: $L > (1-m)/m$ implies $\pi(L) > p(L) > p'(L) > 1/2$, and $L < (1-m)/m$ implies $\pi(L) < p(L) < p'(L) < 1/2$.*

This result is in line with empirical evidence by Verardo (2009) that momentum profits are significantly larger for portfolios characterized by higher heterogeneity of beliefs.

Example. To illustrate our results, suppose that the distribution of subjective prior beliefs over the interval $[0,1]$ is $G(q) = q^\beta / \left[q^\beta + (1-q)^\beta\right]$, where $\beta > 0$ is a parameter that measures the concentration of beliefs. The greater is β, the less spread is this symmetric belief distribution around the average belief $q = 1/2$. For $\beta = 1$ beliefs are uniformly

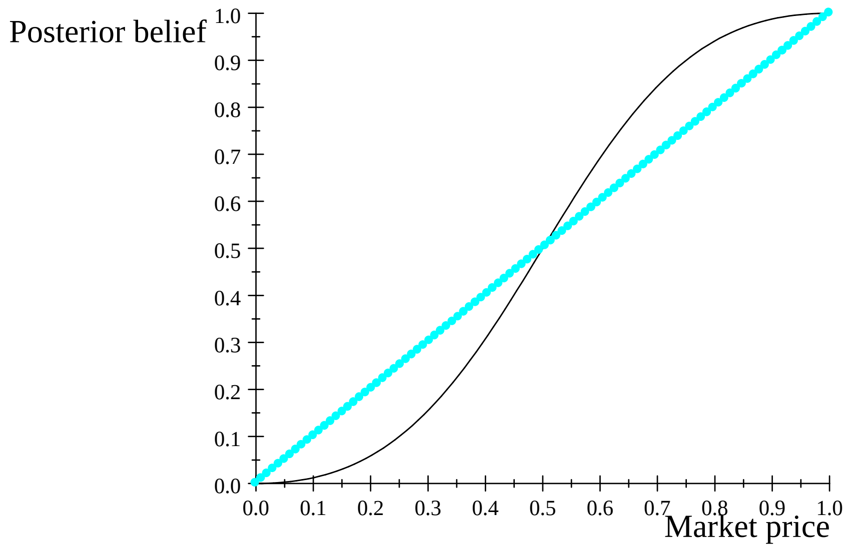

Figure 1: This plot shows the posterior probability for event E as a function of the market price p for the E asset, when the prior beliefs of the risk-neutral traders are uniformly distributed ($\beta = 1$ in the example). The market price is represented by the dotted diagonal.

distributed, as $\beta \to \infty$ beliefs become concentrated near $1/2$, and as $\beta \to 0$ beliefs are maximally dispersed around the extremes of $[0, 1]$. The equilibrium condition (2) becomes

$$\log\left(\frac{p}{1-p}\right) = \log\left(\frac{1 - G\left(\frac{p}{(1-p)L+p}\right)}{G\left(\frac{p}{(1-p)L+p}\right)}\right) = \beta \log\left(\frac{(1-p)\,L}{p}\right),$$

so that the market price $p(L)$ satisfies the linear relation

$$\log\left(\frac{p(L)}{1-p(L)}\right) = \frac{\beta}{1+\beta}\log L.$$

Hence, $\beta/(1+\beta) \in (0, 1)$ measures the extent to which the price reacts to information. Price underreaction is minimal when β is very large, corresponding to the case with nearly homogeneous beliefs. Conversely, there is an arbitrarily large degree of underreaction when beliefs are maximally heterogeneous (i.e. β is close to zero).

Assume that a market observer's prior is $q = 1/2$ for event E, consistent with a symmetric market price of $p(1) = 1/2$ in the absence of additional information. The posterior belief associated with price p then satisfies

$$\log\left(\frac{\pi(L)}{1-\pi(L)}\right) = \log L = \frac{1+\beta}{\beta}\log\left(\frac{p(L)}{1-p(L)}\right).$$

As illustrated in Figure 1 for the case with uniform beliefs ($\beta = 1$), the market price overstates the winning chance of a longshot and understates the winning chance of a favorite by a factor of two.

2.4 Fully Revealing REE with Private Information

Although we have assumed that the signal with likelihood ratio L is publicly observed, our equilibrium is compatible with the fully revealing rational expectations equilibrium (REE) when instead information is private.[19] Such an interpretation is relevant in a prediction market since the designer usually intends for the market to aggregate information that is initially dispersed among market participants. Thus suppose that $s = \{s_i\}_{i \in I}$, where signal s_i is privately observed by trader i before trading. To solve for a fully revealing REE, we must first imagine that traders behave as if the market price p reveals all information. They will then form their individual demand functions as before, and market clearing implies that the price functional must be the solution to (2). Second, it must be verified that this unique candidate for the fully revealing REE price functional is indeed fully revealing. This is the case because the solution to (2) is a strictly increasing function of the sufficient statistic L:

Proposition 6 *In the extension of the model with private information, the price p that solves equation (2) is a strictly increasing function of the information realization L, so that it characterizes the unique fully revealing REE.*

Our analysis could potentially be extended to a partially revealing rational expectations equilibrium. However, the residual private information not revealed by the market then generates private belief heterogeneity that may, in general, depend on the realized L. In principle, there could then be a systematic relationship between favorable information and the extent of belief heterogeneity.[20]

Because the assumption that the market reaches a fully revealing equilibrium is not warranted for some market rules, ours is the most optimistic scenario for information

[19] We focus on the necessary properties of this equilibrium, while Radner (1979) discusses sufficient conditions for its existence.

[20] This is reminiscent of Diamond and Verrecchia's (1987) model of market making. When the realized news is negative for an asset, short-sale constrained informed traders are less frequently encountered in the market, and it takes longer for the market to learn negative news.

aggregation.[21] Indeed, it may be questionable to expect traders to reach the rational expectations equilibrium when priors are heterogeneous.[22] Nevertheless, equilibrium is a useful benchmark, and we find it plausible that individuals make some inferences about the state from the realized price in the presence of asymmetric information. Even though these inferences need not be as correct in reality as they are assumed to be in a REE, it is a strength of our theory that it works under this narrow, standard assumption.

Relaxing the REE assumption, Ottaviani and Sørensen (2009) and (2010) analyze a game-theoretic model of parimutuel betting where traders have a common prior but are unable to condition their behavior on the information that is contained in the equilibrium price.[23] They find that the price underreacts or overreacts to information depending on the amount of information relative to noise that is present in the market. In contrast, in this paper traders are allowed to perfectly share information, but the price underreacts to information nevertheless when the priors are heterogeneous and wealth effects are present.

3 Asset Market Model: Risk Aversion

So far we have assumed that each individual trader is risk neutral, and thus ends up taking as extreme a position as possible on either side of the market. Now, we show that our main result extends nicely to risk-averse individuals, under the empirically plausible assumption that traders' absolute risk aversion is decreasing with wealth. This result does not rely on imposing exogenous constraints on the wealth traders are allowed to invest.

3.1 Model and Equilibrium

Realistically, suppose that each prediction market trader is initially endowed with the same number w_0 of each asset. To properly capture the effect of risk aversion, we suppose that each trader i is also characterized by an initial, state-independent level W_i of additional

[21] Plott and Sunder (1982) and Forsythe and Lundholm (1990) experimentally investigate the conditions leading to equilibrium in settings with differential private information.

[22] Morris (1995) shows that the REE concept in general relies on strong common knowledge assumptions. We are implicitly extending the usual REE assumption that traders commonly know each others' preferences to common knowledge of the heterogeneous prior distribution, G. As argued by Dekel, Fudenberg, and Levine (2004), it is more difficult to justify equilibrium on grounds of learning when players have heterogeneous priors.

