Parimutuel versus Fixed-Odds Markets

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Abstract

This paper compares the outcomes of parimutuel and competitive fixed-odds betting markets. In the model, there is a fraction of privately informed bettors that maximize expected monetary payoffs. For each market structure, the symmetric equilibria are characterized. In parimutuel betting, the return on longshots is driven to zero as the number of insiders grows large. In fixed odds betting instead, this return is bounded below. Conversely, the expected return on longshots is increasing in the amount of insider information in a parimutuel market, but decreasing in a fixed-odds market. The market structure also affects the sign of the comparative statics predictions on the favorite-longshot bias.

Keywords: Parimutuel betting, fixed-odds betting, favorite-longshot bias, private information.

JEL Classification: D82 (Asymmetric and Private Information), D84 (Expectations; Speculations), G13 (Contingent Pricing; Futures Pricing).

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1 Introduction

This paper compares the performance of parimutuel with fixed-odds markets, the two most common market structures used for betting. In *parimutuel* markets, a winning bet pays off a proportional share of the total money bet on all outcomes, so that the odds offered to a bet are determined only after all the bets are placed. In *fixed-odds* markets instead, bookmakers compete to set odds at which they accept bets from the public.

Differently from regular financial markets, in betting markets the uncertainty is resolved unambiguously and the fundamental values are publicly observed. In addition, these values are exogenous with respect to the market prices. Because of these features, betting markets provide an ideal testbed for evaluating theories of price formation (see Thaler and Ziemba 1988, Hausch and Ziemba 1995, Sauer 1998, and Jullien and Salanié 2002 for surveys).

A commonly observed empirical pattern is the *favorite-longshot bias*, according to which horses with short market odds (favorites) offer higher average payoff than horses with long market odds (longshots). In this paper, we show that this bias can result from the presence of privately informed bettors in both the parimutuel and fixed-odds market structures. We further argue that these two structures induce different systematic relations between empirical and market odds, depending on the amount of information present in the market.

In the model, there are two classes of bettors, outsiders and insiders. While outsider are uniformed and place an exogenous amount of bets, insiders are privately informed and maximize their expected monetary payoff. We characterize how the symmetric equilibrium depends on the market structure.

First, consider the outcome of a competitive *fixed-odds* market. Relative to favorites, longshots attract a relatively higher proportion of insiders and pay out more conditional on winning. To counteract this more severe adverse selection problem, competitive bookmarkers quote relatively shorter odds on longshots. This is the equivalent of a larger bid-ask spread in the presence of more insider trading in a standard financial market making mechanism (Glosten and Milgrom 1985). The favorite-longshot bias arises because the adverse selection problem is greater on the longshot than on the favorite.

Consider next the case of *parimutuel markets*. The insiders simultaneously use their private information to decide where to place their bets. As a result of insider cash constraints, the market odds do not move sufficiently far to exhaust the gains revealed by the informed bettors, and the favorite-longshot bias arises. Many (few) informed bets on a horse indicates that this favorite (longshot) is less (more) likely to win than indicated by the realized bet distribution.

Our analysis of fixed-odds markets is closely related to Shin (1991 and 1992). Shin argues that a monopolistic bookmaker sets odds with a favorite-longshot bias in order to limit the subsequent losses to the better informed insiders. We derive a similar bias in a competitive bookmaking market. While our informational assumptions are similar to those made by Shin, we depart from him by considering the case of *ex-post* rather than *ex-ante* competition among bookmakers. Our results on the favorite-longshot bias in competitive fixed-odds markets are therefore new. Our analysis of parimutuel markets builds extensively on the results obtained by Ottaviani and Sørensen (2004b).

To the best of our knowledge, this is the first paper that compares the performance of trading structures used in betting markets. We study these two structures in isolation. When both markets coexist, bettors have the additional choice of participating in either market, possibly depending on their information. The investigation of this selection issue is left to future research.[1]

A number of alternative theories have been formulated to explain the favorite-longshot bias. First, Griffith (1949) suggested that the bias might be due to the tendency of individual decision makers to overestimate small probabilities events. Second, Weitzman (1965) and Ali (1977) hypothesized that individual bettors are risk loving, and so are willing to accept a lower expected payoff when betting on the riskier longshots. Third, Isaacs (1953) noted that an informed monopolist bettor would not bet until the marginal bet has zero value, if the marginal bet reduces the return to the inframarginal bets. Fourth, Hurley and McDonough (1995) noted that the presence of the track take limits the amount

[1] For example, parimutuel and fixed odds markets coexist in the UK. See Gabriel and Marsden (1990) for an empirical investigation of the interaction effects resulting from the bettors' option to select in which system to participate.

of arbitrage by the informed bettors, who are prevented from placing negative bets and so cannot take advantage of negative returns on longshots.

