Online Appendix

When Liability is not Enough: Regulating Bonus Payments in Markets with Advice

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April 2022

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1 An Isomorphic Relation to Mixed Bundling

In Section 6 in the main text we have examined the case of simultaneous arrival of customers. As mentioned in the Introduction, the determination of the optimal nonlinear compensation in this case turns out to be isomorphic to the one of the optimal nonlinear tariffs with mixed bundling in the presence of firms' competition.

Consider again two firms $n = A, B$, each of which sells its own brand of two products, 1 and 2, with a constant per-unit production cost c for each product. Now there is a *single* customer who can purchase one or two products. We model the customer's preferences in the spirit of Hotelling competition as follows. Let u_i be a customer's fixed base utility from purchasing product i and let the sum be $u = u_1 + u_2$. As we consider again the case of full market coverage, u will always be realized. We now model firm horizontal differentiation. Let thus $q_i \in [0, 1]$ be a random variable with CDF $G(q_i)$. The customer is located at (q_1, q_2) and, for each product $i = 1, 2$ and some parameter $t > 0$, he incurs "transport cost" $(1 - q_i)t$ when buying from A and $q_i t$ when buying from B.

Firms can set two individual prices as well as prices for bundles. Imposing symmetry, each firm thus sets the per-unit price p_n and a bundled price of $2p_n - \delta_n$. For instance, when the consumer purchsases product 1 from firm A and product 2 from firm B, he realizes the payoff

$$u - p_A - p_B - (1 - q_1)t - q_2 t.$$

When he purchases instead both products from A, he realizes

$$u - (2p_A - \delta_A) - (1 - q_1)t - (1 - q_2)t.$$

Firm profits are given by

$$\pi_n = \mathrm{S}_n(p_n - c_n) - \mathrm{Pr}_n(2)\delta_n,$$

where we use the same notation as in the main next, notably $\mathrm{S}_n = \mathrm{Pr}(1) + 2\mathrm{Pr}_n(2)$ for the expected total sales for firm n.

Consider now our baseline analysis with advice again. As in our analysis in the main text, take the product prices as given, so that firms' strategic variables are the commission and the bonus. It is now easily seen, that the firm's problem vis-á-vis the advisor in the baseline case is isomorphic to that of optimal (mixed) bundled pricing for the final consumer, both when the consumer's demand arises simultaneously and when the customer buys both products at the same time.

2 Optimality of Non-Linear Incentives without the Full Coverage Assumption

In the main text we imposed a restriction on the advisor's concern levels for suitability of his advice with $w_0 < w_l$, so that he always recommends either product A or B and the market is fully covered (akin to a Hotelling model of horizontal differentiation). In this extension we extend our insight of biased advice to the case where this is no longer the case. Precisely, in the subsequent analysis we consider the case where, when deciding which product to recommend, the advisor's recommendation is ultimately either one of the two products or the option of no purchase at all (see, in particular, the respective advice regions in the subsequent Figures 1 and 2). For this we now specify that $w_h \geq w_0 > w_l$. Let

$$w_1 = w_h - w_0 \geq 0, \quad w_2 = w_0 - w_l > 0, \quad \text{and} \quad w = w_1 + w_2 = w_h - w_l > 0.$$

Advice Again, as in the main analysis, we first consider stage $t = 3$, where the advisor makes recommendations to customers. Suppose the advisor had recommended product A to the first customer, who followed this advice. Consider now the pattern of advice to the second customer. Then, his expected payoff is $f_A + b_A + q_2 w + w_l$ from recommending (again) product A (through sending message $m_2 = A$), $f_B + (1 - q_2)w + w_l$ from recommending product B (through sending message $m_2 = B$), and w_0 from recommending neither product (through sending message $m_2 = \phi$). Comparing the payoff from recommending neither product with the one from recommending either product yields the two thresholds

$$\bar{q}_2^{A\phi_A} = \frac{1}{w}\left(w_2 - (f_A + b_A)\right) \quad \text{and} \quad \bar{q}_2^{A\phi_B} = \frac{1}{w}(w_1 + f_B),$$