[23] See also Shin (1991) and (1992) for a derivation of the favorite-longshot bias when prices are quoted by an uninformed monopolist bookmaker. We refer to Ottaviani and Sørensen (2010) for a discussion of (and references to) explanations of the favorite-longshot bias that are not based on information.

wealth which cannot be brought into the prediction market.[24] Trader i maximizes subjective expected utility of final wealth, $\pi_i u_i(w_i(E)) + (1-\pi_i) u_i(w_i(E^c))$, where π_i is the trader's subjective belief. We suppose that u_i is twice differentiable with $u_i' > 0$ and $u_i'' < 0$, and satisfies the DARA assumption that the de Finetti-Arrow-Pratt coefficient of absolute risk aversion, $-u_i''/u_i'$, is weakly decreasing with wealth, w_i. The cumulative distribution function G describes the distribution of subjective beliefs across traders $i \in [0,1]$, and it is assumed to satisfy the same properties as before. Public information is distributed as in the baseline model (and the same reinterpretation in terms of REE remains valid).

In the market, traders can exchange their asset endowments. To recover the baseline model as a special case (once the coefficient of absolute risk aversion is constant and equal to zero), we allow traders to be subject to a constraint on the wealth they can invest. Note that this constraint does not bind, unless the trader is nearly risk neutral or the difference between π_i and p is sufficiently large. We stress that our underreaction result holds regardless of whether this constraint is binding. In analogy with Proposition 6 we have:

Proposition 7 *There exists a unique competitive equilibrium. The price, p, is a strictly increasing function of the information realization L.*

3.2 Belief Aggregation with CARA Preferences

Suppose first that the traders have constant absolute risk aversion (CARA) utility functions, with heterogeneous degrees of risk aversion, such that $u_i(w) = -\exp(-w/t_i)$, where $t_i > 0$ is the constant coefficient of risk tolerance, the inverse of the coefficient of absolute risk aversion. Denoting the relative risk tolerance of trader i in the population by $\tau_i = t_i / \int_0^1 t_j dG(q_j)$, we have:

Proposition 8 *Suppose traders have CARA preferences and heterogeneous beliefs. Define an average prior belief q by*

$$\log \frac{q}{1-q} = \int_0^1 \tau_i \log \frac{q_i}{1-q_i} dG(q_i), \tag{5}$$

and for each individual let

$$d_i^* = t_i \log\left(\frac{q_i - q q_i}{q - q q_i}\right). \tag{6}$$

[24] See Musto and Yilmaz (2003) for a model in which, instead, traders are subject to wealth risk, because they are differentially affected by the redistribution associated with different electoral outcomes.

Suppose that $1 + w_0/\inf_i d_i^* < w_0/\sup_i d_i^*$*. When the realized information* L *is in the range satisfying*

$$1 + \frac{w_0}{\inf_i d_i^*} \leq \frac{qL}{qL + 1 - q} \leq \frac{w_0}{\sup_i d_i^*} \tag{7}$$

then the equilibrium price satisfies Bayes' rule with market prior q*. When* L *falls outside this range, the price underreacts to changes in* L*.*

Risk aversion allows for the possibility that no trader is bound by the trading constraint. This is more likely to happen when w_0 is large, as also suggested by condition (7). With CARA preferences and when no trader is constrained, trader i chooses net demand d_i^* in equilibrium. Under CARA, wealth effects vanish and heterogeneous beliefs can be aggregated, according to formula (5), consistent with the classic result of Wilson (1968), Lintner (1969), and Rubinstein (1974). The market price thus behaves as a posterior belief and there is no underreaction.

3.3 Underreaction with DARA Preferences

We have seen that CARA preferences lead to an unbiased price reaction to information when the trading constraints are not binding in equilibrium. Now we verify that, for strict DARA preferences, a bias arises in the price, whether traders are constrained or not. When L rises, the rising equilibrium price yields a negative wealth effect on any optimistic individual (with $\pi_i > p$) who is a net demander ($\Delta x_i > 0$). Conversely, pessimistic traders benefit from the price increase. With DARA preferences, the wealth effect implies that optimists become more risk averse while pessimists become less risk averse. Although the price rises with L, it is less reactive than a posterior belief, because pessimists trade more heavily in the market when information is more favorable.[25]

Proposition 9 *Suppose that beliefs are truly heterogeneous and that all individuals have strict DARA preferences. The market price underreacts to information, as for any* $L' \neq L$*,*

$$\left|\log\left(\frac{\pi(L')}{1-\pi(L')}\right) - \log\left(\frac{\pi(L)}{1-\pi(L)}\right)\right| > \left|\log\left(\frac{p(L')}{1-p(L')}\right) - \log\left(\frac{p(L)}{1-p(L)}\right)\right|.$$

[25] Given that CARA is the knife-edge case, reversing the logic of Proposition 9 it can be shown that overreaction results when risk aversion is increasing but not too much (so that demand monotonicity is preserved).

The REE literature typically assumes that traders have a common prior belief (Grossman, 1976). Under the common prior assumption, the price reacts one-for-one to information, regardless of risk attitudes. Our underreaction result thus hold once we allow for *both* heterogeneous priors *and* wealth effects.

Example with Logarithmic Preferences. Suppose that prior beliefs are uniformly distributed over the interval $[0, 1]$, with $G(q) = q$, and traders have logarithmic preferences, $u_i(w) = \log w$, satisfying the DARA assumption. In order to highlight the difference between Propositions 3 and 9, namely the inclusion of individuals with an interior solution to their maximization problem, we remove completely the trading constraint. The well-known solution to the individual demand problem with Cobb-Douglas preferences gives $w_i(E) = \pi_i(W_i + w_0)/p$. The market-clearing price is then an arithmetic average of the posterior beliefs,

$$p(L) = \int_0^1 \pi(L)\, dq = \int_0^1 \frac{qL}{qL + (1-q)}\, dq. \tag{8}$$

For $L \neq 1$, integration by parts of (8) yields $p(L) = L(L - 1 - \log L)/(L-1)^2$. If we keep fixed $p(1) = \int_0^1 q\, dq = 1/2$ as the prior belief of the outside observer, the favorite-longshot bias can be illustrated in a graph with the same qualitative as Figure 1.

Edgeworth Box Illustration. We can graphically illustrate the logic of Proposition 9 for a market with two types of traders (with prior beliefs $q_1 < q_2$) and no trading constraints. As shown in Figure 2, the Edgeworth box is a square because there is no aggregate uncertainty. The initial endowment, e, lies on the diagonal. Traders have convex indifference curves, which are not drawn to avoid cluttering the picture. The slope of the indifference curves at any safe allocation is $-\pi_i/(1-\pi_i) = -q_iL/(1-q_i)$, so along the diagonal trader 2 (optimist) has steeper indifference curves than trader 1 (pessimist). In equilibrium, the marginal rates of substitution are equalized. Thus the equilibrium allocation, w^*, lies above the diagonal, where the optimist buys asset E.