While Isaacs' (1953) market power explanation and Hurley and McDonough's (1995) limited arbitrage explanation are specific to parimutuel markets, the informational explanation of the favorite-longshot bias proposed here applies both to parimutuel and fixed-odds markets. The behavioral and risk loving explanations instead predict the presence of the bias regardless of the market structure, but do not account for the varying extent of the bias under different market institutions.

The paper proceeds as follows. Section 2 formulates the common assumptions on information that are maintained in two models of market structure. Section 3 formulates the competitive fixed-odds model and analyzes its equilibrium. Section 4 turns to parimutuel betting. In Section 5 compares the two outcomes.

2 Setup

We consider a race between two horses. The outcome that horse x wins the race is identified with the *state*, $x \in \{-1, 1\}$.

Unmodeled *outsiders* place bets on the two horses without responding to the market conditions. For simplicity, we assume that the same amount $a \geq 0$ is bet on either horse.

There is a continuum $[0, N]$ of privately informed bettors (or *insiders*). Insiders (as well as the bookmakers in the specification of Section 3) have a common *prior belief* $q = \Pr(x = 1)$, possibly formed after the observation of a common signal. In addition, each insider i privately observed *signal* s_i.[2] The signals are assumed to be identically and independently distributed across insiders, conditional on state x. Since there are only two states, without further loss of generality the likelihood ratio $f(s|x = 1)/f(s|x = -1)$ is monotone. For simplicity, we further assume that the likelihood ratio is strictly increasing in s.

Upon observation of signal s, the prior belief q is updated according to Bayes' rule into the *posterior belief*, $p = \Pr(x = 1|s)$. The posterior belief p is distributed according

[2] Private (or inside) information is believed to be pervasive in horse betting. See e.g., Crafts (1985).

to the continuous distribution function G with density g on $[0,1]$. By the law of iterated expectations, the prior must satisfy $q = E[p] = \int_0^1 pg(p)\,dp$. Bayes' rule yields $p = qg(p|x=1)/g(p)$ and $1-p = (1-q)\,g(p|x=-1)/g(p)$, so that the conditional densities of the posterior are $g(p|x=1) = pg(p)/q$ and $g(p|x=-1) = (1-p)\,g(p)/(1-q)$. Note that $g(p|x=1)/g(p|x=-1) = (p/(1-p))((1-q)/q)$, reflecting the property that high beliefs in outcome 1 are more frequent when outcome 1 is true. Strict monotonicity of the likelihood ratio in p implies that $G(p|x=1)$ first-order stochastically dominates $G(p|x=-1)$ on the support, i.e., $G(p|x=1) - G(p|x=-1) < 0$ for all p such that $0 < G(p) < 1$.

It is convenient to state assumptions on the signal structure in terms of their implications for the conditional distributions of the posterior belief. The signal distribution is said to be *symmetric* if the chance of posterior p conditional on state $x=1$ is equal to the chance of posterior $1-p$ conditional on state $x=-1$, i.e., $G(p|x=1) = 1 - G(1-p|x=-1)$ for all $p \in [0,1]$. The signal distribution is said to be *unbounded* if $0 < G(p) < 1$ for all $p \in (0,1)$.

Example. To illustrate our results we use the *linear signal* example, which can be derived from a binary signal with precision distributed uniformly. In this example, the conditional densities are $f(s|x=1) = 2s$ and $f(s|x=-1) = 2(1-s)$ for $s \in [0,1]$, with corresponding distribution functions $F(s|x=1) = s^2$ and $F(s|x=-1) = 1-(1-s)^2$. The posterior odds ratio is

$$\frac{p}{1-p} = \frac{q}{1-q}\frac{f(s|x=1)}{f(s|x=-1)} = \frac{q}{1-q}\frac{s}{1-s}. \tag{1}$$

Inverting $p/(1-p) = qs/[(1-q)(1-s)]$, we obtain $s = p(1-q)/[p(1-q)+(1-p)q]$. The conditional distribution functions for p are

$$G(p|1) = \left[\frac{p(1-q)}{p(1-q)+(1-p)q}\right]^2 \tag{2}$$

and

$$G(p|-1) = 1 - \left[\frac{(1-p)q}{p(1-q)+(1-p)q}\right]^2. \tag{3}$$

3 Fixed-Odds Betting

We begin by considering fixed-odds betting. After introducing the rules of the market (Section 4.1), we characterize the equilibrium (Section 4.2) and the resulting favorite-longshot bias (Section 4.3).