such that the advisor prefers $m_2 = \phi$ if $\bar{q}_2^{A\phi_B} \leq \bar{q}_2^{A\phi_A}$ and $q_2 \in [\bar{q}_2^{A\phi_B}, \bar{q}_2^{A\phi_A}]$, and $m_2 = A$ or B otherwise. The subscript 2 in both thresholds stands for the advice cutoff applied to the second customer, the first superscript A indicates that the advisor has already sold product A to the first customer, and the second superscript ϕ_A implies that the advisor compares $m_2 = \phi$ with $m_2 = A$ and ϕ_B that he compares $m_2 = \phi$ with $m_2 = B$. To deal with corner solutions, we redefine $\bar{q}_2^{A\phi_A} = 0$ if $w_2 \leq f_A + b_A$ and $\bar{q}_2^{A\phi_A} = 1$ if $w_1 \leq -(f_A + b_A)$.[1] Similarly, we redefine $\bar{q}_2^{A\phi_B} = 1$ if $w_2 \leq f_B$.[2] Next, comparing the payoff from recommending product A with both alternative payoffs yields the threshold

$$\bar{q}_2^{AA} = \max\left\{\frac{1}{2} - \frac{1}{2w}(f_A - f_B + b_A), \bar{q}_2^{A\phi_A}\right\},$$

such that the advisor prefers $m_2 = A$ if $q_2 \geq \bar{q}_2^{AA}$, and $m_2 = B$ or ϕ otherwise. The second superscript A indicates that the advisor recommends product A if $q_2 \geq \bar{q}_2^{AA}$.[3] Comparing

[1]Note that, in general, it could be the case that $f_A + b_A < 0$ with $b_A < 0$, though both $f_A \geq 0$ and $2f_A + b_A \geq 0$ must hold by limited liability.

[2]Note that $\bar{q}_2^{A\phi_B} \geq 0$ always holds true as $f_B \geq 0$ due to limited liability and as $w_1 \geq 0$.

[3]To deal with corner solutions, we redefine $\bar{q}_2^{AA} = 0$ if $w_2 \leq f_A + b_A$, where $w = w_1 + w_2 \leq w_1 + f_A + b_A > f_A - f_B + b_A$, and $\bar{q}_2^{AA} = 1$ if $w \leq -(f_A - f_B + b_A)$ or $w_1 \leq -(f_A + b_A)$.

the payoff from recommending product B with both alternative payoffs yields the threshold

$$\bar{q}_2^{AB} = \min\left\{\frac{1}{2} - \frac{1}{2w}(f_A - f_B + b_A), \bar{q}_2^{A\phi_B}\right\},$$

such that the advisor prefers $m_2 = B$ if $q_2 \leq \bar{q}_2^{AB}$, and $m_2 = A$ or ϕ otherwise. The second superscript B indicates that the advisor recommends product B if $q_2 \leq \bar{q}_2^{AB}$.[4]

Next consider the case that product B was sold to the first customer. Then, the advisor's expected payoff from subsequently recommending A is $f_A + q_2 w + w_l$, $f_B + b_B(1-q_2)w + w_l$ from recommending product B, and w_0 from recommending neither product. Comparing these payoffs now yields the thresholds

$$\bar{q}_2^{BA} = \max\left\{\frac{1}{2} - \frac{1}{2w}(f_A - f_B - b_B), \bar{q}_2^{B\phi_A}\right\}$$

and

$$\bar{q}_2^{BB} = \min\left\{\frac{1}{2} - \frac{1}{2w}(f_A - f_B - b_B), \bar{q}_2^{B\phi_B}\right\},$$

where $\bar{q}_2^{B\phi_A} = (1/w)(w_2 - f_A)$ and $\bar{q}_2^{B\phi_B} = (1/w)(w_1 + f_B + b_B)$. Here the first superscript B indicates that the advisor has sold product B to the first customer.[5]

Finally, consider the case where neither product was recommended to the first customer. Then his expected payoff is $f_A + q_2 w + w_l$ when recommending product A to the second customer, $f_B + (1-q_2)w + w_l$ when recommending product B, and w_0 when recommending neither product. Comparing these payoffs yields the thresholds

$$\bar{q}_2^{\phi A} = \max\left\{\frac{1}{2} - \frac{1}{2w}(f_A - f_B), \bar{q}_2^{\phi\phi_A}\right\}$$

and

$$\bar{q}_2^{\phi B} = \min\left\{\frac{1}{2} - \frac{1}{2w}(f_A - f_B), \bar{q}_2^{\phi\phi_B}\right\},$$

where $\bar{q}_2^{\phi\phi_A} = (1/w)(w_2 - f_A)$ and $\bar{q}_2^{\phi\phi_B} = (1/w)(w_1 + f_B)$. Here the superscript ϕ indicates that the advisor has recommended neither product to the first customer.[6]