How is the equilibrium affected by an exogenous change in information from L to $L' > L$? Marginal rates of substitution are affected such that all indifference curves become steeper by a factor of L'/L. For the sake of argument, imagine that the price were to change as a Bayesian update of market belief $p(L)$ to $p' = p(L)L'/(p(L)L' + (1 - p(L))L)$. Since $p' > p(L)$, the new budget line through e passes above w^*, illustrating the wealth

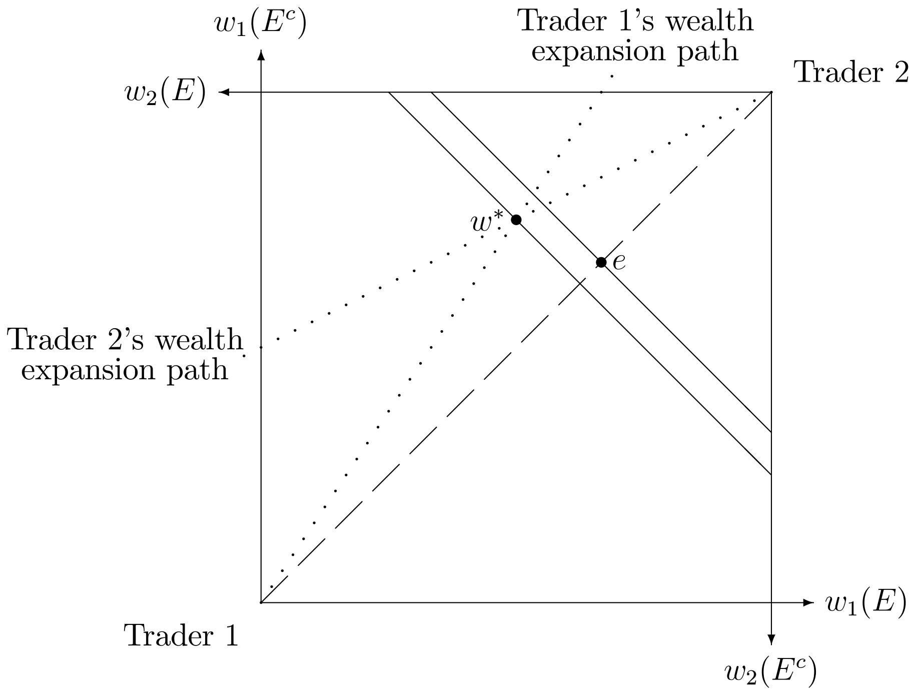

Figure 2: Edgeworth box representation of the underreaction result. Linear wealth expansion paths obtain with logarithmic preferences.

effect which is positive for the pessimistic trader 1. Now, as it has been well known since Arrow (1965), DARA implies that the wealth expansion paths diverge from the diagonal. The richer trader 1 thus demands a riskier bundle further away from the diagonal than at w^*, whereas the poorer trader 2 demands a safer bundle closer to the diagonal. To reach an equilibrium, the price must adjust so as to eliminate the excess demand for asset E^c. This is achieved by a relative reduction in the relative price for asset E, so that $p(L') < p'$. Thus prices must underreact to information.

3.4 Heterogeneous Priors versus Heterogeneous Endowments

In Proposition 9, we have obtained underreaction when motives for trade are generated by the heterogeneity of prior beliefs. It is natural to wonder whether the heterogeneity of priors is essential ingredient of the result. In particular, would underreaction also when traders are motivated by traditional liquidity motives? In this section, we suppose that traders share a common prior belief but have state-dependent endowments. We find that the price reacts one-for-one to information for a broad class of preferences which includes the logarithmic example from Section 3.

We modify the model from that section by assuming that all traders share the common prior q, but allowing trader's i initial asset endowment to vary across states, $w_{i0}(E) \neq w_{i0}(E^c)$. We allow also the aggregate endowment, $w_0(E) = \int_{i \in I} w_{i0}(E)\, di$ and $w_0(E^c) = \int_{i \in I} w_{i0}(E^c)\, di$, to vary across states. For simplicity, we do not constrain the positions traders can take. Suppose that there exists constants α_i and β such that trader i has Hyperbolic Absolute Risk Aversion (HARA), $-u_i''(w)/u_i'(w) = 1/(\alpha_i + \beta w)$. The fact that β is constant across traders means that traders are equally cautious.[26]

Proposition 10 *If all traders have HARA preferences with common cautiousness parameter, then the market price reacts as a Bayesian posterior belief to information.*

The result follows from Rubinstein's (1974) observation that in this case there exists a representative trader. In equilibrium, this trader must demand the aggregate endowment for *any* posterior belief, so that the price reacts one-for-one to information. In the special case without aggregate uncertainty ($w_0(E) = w_0(E^c)$), the market price is equal to the common posterior.

We can briefly revisit the example with logarithmic preferences which belong to the HARA class with $\alpha_i = 0$ and $\beta = 1$. Denoting by π the common posterior belief, the demand of trader i satisfies $pw_i(E) = \pi(pw_{i0}(E) + (1-p)w_{i0}(E^c))$. Market clearing implies $pw_0(E) = \pi(pw_0(E) + (1-p)w_0(E^c))$ so that the equilibrium price is $p = \pi w_0(E^c)/(\pi w_0(E^c) + (1-\pi)w_0(E)) = qw_0(E^c)L/(qw_0(E^c) + (1-q)w_0(E))$. This is in accordance with Bayes' rule, except that the market belief p is derived from a risk-adjusted prior belief. Trader i's net asset trade can be measured by

$$w_i(E) - w_{i0}(E) = (1-\pi)\left(\frac{w_0(E)}{w_0(E^c)} w_{i0}(E^c) - w_{i0}(E)\right).$$

In this risk-sharing model, trade moves individual asset positions closer to the average. The size of the net trade, $|w_i(E) - w_{i0}(E)|$, is monotone in π. In this setting, when the price responds to information as a posterior belief, buyers buy more aggressively when sellers sell more aggressively. Thus there is no reweighting across traders and no countervailing

[26] In the special case with $\beta \geq 0$, the absolute risk aversion $1/T$ is decreasing in w. In this case, these preferences are a special case of DARA preferences. CARA results when $\beta = 0$.

adjustment in the price depending on the realized information. With heterogeneous beliefs, instead, we found that optimists (who buy) buy less, while pessimists (who sell) sell more when information favors the assets' underlying event—so the price had to equilibrate against the direction of the information.[27]

For more general preferences outside the HARA class the price needs not react as a posterior belief, but it is not immediate whether under or overreaction results. To understand why the logic of Section 3 does not carry over, reconsider its Edgeworth box illustration. The box is no longer a square when $w_0(E) \neq w_0(E^c)$. Without loss of generality, suppose $w_0(E) > w_0(E^c)$. Unlike in Figure 2, the two risk-averse traders with common prior have an equilibrium bundle on the same side of their respective diagonal (i.e., $w_1(E) > w_1(E^c)$ and $w_2(E) > w_2(E^c)$). The DARA income expansion paths no longer contradict the possibility of staying in equilibrium when the price is Bayes updated to information. In conclusion, heterogeneity in priors cannot be replaced by heterogeneity in endowments to obtain underreaction to information.

4 Dynamic Model: Momentum

In this section we extend our model to a dynamic setting in which information arrives sequentially to the market after the initial round of trade. We verify that there exists an equilibrium where the initial round of trade is captured by our baseline model, and where there is no trade in subsequent periods, consistent with Milgrom and Stokey's (1982) no trade theorem. We then show that the initial under-reaction of the price to information implies momentum of the price process in subsequent periods—if the initial price movement is upwards, prices subsequently move up on average. Intuitively, first-round information is swamped by the information revealed in subsequent rounds, and hence over time the price comes to approximate the correctly updated prior belief.

In this model, the constant set of traders I is initially in the same situation as in our baseline model. Each trader is allowed to trade at every time date $t \in \{1, \ldots, T\}$ at the competitive price p_t. The joint information received (either publicly or privately) by traders up until period t has likelihood ratio L_t. The asset position of trader i after trade at period t is summarized by Δx_{it}. At time $T+1$ the true event is revealed, and the asset

[27] On the optimal allocation of risk with heterogeneous prior beliefs and risk preferences but without private information, see also the recent developments by Gollier (2007).

pays out. Each trader aims to maximize the expected utility of period $T+1$ wealth, which consists of other wealth W_i plus prediction market wealth w_{iT}.