3.1 Market Rules

We consider a market with competitive bookmakers operating as follows. First, $M \geq 2$ bookmakers simultaneously quote odds. We denote by $1 + \rho_x = 1/\pi_x$ the return to every dollar bet on horse x offered by the bookmaker(s) making the most advantageous offer. Second, bettors simultaneously place bets.[3] The insiders are allowed to bet one dollar on either horse, or abstain from betting. In addition, there is a given amount of bets placed by unmodeled outsiders, and these outside bets are equal to $a > 0$ on each of the two horses regardless of the state. Third, the state x is realized, and bookmakers pay out the promised returns on the winning tickets.

Note the difference of our model with Shin's (1991 and 1992) model. In his setting, bookmarkers compete ex-ante for the position to be a monopolist. The monopolist bookmaker then set odds on the different horses at the same time and so cross-subsidizes across the corresponding markets. In Shin's model, bookmakers make zero expected profits in equilibrium of the full game, even if they do make non-zero profits on the markets for bets on individual horses. We instead allow for competition for bets on each separate horse, as in Glosten and Milgrom's (1985) model of competitive market making.

3.2 Equilibrium Characterization

When odds ρ_1 are offered on horse 1, every insider with beliefs above the cutoff belief p_1 prefers to bet on horse 1 rather than to abstain, where p_1 is defined by the indifference $p_1 \rho_1 = 1 - p_1$. Thus, the solution $p_1 = 1/(1 + \rho_1) = \pi_1$ is precisely the implied market probability.

[3] Ottaviani and Sørensen (2004b) allow insiders to decide when to bet. They show that in equilibrium insiders bet at the end of the betting period without access to the information of the others. In reality, a large amount of bets are places in the last few minutes, as in the static game analyzed here.

By Bertrand competition, in equilibrium each bookmaker make zero expected profits in the market corresponding to each horse. Taking into account the optimal response of the informed bettors, the bookmaker makes zero expected profits on horse 1 when

$$q\left(a+N\left[1-G\left(\frac{1}{1+\rho_1}|1\right)\right]\right)\rho_1=(1-q)\left(a+N\left[1-G\left(\frac{1}{1+\rho_1}|-1\right)\right]\right). \quad (4)$$

To understand this, note that the bookmaker believes that horse 1 wins with probability q, in which case the bookmaker makes a net payment equal to ρ_1 to a outsiders and to the insiders with a belief above π_1. If instead horse -1 wins, the informed place a lower amount of bets on the horse 1, since $1-G(\pi_1|-1)<1-G(\pi_1|1)$ by the stochastic dominance property of beliefs.[4]

An equilibrium on the market for horse 1 is defined by any $\rho_1>0$ that solves equation (4). Observe that for $\rho_1=0$, the left-hand side is strictly lower than the right hand side, since $0<(1-q)\,a$. As $\rho_1\to+\infty$, the left-hand side increases without bound and so exceeds the right-hand side, equal to the bounded $(1-q)(a+N)$. Thus there exists an equilibrium, that is shown below to be unique.

Equation (4) may be rewritten as

$$\rho_1=\frac{1-q}{q}\frac{a+N\left(1-G\left(\frac{1}{1+\rho_1}|-1\right)\right)}{a+N\left(1-G\left(\frac{1}{1+\rho_1}|1\right)\right)}<\frac{1-q}{q} \quad (5)$$

where the inequality is due to $G(p|1)<G(p|-1)$. The left-hand side is equal to the market odds, while the right-hand side gives the prior odds. Systematically, bookmakers quote market odds shorter than the prior odds in order to protect their profits against the informational advantage of the insiders. This difference may be loosely interpreted as a bid-ask spread. It implies that a bet based on the prior belief q results in a negative expected return.

3.3 Favorite-Longshot Bias

Note that the implied market probability $\pi_1=1/(1+\rho_1)$ that results in equilibrium departs systematically from the prior belief q that horse 1 wins. The corresponding empirical

[4] In Glosten and Milgrom's model, the bookmakers earn zero expected profits conditional on the news that the next bettor wants to bet on horse 1. Indeed, (4) can be rewritten to confirm that π_1 is the posterior probability of state 1 given a bet on horse 1.

average return to a bet on horse 1 is then $q/\pi_1 - 1 < 0$. In accordance with Shin's (1991 and 1992) definition, we say that the favorite-longshot bias arises if the ratio π_1/q is a decreasing function of q (or, equivalently, if the empirical average return to a bet on horse 1 is increasing in q).

We now establish that π_1 is an increasing function of q and π_1/q is a decreasing function of q:

Proposition 1 *There exists a unique constant factor $\mu > 1$, such that the market implied probability can be written as $\pi_1 = \mu q / [\mu q + (1-q)]$. There is a favorite-longshot bias.*

Proof. Defining $\alpha = a/N$, and using the variable $\pi_1 = 1/(1+\rho_1)$, the equilibrium condition (4) is

$$\frac{(1-q)\,\pi_1}{q\,(1-\pi_1)} = \frac{\alpha + 1 - G(\pi_1|1)}{\alpha + 1 - G(\pi_1|-1)}. \tag{6}$$

Here, the left hand side is continuous and strictly increasing in π_1, being 0 at $\pi_1 = 0$ and tending to infinity as $\pi_1 \to 1$. The right-hand side is continuous with value 1 at both ends $\pi_1 = 0, 1$. Thus, for every given $q \in (0,1)$ there exists a solution π_1 to (6).