A dynamic competitive equilibrium is defined as follows. First, for every $t = 1, \ldots, T$ there is a price function $p_t(L_t)$. By convention, $p_{T+1} = 1$ when E is true, and $p_{T+1} = 0$ when E^c is true. Second, given these price functions, every trader i chooses a contingent strategy of asset trades in order to maximize expected utility of final wealth. If constrained, the trader's prediction market wealth must always stay non-negative. Finally, in every period t at any information L_t realization, the market clears.[28]

Proposition 11 *There exists a dynamic competitive equilibrium with the following properties. In the first round of trade, the price $p_1(L_1)$ is equal to the static equilibrium price $p(L_1)$ characterized before. In all subsequent periods there is no trade, and the price satisfies Bayes' updating rule,*

$$\frac{p_t(L_t)}{1 - p_t(L_t)} = \frac{L_t}{L_1} \frac{p_1(L_1)}{1 - p_1(L_1)}. \tag{9}$$

The marginal trader, who holds belief $\pi_i(L_1) = p(L_1)$ after the first round of trading, remains the marginal trader in future rounds. The market price in future periods is the Bayesian posterior of this belief updated with the newly arriving information. From this trader's point of view prices follow a martingale, i.e., $E[p_t(L_t)|L_{t'}] = p_{t'}(L_{t'})$ for all $t' < t$.

Every trader who is initially more optimistic than this marginal trader, and hence has first-round posterior $\pi_i(L_1) > p(L_1)$ and has chosen $\Delta x_i > 0$, believes that the price is a sub-martingale (trending upwards). Despite this belief, the no-trade theorem establishes that such a trader does not wish to alter the position away from the initial Δx_i. The position already reflects a wealth- or risk-constrained position on the asset eventually rising in price, and there is no desire to further speculate on the upward trend in future asset prices.

The underreaction to the first-period information implies momentum in returns, consistent with the findings of Jagadeesh and Titman (1993). If the information revealed in the first period is favorable, i.e. if the realized likelihood ratio satisfies $L_1 > 1$, then the trader with neutral prior belief $q = p(1)$ joins the group of optimists taking long positions on E.

[28] The equilibrium of this Proposition is also a fully revealing REE when information is spread across traders as discussed before. Namely, we know from before that $p_1(L_1)$ is injective, and it follows from (9) that also $p_t(L_t)$ is injective.

From the natural point of view of an observer with a prior belief equal to this neutral prior, how are prices expected to behave? We show that prices trend upwards (downwards) in expectation following the initial realization of favorable (unfavorable) information.

Proposition 12 *Suppose that beliefs are truly heterogeneous and that all individuals are constrained or have strictly decreasing absolute risk aversion (DARA). Fix the market prior at the natural level* $q = p(1)$. *When non-degenerate new information* L_t *arrives to the market in later trading periods, then prices exhibit momentum, in the sense that changes in prices in later periods are positively correlated with early changes in prices. For any* $t_1 > t_2 > 0$,

$$E\left[\left(p_{t_1}\left(L_{t_1}\right) - p_{t_2}\left(L_{t_2}\right)\right)\left(p_{t_2}\left(L_{t_2}\right) - p\left(1\right)\right)\right] > 0. \tag{10}$$

Price changes are positively correlated with the opening price; in a regression of price changes on earlier price changes there should be a positive coefficient. Thus the initial underreaction must be followed by a correcting price momentum.[29]

Proposition 12 is consistent with the seemingly conflicting findings on price drift recently documented by Levitt and Gil (2007) and Croxson and Reade (2008) in the context of sport betting markets. On the one hand, Levitt and Gil (2007) find that the immediate price reaction to goals scored in the 2002 World Cup games is sizeable but incomplete and that price changes tend to be positively correlated, as predicted by our model. On the other hand, Croxson and Reade (2008) find no drift during the half-time break, thus challenging the view that the positive correlation of price changes during play time indicates slow incorporation of information. Consistent with this second bit of evidence, our model predicts the absence of drift when no new information arrives to the market, as it is realistic to assume during the break when the game is not played. This results follows immediately from Proposition 12 when $L_t = L_{t'} = L_{t''}$ for all periods t in the break $\{t', ..., t''\}$.

Allen, Morris, and Shin (2006) analyze a dynamic noisy REE model with overlapping generations of traders. In their model, traders with short horizons are subject to noise, which makes prices partially revealing. Because of short-term bias, traders must forecast

[29] Although the present analysis focuses on a period of trade opening, and subsequent periods, our results apply more broadly to trading environments in which the arrival of new information coincides with trade—either because of added liquidity reasons or differential interpretation of information, from which the present analysis abstracts.

the next period average forecasts and so end up overweighting the common public information. The mechanism behind our underreaction result is instead driven by wealth effects that are absent in Allen, Morris, and Shin's (2006) model with CARA preferences. Banerjee, Kaniel, and Kremer (2009) further investigate the conditions for momentum with CARA preferences—they obtain momentum by assuming that traders do not recognize the information of other traders and thus do not react to the information contained in the equilibrium price, deviating from the REE framework. Instead, we analyze the implications of belief heterogeneity within the REE setting with concordant beliefs when traders fully incorporate the information available to them and to other traders.[30]

5 Conclusion

When viewed as special asset markets, a defining feature of prediction markets is that traders have constant endowments across states of the world. Therefore, prediction markets offer an ideal setting to investigate how market prices react to information when traders have heterogeneous beliefs. We have shown that information results in a redistribution of wealth across traders with different beliefs. Thus prices tend to underreact to information when traders are subject to wealth effects—either because they are allowed to invest a limited amount or because their absolute risk aversion decreases with wealth. This result is driven by a wealth effect arising because traders with heterogeneous beliefs take speculative net positions. Even in the absence of any exogenous bound on positions or without borrowing constraints, underreaction holds under the realistic assumption that traders become less risk averse when their wealth increases.

To recap, our contribution builds on the literature on trade with heterogeneous priors and combines it with information updating that is typical of the REE literature. In our binary state model, we can allow for wealth effects in a tractable way without making parametric or distributional assumptions. Our model's three key ingredients (heterogeneous beliefs, information, and wealth effects) are characteristically present not only in prediction markets, but also in more general financial markets (see Hong and Stein, 2007). While previous literature has considered the effect of these ingredients either in isolation or

[30] We assume that traders extract information from the market price. To further appreciate the fundamental difference between the two settings, note that in Banerjee, Kaniel, and Kremer's (2009) static model with unconstrained traders, underreaction immediately results even with CARA preferences, in contrast to our Proposition 8.

in partial combination—our underreaction result crucially relies on the simultaneous presence of all three ingredients. As we have argued, this result can shed light on empirical findings from prediction and financial markets.

We see our analysis as a first step toward understanding price reaction to information in the presence of heterogeneous priors and wealth effects. Beyond the setting with two states on which we concentrate in this paper, the wealth effect underlying our results introduces an additional channel through which information affects prices: information about the likelihood of a state relative to a second state can impact the price of the asset for a third state. In turn, the adjustment related to this contagion effect impacts the prices of the assets for the first two states. A general analysis of how prices react to information in the presence of wealth effects and heterogeneous priors is a challenging but promising problem for future research.

Appendix

Proof of Proposition 1. For a given likelihood ratio L, the prior of an individual with posterior belief π_i is, using (1), $q_i = \pi_i / [(1-\pi_i) L + \pi_i]$. The E^c asset is demanded in amount $w_{i0}/(1-p)$ by every individual with $\pi_i < p$, or equivalently $q_i < p/[(1-p) L + p]$. The aggregate demand for this asset is then $G(p/[(1-p)L+p])/(1-p)$. In equilibrium, aggregate demand is equal to aggregate supply, equal to 1, resulting in equation (2).

Next, we establish that the price defined by (2) is a strictly increasing function of L. The left-hand side of (2) is a strictly increasing continuous function of p, which is 0 at $p = 0$ and 1 at $p = 1$. For any $L \in (0, \infty)$, the right-hand side is a weakly decreasing continuous function of p, for the cumulative distribution function G is non-decreasing. The right-hand side is equal to 1 at $p = 0$, while it is 0 at $p = 1$. Thus there exists a unique solution, such that $G \notin \{0, 1\}$. When L rises, the left-hand side is unaffected, while the right-hand side rises for any p, strictly so near the solution to (2) by the assumptions on G. Hence, the solution p must be increasing with L.

Proof of Proposition 2. The market price p is the posterior belief given information L and market prior belief $p/[(1-p)L+p]$. When L increases, so does p. By equation (2), when p increases, $p/[(1-p)L+p]$ must fall, because the cumulative distribution function G is non-decreasing.

Proof of Proposition 3. By Proposition 1, $p(L') > p(L)$. By (3), (4) is equivalent to

$$\log\left(\frac{p(L')}{1-p(L')}\right) - \log\left(\frac{p(L)}{1-p(L)}\right) < \log L' - \log L,$$

or

$$\frac{p(L')}{1-p(L')}\frac{1}{L'} < \frac{p(L)}{1-p(L)}\frac{1}{L}.$$

Using the strictly increasing transformation $z \to z/(1+z)$ on both sides of this inequality, it is equivalent to

$$\frac{p(L')}{[1-p(L')]L' + p(L')} < \frac{p(L)}{[1-p(L)]L + p(L)},$$

which is true by Proposition 2.

Proof of Proposition 4. By Proposition 3, the function

$$\Psi(L) = \log\left(\frac{\pi(L)}{1-\pi(L)}\right) - \log\left(\frac{p(L)}{1-p(L)}\right)$$

is strictly increasing in L. Hence, one of the following three cases will hold. In the first case, there exists an $L^* \in (0,\infty)$ such that $\Psi(L)$ is negative for $L < L^*$ and positive for $L > L^*$—in this case, the result follows with $p^* = p(L^*)$. In the second case, $\Psi(L)$ is negative for all L, and the result holds for $p^* = 1$. In the third case, $\Psi(L)$ is positive for all L, and the result is true with $p^* = 0$.

Proof of Proposition 5. Note first that $p((1-m)/m) = 1/2$ by (2). Consider now $L > (1-m)/m$ such that the equilibrium prices satisfy $\pi(L) > p(L), p'(L) > 1/2$. If, contrary to the claim, $p(L) < p'(L)$, then (2) implies that

$$G\left(\frac{p(L)}{(1-p(L))L + p(L)}\right) = 1 - p(L) > 1 - p'(L) = G'\left(\frac{p'(L)}{(1-p'(L))L + p'(L)}\right).$$

Further,

$$\frac{p'(L)}{(1-p'(L))L + p'(L)} > \frac{p(L)}{(1-p(L))L + p(L)},$$

while $p'(L) > 1/2$ in equilibrium implies

$$\frac{p'(L)}{(1-p'(L))L + p'(L)} < m.$$

Thus the median preserving spread property implies the contradiction,

$$G\left(\frac{p(L)}{(1-p(L))L + p(L)}\right) < G'\left(\frac{p'(L)}{(1-p'(L))L + p'(L)}\right).$$

A similar argument applies when $L < (1-m)/m$.

Proof of Proposition 6. Given that the equilibrium price is a strictly increasing function of L by Proposition 1, the realization of p fully reveals L. Thus (2) characterizes the unique fully revealing REE.

Proof of Proposition 7. Let Δx_i denote the choice variable of trader i, such that $p\Delta x_i$ units of the E^c asset are exchanged for $(1-p)\Delta x_i$ units of the E asset. Note that this is a zero net value trade, since the asset sale generates $(1-p)p|\Delta x_i|$ of cash that is spent on buying the other asset. The final wealth levels in the two states are:

$$w_i(E) = W_i + w_0 + (1-p)\Delta x_i, \tag{11}$$

$$w_i\left(E^c\right) = W_i + w_0 - p\Delta x_i. \tag{12}$$

The trading constraints are that $\Delta x_i \in \left[-w_0/\left(1-p\right), w_0/p\right]$.

The individual trader solves the problem

$$\max_{\Delta x_i \in [-w_0/(1-p), w_0/p]} \pi_i u_i\left(w_i\left(E\right)\right) + \left(1-\pi_i\right) u_i\left(w_i\left(E^c\right)\right).$$

Strict concavity of u_i ensures that the maximand Δx_i is unique. By the Theorem of the Maximum, Δx_i is a continuous function of π_i and p. We first show that the optimizer Δx_i is strictly decreasing in p and weakly increasing in π_i, strictly so when $\Delta x_i \in \left(-w_0/\left(1-p\right), w_0/p\right)$.

The constraint set $\left[-w_0/\left(1-p\right), w_0/p\right]$ does not depend on π_i and falls in Veinott's set order when p rises. The trader's objective function $\pi_i u_i\left(W_i + w_0 + \left(1-p\right)\Delta x_i\right) + \left(1-\pi_i\right) u_i\left(W_i + w_0 - p\Delta x_i\right)$ has first derivative

$$\pi_i\left(1-p\right) u_i'\left(w_i\left(E\right)\right) - \left(1-\pi_i\right) p u_i'\left(w_i\left(E^c\right)\right)$$

with respect to Δx_i. Since $u_i' > 0$, the cross-partial of the objective with respect to the choice variable Δx_i and the exogenous π_i is strictly positive, and hence Δx_i is weakly increasing in π_i, strictly so when the unique Δx_i optimizer satisfies the interior first-order condition

$$\frac{\pi_i}{1-\pi_i}\frac{u_i'\left(w_i\left(E\right)\right)}{u_i'\left(w_i\left(E^c\right)\right)} = \frac{p}{1-p}. \tag{13}$$

A sufficient condition for a strictly negative cross-partial with respect to Δx_i and p is

$$\Delta x_i\left[\pi_i\left(1-p\right) u_i''\left(w_i\left(E\right)\right) - \left(1-\pi_i\right) p u_i''\left(w_i\left(E^c\right)\right)\right] > 0. \tag{14}$$

Using the first-order condition for interior optimality, the second factor of (14) is positive if and only if

$$-\frac{u_i''\left(w_i\left(E^c\right)\right)}{u_i'\left(w_i\left(E^c\right)\right)} > -\frac{u_i''\left(w_i\left(E\right)\right)}{u_i'\left(w_i\left(E\right)\right)}.$$

By the DARA assumption, this inequality holds if and only if $w_i\left(E\right) > w_i\left(E^c\right)$, i.e., $\Delta x_i > 0$. Thus the cross-partial is strictly negative, so that Δx_i is strictly decreasing in p.

Equilibrium is characterized by the requirement that the aggregate purchase of asset E must be zero, i.e., $\int_0^1 \Delta x_i\left(p, q_i, L\right) dG\left(q_i\right) = 0$. When $p = 0$, every trader has $\pi_i > p$ and hence $\Delta x_i > 0$, while the opposite relation holds when $p = 1$. Individual demands are continuous and strictly decreasing in p, so there exists a unique equilibrium price in $(0, 1)$. When L is increased, $\pi_i\left(L\right)$ rises, and hence Δx_i rises for every trader. The price must then be strictly increased, in order to restore equilibrium.