This solution is unique, for whenever equation (6) holds, the right-hand side intersects the left hand side from above. This follows, since the slope of the right-hand side at an intersection is

$$\frac{g(\pi_1|-1)(\alpha+1-G(\pi_1|1)) - g(\pi_1|1)(\alpha+1-G(\pi_1|1))}{(\alpha+1-G(\pi_1|-1))^2} = \frac{g(\pi_1|-1) - g(\pi_1|1)\frac{(1-q)\pi_1}{q(1-\pi_1)}}{(\alpha+1-G(\pi_1|-1))}$$

where the equality followed from (6). By $g(\pi_1|1)/g(\pi_1|-1) = (\pi_1/(1-\pi_1))((1-q)/q)$, we see that this slope is in fact zero. Since the left hand side is increasing, the intersection is from above, as claimed.

So far we have established that for every q there exists a unique solution π_1 to (6). We now describe how π_1 depends on q. Recall that the distribution G of the posterior belief depends on q, so we aim first to rewrite (6) in terms of the fixed signal distribution F. Bayesian updating gives $\pi_1(1-q)/[(1-\pi_1)\,q] = f(s|1)/f(s|-1)$, and since the likelihood ratio $f(s|1)/f(s|-1)$ is strictly monotone, we can recover s as a function σ

of the variable $\mu = \pi_1 (1-q) / [(1-\pi_1) q]$. Using the relationship between s and π_1, for given q we defined $G(\pi_1|x) = F(s|x)$. Now (6) reduces to

$$\mu = \frac{\alpha + 1 - F(\sigma(\mu)|1)}{\alpha + 1 - F(\sigma(\mu)|-1)} \tag{7}$$

The existence of the unique solution for π_1 given q to equation (6) implies that there exists a unique solution in μ to equation (7). Namely, consider $q = 1/2$ and take the π_1 that solved (6), let $\mu = \pi_1 / (1-\pi_1)$ and note that this is a solution to (7). On the other hand, if $\mu' \neq \mu$ also solved (7), then $\pi_1' = \mu' / (1+\mu') \neq \pi_1$ would also solve (6) for $q = 1/2$, in contradiction to its uniqueness.

As noted in (5), the market odds are shorter than the prior odds, so $\mu > 1$. Rewriting, we see that $\pi_1 = \mu q / [\mu q + (1-q)]$. Thus π_1 is an increasing function of q, while $\pi_1/q = \mu / [\mu q + (1-q)]$ is a decreasing function of q since $\mu > 1$. Thus, there is a favorite-longshot bias. □

Note that the fact that π_1/q is a decreasing function of q combined with the fact that π_1 is an increasing function of q imply that the empirical average return to a bet on horse 1 is increasing in π_1, or equivalently decreasing in the market odds $\rho_1 = (1-\pi_1)/\pi_1$.

As the chance q that horse 1 wins is increased, the bookmaker naturally sets shorter odds for horse 1. This drives away some of the informed bettors, but we have assumed that the outsiders keep betting the same amount. Thus, the bookmaker's adverse selection problem is reduced, and in equilibrium the implied market probability π_1 is brought closer to the prior probability q. Yet, we showed that the ratio π_1/q falls. The subtlety of the result is underlined by the fact that the ratio of market odds to prior odds is constant with respect to q and equal to $1/\mu$.

The factor μ is a decreasing function of the ratio a/N of outsider to insider population sizes, since the bookmaker's adverse selection problem is smaller when the outsiders place more bets in comparison to the insiders. This implies that the favorite-longshot bias is more pronounced when there are more insiders.

Proposition 2 *The factor μ, and thus the extent of the favorite-longshot bias, is a decreasing function of a/N.*

Proof. An increase in $\alpha = a/N$ serves to decrease the right-hand side of (6) for every $\pi_1 \in (0,1)$. This follows from the first-order stochastic dominance property $1 - G(\pi_1|1) > 1 - G(\pi_1|-1)$. The strictly increasing left-hand side is not affected by the change in α, so the equilibrium value of π_1 must be smaller than before for every q. This implies that $\mu = \pi_1 (1-q) / [(1-\pi_1) q]$ is smaller, and closer to one, than before. □

Example. In the special symmetric signal distribution example, G is defined by (2) and (3). Defining $\alpha = a/N$, and using the variable $\pi_1 = 1/(1+\rho_1)$, the equilibrium condition (4) is then

$$\frac{(1-q)\pi_1}{q(1-\pi_1)} = \frac{\alpha + 1 - \left(\frac{\pi_1(1-q)}{\pi_1(1-q)+(1-\pi_1)q}\right)^2}{\alpha + \left(\frac{(1-\pi_1)q}{\pi_1(1-q)+(1-\pi_1)q}\right)^2}. \tag{8}$$

Using $\mu = (1-q)\pi_1/[q(1-\pi_1)]$, this equation further reduces to the cubic $\alpha\mu^3 + \alpha\mu^2 - (1+\alpha)\mu - (1+\alpha) = 0$. Eliminating the irrelevant root $\mu = -1$, this reduces to the quadratic equation $\alpha\mu^2 - (1+\alpha) = 0$ and is solved by $\mu = \sqrt{(1+\alpha)/\alpha} > 1$. Note that μ decreases in α, confirming Proposition 2. Figure 1 displays the expected return $q/\pi_1 - 1$ against market odds $(1-\pi_1)/\pi_1$, for five values of α. Notice the similarity of this figure with Jullien and Salanié's (2002) Figure 1, derived from bookmakers' odds data for horse races run in Britain between 1986 and 1995 (see also Jullien and Salanié 2000).

4 Parimutuel Betting

This section reviews how the favorite-longshot bias arises from simultaneous informed betting, as derived in Ottaviani and Sørensen (2004b). We first introduce the rules of the market (Section 4.1). We then characterize the equilibrium and derive the bias (Section 4.2). Finally, we illustrate how the bias varies with the number of insiders (Section 4.3).

4.1 Market Rules

We assume that each insider can bet a limited amount, set equal to 1, and bets in order to maximize the expected return. The total amount bet by these insiders on outcome y

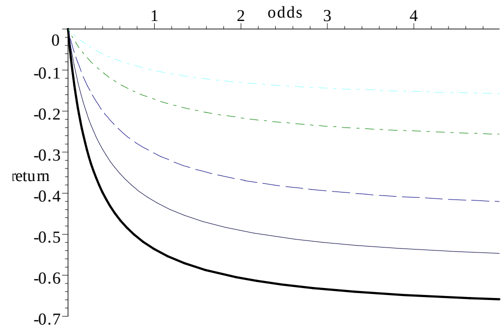

Figure 1: The expected return to a bet on outcome 1 in the fixed-odds market is plotted against the market odds ratio in the linear signal example. The five curves show the cases $a/N = 2, 1, .4, .2, .1$, in progressively thicker shade.

in state x is denoted by $b_{y|x}$. All bets on both outcomes are placed in a common pool, from which the *track take* $\tau \in [0, 1)$ is subtracted. The remaining amount in the pool is returned to the winning bets. Thus, if x is the winner, each unit bet on outcome x yields

$$W(x|x) = (1-\tau)\frac{2a + b_{x|x} + b_{-x|x}}{a + b_{x|x}}. \tag{9}$$

Notice the important difference to the case of fixed-odds betting, that each horse cannot be studied in isolation — the return on horse x is influenced by the amount bet also on horse $-x$. With a continuum of small informed bettors, we assume each of them takes the returns $W(x|x)$ as given.[5]

4.2 Equilibrium Characterization

With several bettors possessing private information, we solve the model for a Bayes-Nash equilibrium. Each bettor takes the correct equilibrium numbers $W(x|x)$ as given, and chooses a best reply. Naturally, the greater is the belief in an outcome, the more attractive it is to bet on that outcome. This leads to the following result.

Proposition 3 *Assume that the private beliefs distribution is unbounded, and that* $0 \leq \tau < 1/2$. *There exists a unique equilibrium. An insider bets on* -1 *when* $p < \hat{p}_{-1}$, *abstains*

[5] Ottaviani and Sørensen (2004a) consider finitely many insiders.

when $\hat{p}_{-1} < p < \hat{p}_1$, *and bets on* 1 *when* $p > \hat{p}_1$, *where the thresholds* $0 < \hat{p}_{-1} < \hat{p}_1 < 1$ *are the unique solutions to the two indifference conditions:*

$$\hat{p}_1 = \frac{1}{1-\tau}\frac{a + N\left(1 - G\left(\hat{p}_1|1\right)\right)}{2a + N\left(1 - G\left(\hat{p}_1|1\right)\right) + NG\left(\hat{p}_{-1}|1\right)} \tag{10}$$

and

$$1 - \hat{p}_{-1} = \frac{1}{1-\tau}\frac{a + NG\left(\hat{p}_{-1}|-1\right)}{2a + N\left(1 - G\left(\hat{p}_1|-1\right)\right) + NG\left(\hat{p}_{-1}|-1\right)}. \tag{11}$$

Proof. See Proposition 1 of Ottaviani and Sørensen (2004b). □

Once the bettors use threshold strategies, we obtain $b_{1|x} = 1 - G\left(\hat{p}_1|x\right)$ and $b_{-1|x} = G\left(\hat{p}_{-1}|x\right)$. Conditions (10) and (11) require that the bettors with threshold beliefs are indifferent between betting and abstaining, i.e., $\hat{p}_1 W\left(1|1\right) = \left(1 - \hat{p}_{-1}\right) W\left(-1|-1\right) = 1$.