Proof of Proposition 8. Suppose for a moment that no trader is constrained in equilibrium. The necessary and sufficient first-order condition (13) for the unconstrained optimum is solved by

$$\Delta x_i = t_i \log\left(\frac{1-p(L)}{p(L)}\frac{\pi_i(L)}{1-\pi_i(L)}\right). \tag{15}$$

Market clearing occurs when $\int_0^1 \Delta x_i dG(q_i) = 0$. By (15) and using $\pi_i(L)/(1-\pi_i(L)) = q_i L/(1-q_i)$ this is solved by $p(L) = qL/(qL+1-q)$. Inserting this market price in the individual demand (15), the resulting equilibrium demand is d_i^*, as given in (6). This analysis describes the equilibrium, provided no individual is constrained. The lower bound constrains no individual when $0 > \inf_i d_i^* \geq -w_0/(1-p(L))$, or equivalently $p(L) \geq 1 + w_0/\inf_i d_i^*$. Likewise, the upper bound is equivalent to $p(L) \leq w_0/\sup_i d_i^*$.

When a positive mass of traders are constrained, the bias follows from the argument of Proposition 9 reported below.

Proof of Proposition 9. The result follows as in the proof of Proposition 3, once we establish that $\log[p(L)/(1-p(L))] - \log(L)$ is strictly decreasing in L. Suppose, for a contradiction, that $\log[p(L)/(1-p(L))] - \log(L)$ is non-decreasing near some L. Traders at the boundary $\Delta x_i = -w_0/(1-p)$ have their demand decreasing in p, and hence $d\Delta x_i/dL < 0$. Likewise, $d\Delta x_i/dL < 0$ at the other boundary $\Delta x_i = w_0/p$. We will argue in the next paragraph that the same effect holds for traders satisfying (13). Since market clearing $\int_0^1 \Delta x_i(p(L), q_i, L)\, dG(q_i) = 0$ implies $\int_0^1 [d\Delta x_i(p, q_i, L)/dL]\, dG(q_i) = 0$, we will then obtain a contradiction establishing the claim.

Since $\log[\pi_i(L)/(1-\pi_i(L))] - \log(L)$ is constant, (13) implies that $u_i'(w_i(E))/u_i'(w_i(E^c))$ is non-decreasing in L. Using the expressions for the final wealth levels (11) and (12), non-negativity of the derivative of $u_i'(w_i(E))/u_i'(w_i(E^c))$ implies that

$$u_i''(w_i(E))\, u_i'(w_i(E^c)) \left[(1-p)\tfrac{d\Delta x_i}{dL} - \Delta x_i \tfrac{dp}{dL}\right] \geq -u_i''(w_i(E^c))\, u_i'(w_i(E)) \left[p\tfrac{d\Delta x_i}{dL} + \Delta x_i \tfrac{dp}{dL}\right].$$

The second derivative of the utility function is negative, so this implies

$$\frac{d\Delta x_i}{dL} \leq \Delta x_i \frac{dp}{dL} \frac{u_i''(w_i(E))\, u_i'(w_i(E^c)) - u_i''(w_i(E^c))\, u_i'(w_i(E))}{(1-p)\, u_i''(w_i(E))\, u_i'(w_i(E^c)) + p u_i''(w_i(E^c))\, u_i'(w_i(E))}. \tag{16}$$

On the right-hand side of (16), $dp/dL > 0$ by Proposition 7, and the denominator is negative. Recall that $\Delta x_i > 0$ if and only if $w_i(E) > w_i(E^c)$. By DARA, this implies that

$$-\frac{u_i''(w_i(E))}{u_i'(w_i(E))} < -\frac{u_i''(w_i(E^c))}{u_i'(w_i(E^c))}$$

or that the numerator is positive. Likewise, when $\Delta x_i < 0$, the numerator is negative. In either case, the right-hand side of (16) is strictly negative. Hence, $d\Delta x_i/dL < 0$ for every trader who satisfies the first-order condition (13).

Proof of Proposition 10. According to Rubinstein's (1974) Aggregation Theorem (ii), in this case there exists a representative trader. In equilibrium, for any posterior belief, this trader must demand the constant aggregate endowment $(w_{i0}(E), w_{i0}(E^c))$. Denoting the utility function of the representative trader by U, the equilibrium price $p(L)$ must satisfy the equivalent of (13),

$$\frac{qL}{1-q}\frac{U'(w_0(E))}{U'(w_0(E^c))} = \frac{p(L)}{1-p(L)}.$$

Hence, $\log[p(L)/(1-p(L))] - \log(L)$ is constant in L.

Proof of Proposition 11. We verify that the described outcome is an equilibrium. For the final equilibrium condition, note that the market will clear because trader positions are the same as in the static equilibrium. The remainder of the proof verifies that this constant position is indeed optimal in the individual dynamic optimization problem.

Let $\Delta x_{it}(L_t)$ denote the contingent net position of trader i in period t after information realization L_t. By convention, $\Delta x_{i0} = 0$. The trader's prediction market wealth evolves randomly over time as $w_{it}(L_t) = w_{it-1}(L_{t-1}) + (p_t(L_t) - p_{t-1}(L_{t-1}))\Delta x_{it-1}(L_{t-1})$ for $t = 1, \ldots, T+1$, with $w_{i0} > 0$ given as before. If constrained, the trader's net position choice at $t-1$ must satisfy $\Delta x_{it-1}(L_{t-1}) \in [-w_{it-1}(L_{t-1})/(1-p_{t-1}(L_{t-1})), w_{it-1}(L_{t-1})/p_{t-1}(L_{t-1})]$.

Suppose at period t, information L_t has been realized. To save notation, write p_t for the realization of $p_t(L_t)$ and w_{it} for the realization of $w_{it}(L_t)$. Two observations are essential. First, Δx_{it} is at the upper bound (interior, lower bound) of the constraint set $[-w_{it}/(1-p_t), w_{it}/p_t]$ if and only if, for all L_{+1}, Δx_{it} is on the upper bound (interior, lower bound) of the constraint set $[-w_{it+1}(L_{t+1})/(1-p_{it+1}(L_{t+1})), w_{it+1}(L_{t+1})/p_{it+1}(L_{t+1})]$. Second, for all realizations of the string $(L_{t+1,}\ldots, L_T)$, the feasible choice $\Delta x_{iT}(L_T) = \ldots = \Delta x_{it+1}(L_{t+1}) = \Delta x_{it}$ implies

$$\frac{u_i'(W_i + w_{iT}(E))}{u_i'(W_i + w_{iT}(E^c))} = \frac{u_i'(W_i + w_{it} + (1-p_t)\Delta x_{it})}{u_i'(W_i + w_{it} - p_t\Delta x_{it})}.$$

Both observations follow from the wealth evolution equation $w_{i\tau}(L_t) = w_{i\tau-1}(L_{\tau-1}) + (p_\tau(L_\tau) - p_{\tau-1}(L_{\tau-1}))\Delta x_{i\tau-1}(L_{\tau-1})$ for periods $\tau = t+1, \ldots, T$.

To prove our claim that the trader in every period selects the same position $\Delta x_{it} = \Delta x_{i1}(L_1)$ as in the static model given price $p_1(L_1)$, we proceed by backwards induction. The induction hypothesis t states that the agent in period t given price $p_t(L_t)$ (i) chooses Δx_{it} to satisfy the static first-order condition

$$\frac{p_t(L_t)}{1-p_t(L_t)} = \frac{\pi_i(L_t)}{1-\pi_i(L_t)} \frac{u_i'(W_i + w_{it}(L_t) + (1-p_t(L_t))\Delta x_{it})}{u_i'(W_i + w_{it}(L_t) - p_t(L_t)\Delta x_{it})}$$

if feasible, or (ii) chooses $\Delta x_{it} = w_{it}(L_t)/p_t(L_t)$ if the left-hand side of this static condition is below the right-hand side at this choice, and (iii) chooses $\Delta x_{it} = -w_{it}(L_t)/(1-p_t(L_t))$ if the left-hand side of this static condition exceeds the right-hand side at this choice. Note from the previous two essential observations, that once we have proved the induction hypothesis for all t, we have $\Delta x_{iT}(L_T) = \ldots = \Delta x_{i1}(L_1)$, and $\Delta x_{i1}(L_1)$ is the solution to the individual problem in Proposition 7.