4.3 Favorite-Longshot Bias

In the parimutuel market, by definition, the implied market probability for outcome x is $\left(1-\tau\right)/W\left(x|x\right)$, equal to the fraction of money placed on outcome x. The favorite-longshot bias claims that the greater is this implied market probability, the greater the expected return to a dollar bet on x. Conditioning on the realization of this implied market probability, the empirical researcher can estimate the expected return to a bet on x. In equilibrium of our model with the continuum of privately informed bettors, strictly more bets are placed on outcome x when it is true than when it is false. Thus, the realized bets fully reveal the true outcome. We conclude that the equilibrium outcome exhibits the favorite-longshot bias. The insiders' bets $\left(b_{1|x}, b_{-1|x}\right)$ reveal the true winner, and although horse x is more of a favorite $\left(b_{x|x} > b_{x|-x}\right)$ when it wins, the market implied probability for the winner is less than one.[6]

4.4 Comparative Statics

An increase in the track take reduces the profitability of betting and results in more extreme equilibrium thresholds.

[6] Ottaviani and Sørensen (2004a) investigate more generally the conditions for the occurrence of the favorite-longshot bias with a *finite* number of players. The sign and extent of the bias depends on the interaction of noise and information.

Proposition 4 *When $0 < \tau < 1/2$, a marginal increase τ implies that $\hat{p}_1$ strictly increases and $\hat{p}_{-1}$ strictly decreases, and that $W(1|1)$ and $W(-1|-1)$ both strictly decrease.*

Proof. The equilibrium is determined by the thresholds $(\hat{p}_{-1}, \hat{p}_1) \in (0,1)^2$ at which the two downward sloping curves defined by (10) and (11) intersect. The (11)-curve is steeper than the (10)-curve at the intersection. Rewrite (10) as

$$G(\hat{p}_{-1}|1) = \frac{a}{N(1-\tau)\hat{p}_1} - \frac{2a}{N} + \left[\frac{1}{(1-\tau)\hat{p}_1} - 1\right][1 - G(\hat{p}_1|1)]. \qquad (12)$$

Consider the effect of an increase in τ on the equilibrium. For any $\hat{p}_1 \in (0,1)$, the right-hand side of (12) is strictly increased. Thus, the left-hand side must strictly increase, in order to equilibrate. Thus, the (10)-curve shifts outwards: for any $\hat{p}_1$, the corresponding $\hat{p}_{-1}$ is strictly greater. Similarly, the (11)-curve shifts inwards: for any $\hat{p}_{-1}$, the corresponding $\hat{p}_1$ is strictly smaller. Since the steeper curve shifts inwards, and the flatter curve shifts outwards, we can conclude that the unique equilibrium must shift to the north-west in the $(\hat{p}_{-1}, \hat{p}_1)$-space. Thus, $\hat{p}_{-1}$ strictly decreases, and $\hat{p}_1$ strictly increases. Since the indifference conditions $\hat{p}_1 W(1|1) = 1 = (1-\hat{p}_{-1}) W(-1|-1)$ continue to hold after the change, we can also conclude that $W(1|1)$ and $W(-1|-1)$ both strictly decrease. □

If the distributions of the private beliefs and the initial bets are symmetric, we can derive additional comparative statics results. When the number of insiders increases (holding fixed the amount a bet by the outsiders), their bets have a greater impact on the market odds. This tends to make informed betting less attractive for individuals with a given signal. The equilibrium must have more extreme thresholds. More extreme indifference thresholds imply that the winner's odds ratio $W(x|x) - 1$ is lower, and thus the favorite-longshot bias is reduced. However, even as N grows arbitrarily large, the unique equilibrium remains interior, implying that $W(x|x) > 1$. The market implied probability $(1-\tau)/W(x|x)$ can thus never exceed $1-\tau$, since the track take prevents the informed population from fully correcting the odds resulting from the outsiders.

Proposition 5 *Assume that the distribution of private posterior beliefs is symmetric and unbounded, and that $0 < \tau < 1/2$. The unique equilibrium of Proposition 3 satisfies*

$\hat{p}_1 = 1 - \hat{p}_{-1} \in (1/2, 1)$. *The threshold* $\hat{p}_1$ *is increasing in* τ *and* N/a. *The favorite-longshot bias is reduced when either* a/N *or* τ *is decreased.*

Proof. Using the assumptions, it is easy to verify that if the pair $(\hat{p}_{-1}, \hat{p}_1)$ solves (10) and (11), then $(1 - \hat{p}_1, \hat{p}_{-1})$ also solves. Since the solution is unique by Proposition 3, the equilibrium must satisfy $\hat{p}_1 = 1 - \hat{p}_{-1}$. Given this, condition (11) reduces to condition (10), and either condition can be rewritten as

$$(1 - \tau)\,\hat{p}_1 = \frac{a/N + 1 - G(\hat{p}_1|1)}{2a/N + 1 - G(\hat{p}_1|1) + 1 - G(\hat{p}_1|-1)}. \tag{13}$$

The right-hand side of (13) is continuous in $\hat{p}_1$. At 1/2 it strictly exceeds the left-hand side, while the opposite is true at 1. Thus the unique solution belongs to $(1/2, 1)$.

The left-hand side of (13) is strictly increasing in $\hat{p}_1$, while, at any solution, the right-hand side is a weakly decreasing function of $\hat{p}_1$. To see the latter claim, take the logarithm of the right-hand side, differentiate and use symmetry of G to arrive at the desired inequality

$$\frac{a/N + 1 - G(\hat{p}_1|1)}{a/N + 1 - G(\hat{p}_1|-1)} \leq \frac{g(\hat{p}_1|1)}{g(\hat{p}_1|-1)} = \frac{\hat{p}_1}{1 - \hat{p}_1},$$

i.e.,

$$\hat{p}_1 \geq \frac{a/N + 1 - G(\hat{p}_1|1)}{2a/N + 1 - G(\hat{p}_1|1) + 1 - G(\hat{p}_1|-1)},$$

which is implied by (13).

An increase in τ has a negative direct effect on the left-hand side of (13), so it results in an increase in $\hat{p}_1$. In turn, the market odds on the right-hand side is decreased. An increase in a/N reduces the right-hand side, so $\hat{p}_1$ falls. Since the left-hand side falls, the market odds ratio on the right-hand side also falls. □

The symmetric setting has the appealing property that the initial market belief in outcome 1, $a/[a + a]$, is equal to the prior belief $q = 1/2$. A priori, then, the market odds are correct, and there is no scope for betting on the basis of public information alone. Nevertheless, privately informed individuals can profit from betting. In the symmetric model we have $b_{1|-1} = b_{-1|1} < b_{1|1} = b_{-1|-1}$, so the final implied market probabilities satisfy $[a + b_{1|1}] / [a + a + b_{-1|1} + b_{1|1}] > 1/2 > [a + b_{1|-1}] / [a + a + b_{-1|-1} + b_{1|-1}]$. When the market's implied probability of an outcome exceeds 1/2, but remains well below 1, the true (and empirical) probability of the outcome is 1. The favorite-longshot bias is evident.

Example. In the linear signal example with fair prior ($q = 1/2$) and track take $\tau \leq 1/2$, the unique symmetric-policy Nash equilibrium has an explicit expression, with cutoff belief

$$\hat{p}_1 = \frac{(1-\tau)(1+a/n) - \sqrt{(1+a/N)\left(\tau^2 + (1-\tau)^2 a/N\right)}}{(1-2\tau)} \in [1/2, 1)\,.$$

5 Comparison of Market Structures

We have established that the favorite-longshot bias can arise as the result of informed betting in both parimutuel and fixed-odds markets. The aim of this section is to compare the extent of the bias in these two market structures. The main challenge for this comparison is that the market odds have different meaning in the two markets. In this section, we propose a method for overcoming this difficulty.

In the case of fixed-odds betting, the bookmaker uses the prior probability q as a basis for quoting odds ρ_1. The corresponding market implied probability $\pi_1 = 1/(1+\rho_i)$, are then associated with the prior probability q. Our aim is to obtain a relation between the prior probability and the market odds for the parimutuel market.

In the parimutuel system, for any given prior belief q there is a distribution of market implied probabilities and associated posterior probabilities. This is because the market implied probability $\pi = k/N$ results from the volume of bets placed on the two horses that is realized in the market. Since this random volume reveals information, to each market implied probability $\pi = k/N$ we associate the corresponding (Bayesian) posterior probability $1/(1+\beta)$. Note that in our model with a continuum of bettors, the law of large number applies so that for any prior q there are two possible realization of the market implied probability, depending on the outcome of the race.