The induction hypothesis T is satisfied because the static first-order condition characterizes the solution to the remaining one-period problem. We now assume that the induction hypothesis is true at $t+1, \ldots, T$, and will prove that induction hypothesis $t < T$ is true. Suppose at period t, information L_t is realized. Final wealth levels are then

$$w_{iT}(E) = W_i + w_{it} + (p_{t+1}(L_{t+1}) - p_t)\Delta x_{it} + (1 - p_{t+1}(L_{t+1}))\Delta x_{it+1}(L_{t+1})$$

and

$$w_{iT}(E^c) = W_i + w_{it} + (p_{t+1}(L_{t+1}) - p_t)\Delta x_{it} - p_{t+1}(L_{t+1})\Delta x_{it+1}(L_{t+1})$$

where $\Delta x_{it+1}(L_{t+1})$ is the reaction prescribed by induction hypothesis $t+1$. The time t problem is

$$\max_{\Delta x_{it} \in [-w_{it}/(1-p_t), w_{it}/p_t]} \pi_i(L_t) E[u_i(w_{iT}(E)) | E] + (1-\pi_i(L_t)) E[u_i(w_{iT}(E^c)) | E^c]$$

where the expectations are taken over the realization of L_{t+1}. In case (i), the static first-order condition can be satisfied with an interior choice of Δx_t. Evaluated at this choice, the derivative of the time t objective function is, by the envelope theorem,

$$\begin{aligned}
& \pi_i(L_t) E[(p_{t+1}(L_{t+1}) - p_t) u_i'(w_{iT}(E)) | E] \\
& + (1-\pi_i(L_t)) E[(p_{t+1}(L_{t+1}) - p_t) u_i'(w_{iT}(E^c)) | E^c] \\
= \; & p_t E\left[\tfrac{\pi_i(L_t) u_i'(w_{iT}(E))}{p_t} (p_{t+1}(L_{t+1}) - p_t) | E\right] \\
& + (1-p_t) E\left[\tfrac{(1-\pi_i(L_t)) u_i'(w_{iT}(E^c))}{1-p_t} (p_{t+1}(L_{t+1}) - p_t) | E^c\right].
\end{aligned}$$

Here $w_{iT}(E)$ and $w_{iT}(E^c)$ are constant across realizations of L_{t+1}. The static first-order condition then allows us to rewrite the derivative with respect to the control variable as

$$\frac{\pi_i(L_t)\,u_i'(w_{iT}(E))}{p_t}\left\{p_t E[p_{t+1}(L_{t+1})-p_t|E]+(1-p_t)\,E[(p_{t+1}(L_{t+1})-p_t)\,|E^c]\right\}.$$

By the martingale property of Bayes-updated prices at market belief p_t, we have

$$p_t E[p_{t+1}(L_{t+1})-p_t|E]+(1-p_t)\,E[(p_{t+1}(L_{t+1})-p_t)\,|E^c]=0.$$

Thus the first-order condition for optimality of Δx_{it} is satisfied at the choice resulting from the static model. The two other cases (with constrained choices) follow likewise.

Proof of Proposition 12. For a given realization of L_{t_2}, we denote for simplicity the resulting price by $p = p_{t_2}(L_{t_2})$ and the natural posterior by $\pi = qL_{t_2}/(qL_{t_2}+1-q)$. Under the natural posterior, we have

$$\begin{aligned} E[p_{t_1}(L_{t_1})-p|L_{t_2}] &= \pi E[p_{t_1}(L_{t_1})-p|E]+(1-\pi)\,E[p_{t_1}(L_{t_1})-p|E^c] \\ &= (\pi-p)\left\{E[p_{t_1}(L_{t_1})-p|E]-E[p_{t_1}(L_{t_1})-p|E^c]\right\}, \end{aligned}$$

using the martingale property of prices at the market belief p. Next,

$$p_{t_1}(L_{t_1})=\frac{pL_{t_1}}{pL_{t_1}+(1-p)\,L_{t_2}}=p+\frac{(1-p)\,p\,(L_{t_1}-L_{t_2})}{pL_{t_1}+(1-p)\,L_{t_2}}$$

follows from equation (9). At t_2, there is uncertainty about the realization of the future L_{t_1}. Bayes' rule implies $L_{t_1}/L_{t_2}=f_{t_2}(L_{t_1}|E)/f_{t_2}(L_{t_1}|E^c)$ where f_{t_2} denotes the p.d.f. for L_{t_1}. Collecting these pieces, we obtain

$$E[p_{t_1}(L_{t_1})-p|L_{t_2}]=(\pi-p)\int_0^\infty \frac{(1-p)\,p\,(L_{t_1}-L_{t_2})}{pL_{t_1}+(1-p)\,L_{t_2}}\,(L_{t_1}-L_{t_2})\,f_{t_2}(L_{t_1}|E^c)\,dL_{t_1}.$$

Now, averaging over realizations of L_{t_2}, we find

$$\begin{aligned} & E\left[(p_{t_1}(L_{t_1})-p_{t_2}(L_{t_2}))\,(p_{t_2}(L_{t_2})-p(1))\right] \\ =\; & E\left[E\left[(p_{t_1}(L_{t_1})-p_{t_2}(L_{t_2}))\,(p_{t_2}(L_{t_2})-q)\,|L_{t_2}\right]\right] \\ =\; & E\left[(p_{t_2}(L_{t_2})-q)\,(\pi(L_{t_2})-p_{t_2}(L_{t_2}))\int_0^\infty \frac{(1-p_{t_2}(L_{t_2}))p_{t_2}(L_{t_2})(L_{t_1}-L_{t_2})^2}{p_{t_2}(L_{t_2})L_{t_1}+(1-p_{t_2}(L_{t_2}))L_{t_2}}\,f_{t_2}(L_{t_1}|E^c)\,dL_{t_1}\right]. \end{aligned}$$

Underreaction states that $(p_{t_2}(L_{t_2})-q)\,(\pi(L_{t_2})-p_{t_2}(L_{t_2}))>0$ for all $L_{t_2}\neq 1$. Since all terms in the expectation are positive, we have proved (10).

References

Ali, Mukhtar M., "Probability and Utility Estimates for Racetrack Bettors," *Journal of Political Economy*, 1977, 85(4), 803–815.

Allen, Franklin, Stephen Morris, and Hyun Shin, "Beauty Contests and Iterated Expectations in Asset Markets," *Review of Financial Studies*, 2006, 19(3), 719–752.

Arrow, Kenneth J., *Aspects of the Theory of Risk-Bearing*, Yrjö Jahnsson Lectures, Academic Bookstore, Helsinki, 1965.

Aumann, Robert J., "Agreeing to Disagree," *Annals of Statistics*, 1976, 4(6), 1236–1239.

Banerjee, Snehal, Ron Kaniel, and Ilan Kremer, "Price Drift as an Outcome of Differences in Higher Order Beliefs," *Review of Financial Studies*, 2009, 22(9), 3707–3734.

Barberis, Nicholas, Andrei Shleifer, and Robert Vishny, "A Model of Investor Sentiment," *Journal of Financial Economics,* 1998, 49(3), 307–343.

Berg, Joyce, Robert Forsythe, Forrest Nelson, and Thomas Rietz, "Results from a Dozen Years of Election Futures Markets Research," in: *Handbook of Experimental Economics Results*, edited by Charles Plott and Vernon Smith, Elsevier, Amsterdam, 2008, 742–751.