In order to compare the extent of the favorite longshot bias in the two models, we perform the following transformation of the odds generated in the parimutuel market. For any value of the prior probability q, in the parimutuel market we determine the *average*

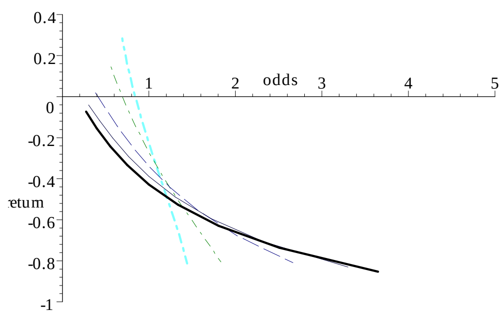

Figure 2: The average return to a parimutuel bet on horse 1 is plotted against the average market odds ratio in the linear signal example. All plots have $\tau = .15$, and show the results as q varies in the interval $(.1, .9)$. The five curves show the cases $a/N = 2, 1, .4, .2, .1$, in progressively thicker shade.

market implied probability for horse 1 and the *average* return to an extra bet on horse 1. Varying $q \in (0, 1)$ parametrically, we then obtain a plot of returns against market implied probabilities, directly comparable to Figure 1 obtained for the fixed-odds market.

The amount of bets on horse 1 is $a + N(1 - G(\hat{p}_1|x))$ when x is the winner. The average market implied probability for horse 1 is then

$$q\frac{a + N(1 - G(\hat{p}_1|1))}{2a + N(1 - G(\hat{p}_1|1)) + NG(\hat{p}_{-1}|1)} + (1-q)\frac{a + N(1 - G(\hat{p}_1|-1))}{2a + N(1 - G(\hat{p}_1|-1)) + NG(\hat{p}_{-1}|-1)}.$$

Ex ante, the expected payoff of a marginal extra bet on horse 1 is

$$qW(1|1) - 1 = q\frac{2a + N(1 - G(\hat{p}_1|1)) + NG(\hat{p}_{-1}|1)}{a + N(1 - G(\hat{p}_1|1))} - 1.$$

Using the equilibrium conditions (10) and (11), we can solve for $(\hat{p}_1, \hat{p}_{-1})$ as a function of q and compute these quantities. Figure 2 shows the resulting plot in our linear example, when the track take is $\tau = .15$. We have solved the system numerically, using Maple.[7]

In the parimutuel system, the amount bet on the longshot is almost completely lost. This is because the limited interest obtained by longshots is bad news about their chance of winning. In comparison, Figure 1 reveals that the loss to a bet on a longshot is bounded below. In fixed-odds betting, an ex-ante longshot is unlikely to attract insiders, so competition among bookmakers leads them to set more attractive odds on longshots.

[7] Our Maple worksheet is available upon request.

A related striking feature of Figure 2 is that favorites present a strictly positive return, when the number of insiders is small. This arbitrage opportunity arises since the cash-constrained insider population cannot correct the mis-pricing inherent in the bets placed by the outsiders.

As seen in Figure 2, in parimutuel betting the expected loss to a given long market odds ratio is decreasing in the fraction of money placed by insiders relative to outsiders. To understand this, notice that a long odds ratio $(a + b_{-1}) / (a + b_1) > 1$ arises for a smaller information ratio b_{-1}/b_1 when the total amount of insiders bets (both b_{-1} and b_1) are larger. Thus, the same ratio is actually less informative news against the longshot, when the insider population is larger. As a result, the increased presence of insiders serves to limit the losses an uninformed bettor who bets on longshots, and so results in a reduction of the favorite-longshot bias. In contrast, in fixed-odds markets the bookmaker must always increase the spread in the presence of more insiders.

Intuitively, the parimutuel payoff structure has a built-in insurance against adverse selection — when bettors have unfavorable information about a horse, this horse attracts few bets and so becomes automatically more attractive. This is because with parimutuel payoffs, the odds must always adjust to balance the budget.

The comparative statics of the favorite-longshot bias with respect to the fraction of insiders betting depends crucially on the market structure. In parimutuel markets an increase in the number of informed bettors results in an increase in the bias for any given market odds realization, but at the same time drives market odds to be more extreme and so reduces the bias. In fixed-odds markets instead, an increase in the fraction of informed bettors worsens the adverse selection problem, resulting in a higher markup and a stronger favorite-longshot bias.

These findings call for further empirical work in the area, in which controls are added for the market rules. The cross-country and cross-market variation in the extent of the favorite-longshot bias points to the relevance of the market rules in determining the behavior of participants on the supply and demand side.[8] Persistent differences in the observed

[8] Interesting selection issues arise when different betting schemes (fixed odd and parimutuel) coexist and compete to attract bets, as in the UK. As also suggested by Gabriel and Marsden (1990) and Bruce

biases could be attributed to varying degrees of market participation, informational asymmetry, amount of information available on bets placed, and degrees of randomness in the post time.

and Johnson (2000), wagerers might have different incentives to place their bets on the parimutuel system rather than with the bookmakers, depending on the quality of their information.

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