Brunnermeier, Markus K., and Jonathan A. Parker, "Optimal Expectations," *American Economics Review*, 2005, 95(4), 1092–1118.

Cespa, Giovanni, and Xavier Vives, "Dynamic Trading and Asset Prices: Keynes vs. Hayek," 2009, mimeo, Queen Mary University of London and IESE Business School.

Croxson, Karen, and J. James Reade, "Information and Efficiency: Goal Arrival in Soccer Betting," 2008, mimeo, Oxford University.

Dekel, Eddie, Drew Fudenberg, and David K. Levine, "Learning to Play Bayesian Games," *Games and Economic Behavior*, 2004, 46(2), 282–303.

Diamond, Douglas W., and Robert E. Verrecchia, "Constraints on Short-Selling and Asset Price Adjustment to Private Information," *Journal of Financial Economics,* 1987, 18(2), 277–311.

Forsythe, Robert, Forrest Nelson, George R. Neumann, and Jack Wright, "Anatomy of an Experimental Political Stock Market," *American Economic Review*, 1992, 82(5), 1142–1161.

Forsythe, Robert, and Russell Lundholm, "Information Aggregation in an Experimental Market," *Econometrica*, 1990, 58(2), 309–347.

Fostel, Ana, and John Geanakoplos, "Leverage Cycles and the Anxious Economy," *American Economic Review*, 2008, 98(4), 1211–1244.

Gil, Ricard, and Steven D. Levitt, "Testing the Efficiency of Markets in 2002 World Cup," *Journal of Prediction Markets*, 2007, 3(1), 255–270.

Gjerstad, Steven, "Risk Aversion, Beliefs, and Prediction Market Equilibrium," 2005, mimeo, University of Arizona.

Gollier, Christian, "Whom Should We Believe? Aggregation of Heterogeneous Beliefs," *Journal of Risk and Uncertainty*, 2007, 35(2), 107–127.

Grossman, Sanford J., "On the Efficiency of Competitive Stock Markets Where Traders Have Diverse Information," *Journal of Finance*, 1976, 31(2), 573–585.

Grossman, Sanford J., *The Informational Role of Prices*, MIT Press, 1989.

Hahn, Robert W., and Paul C. Tetlock, "Using Information Markets to Improve Public Decision Making," *Harvard Journal of Law and Public Policy*, 2005, 29(1), 213–289.

Hanson, Robin D., "Decision Markets," *IEEE Intelligent Systems*, 1999, 14(3), 16–19.

Harrison, J. Michael, and David M. Kreps, "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations," *Quarterly Journal of Economics*, 1978, 92(2), 323–336.

Hong, Harrison, and Jeremy C. Stein, "A Unified Theory of Underreaction, Momentum Trading, and Overreaction in Asset Markets," *Journal of Finance*, 1999, 54(6), 2143–2184.

Hong, Harrison, and Jeremy C. Stein, "Disagreement and the Stock Market," *Journal of Economic Perspectives*, 2007, 21(2), 109–128.

Jagadeesh, Narasimhan, and Sheridan Titman, "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency," *Journal of Finance*, 1993, 48(1), 65–91.

Lieli, Robert P., and Augusto Nieto-Barthaburu, "Feedback in Prediction Markets," 2009, University of Texas, Austin and East Carolina University mimeo.

Lintner, John, "The Aggregation of Investor's Diverse Judgments and Preferences in Purely Competitive Markets," *Journal of Financial and Quantitative Analysis*, 1969, 4(4), 347–400.

Manski, Charles F., "Interpreting the Predictions of Prediction Markets," *Economics Letters*, 2006, 91(3), 425–429.

Milgrom, Paul, and Nancy Stokey, "Information, Trade and Common Knowledge," *Journal of Economic Theory*, 1982, 26(1), 17–27.

Miller, Edward M., "Risk, Uncertainty, and Divergence of Opinion," *Journal of Finance,* 1977, 32(4), 1151–1168.

Morris, Stephen, "Justifying Rational Expectations," 1995, mimeo, University of Pennsylvania.

Morris, Stephen, "Speculative Investor Behavior and Learning," *Quarterly Journal of Economics*, 1996, 111(4), 1111–1133.

Musto, David K., and Bilge Yilmaz, "Trading and Voting," *Journal of Political Economy*, 2003, 111(5), 990–1003.

Ottaviani, Marco, and Peter N. Sørensen, "Outcome Manipulation in Corporate Prediction Markets," *Journal of the European Economic Association, Papers and Proceedings*, 2007, 5(2–3), 554–563.

Ottaviani, Marco, and Peter N. Sørensen, "Surprised by the Parimutuel Odds?" *American Economic Review*, 2009, 99(5), 2129–2134.

Ottaviani, Marco, and Peter N. Sørensen, "Noise, Information, and the Favorite-Longshot Bias in Parimutuel Predictions," *American Economic Journal: Microeconomics*, 2010, 2(1), 58–85.

Palfrey, Thomas R., and Stephanie Wang, "Speculative Overpricing in Asset Markets with Information Flows," 2009, Social Science Working Paper 1309, California Institute of Technology.

Plott, Charles R., and Shyam Sunder, "Efficiency of Experimental Security Markets with Insider Information: An Application of Rational-Expectations Models," *Journal of Political Economy*, 1982, 90(4), 663–698.

Radner, Roy, "Rational Expectations Equilibrium: Generic Existence and the Information Revealed by Prices," *Econometrica*, 1979, 46(3), 655–678.

Rothschild, Michael, and Joseph E. Stiglitz, "Increasing Risk: I. A Definition," *Journal of Economic Theory*, 1970, 2(3), 225–243.

Rubinstein, Mark, "An Aggregation Theorem for Securities Markets," *Journal of Financial Economics*, 1974, 1, 225–244.

Scheinkman, Jose A., and Wei Xiong, "Heterogeneous Beliefs, Speculation and Trading in Financial Markets," in: *Paris-Princeton Lectures on Mathematical Finance 2003*, Lecture Notes in Mathematics 1847, Springer-Verlag, Berlin, 2004.

Serrano-Padial, Ricardo, "Information Aggregation in Common Value Asset Markets and the Efficient Markets Hypothesis," 2010, mimeo, University of Wisconsin-Madison.

Shin, Hyun Song, "Optimal Betting Odds Against Insider Traders," *Economic Journal*, 1991, 101(408), 1179–1185.

Shin, Hyun Song, "Prices of State Contingent Claims with Insider Traders, and the Favourite-Longshot Bias," *Economic Journal*, 1992, 102(411), 426–435.

Snowberg, Erik, and Justin Wolfers, "Explaining the Favorite-Longshot Bias: Is It Risk-Love or Misperceptions?" 2005, mimeo, Stanford GSB and Wharton School.

Thaler, Richard H., and William T. Ziemba, "Anomalies: Parimutuel Betting Markets: Racetracks and Lotteries," *Journal of Economic Perspectives*, 1988, 2(2), 161–174.

Varian, Hal R., "Differences of Opinion in Financial Markets," in: *Financial Risk: Theory, Evidence and Implications,* edited by Courtenay C. Stone, Kluwer, 1989.

Verardo, Michela, "Heterogeneous Beliefs and Momentum Profits," *Journal of Financial and Quantitative Analysis*, 2009, 44(4), 795–822.

Wilson, Robert, "The Theory of Syndicates," *Econometrica*, 1968, 36(1), 119–132.

Wolfers, Justin, and Eric Zitzewitz, "Prediction Markets," *Journal of Economic Perspectives*, 2004, 18(2), 107–126.

Wolfers, Justin, and Eric Zitzewitz, "Interpreting Prediction Market Prices as Probabilities," 2005, mimeo, Wharton, University of Pennsylvania.