With these thresholds, consider now the pattern of advice to the first customer. For the sake of brevity, we write $\bar{\mathbf{q}}_2^k = (\bar{q}_2^{kA}, \bar{q}_2^{kB})$ for any given $k \in \{A, B, \phi\}$. Note that $\bar{q}_2^{kB} \leq \bar{q}_2^{kA}$

[4]To deal with corner solutions, we redefine $\bar{q}_2^{AB} = 0$ if $w \leq f_A - f_B + b_A$ and $\bar{q}_2^{AB} = 1$ if $w \leq \max\{-(f_A - f_B + b_A), w_1 + f_B\}$. Note that $-(f_A - f_B + b_A) = -(f_A + b_A) + f_B$ could be larger or smaller than $w_1 + f_B$, depending on the sign of $f_A + b_A$.

[5]To deal with corner solutions, we redefine the thresholds as follows: $\bar{q}_2^{B\phi_A} = 0$ if $w_2 \leq f_A$, whereas $\bar{q}_2^{B\phi_A} \leq 1$ always holds true as $w_2 < w$ and $f_A \geq 0$ due to limited liability; $\bar{q}_2^{B\phi_B} = 0$ if $w_1 \leq -(f_B + b_B)$ and $\bar{q}_2^{B\phi_B} = 1$ if $w_2 \leq f_B + b_B$; $\bar{q}_2^{BA} = 0$ if $w \leq f_A - f_B - b_B$ or $w_2 \leq f_A$, and $\bar{q}_2^{BA} = 1$ if $w < -(f_A - f_B - b_B)$; $\bar{q}_2^{BB} = 0$ if $w \leq f_A - f_B - b_B$ or $w_1 \leq -(f_B + b_B)$, and $\bar{q}_2^{BB} = 1$ if $w_2 \leq f_B + b_B$.

[6]To deal with corner solutions, we redefine the thresholds as follows: $\bar{q}_2^{\phi\phi_A} = 0$ if $w_2 \leq f_A$; $\bar{q}_2^{\phi\phi_B} = 1$ if $w_2 \leq f_B$; $\bar{q}_2^{\phi A} = 0$ if $w_2 \leq f_A$ and $\bar{q}_2^{\phi A} = 1$ if $w < -(f_A - f_B)$; $\bar{q}_2^{\phi B} = 0$ if $w \leq f_A - f_B$ and $\bar{q}_2^{\phi B} = 1$ if $w_2 \leq f_B$.

always holds true by definition. Using this, we define

$$
\begin{aligned}
Z(\bar{\mathbf{q}}_2^k) \; &= \; \int_0^{\bar{q}_2^{kB}} (f_B + (1-q)w + w_l)\, g(q)dq + w_0 \left(G(\bar{q}_2^{kA}) - G(\bar{q}_2^{kB}) \right) + \int_{\bar{q}_2^{kA}}^1 (f_A + qw + w_l) g(q)dq \\
&\quad + \begin{cases} b_A \left(1 - G(\bar{q}_2^{AA})\right) & \text{if } k = A, \\ b_B G(\bar{q}_2^{BB}) & \text{if } k = B, \\ 0 & \text{if } k = \phi, \end{cases}
\end{aligned}
$$

where $Z(\bar{\mathbf{q}}_2^k) \geq 0$ holds for any given $k \in \{A, B, \phi\}$. Sending message $m_1 = A$ to the first customer yields the expected payoff

$$f_A + q_1 w + w_l + Z(\bar{\mathbf{q}}_2^A),$$

sending $m_1 = B$ the expected payoff

$$f_B + (1 - q_1)w + w_l + Z(\bar{\mathbf{q}}_2^B),$$

and sending $m_1 = \phi$ the expected payoff $w_0 + Z(\bar{\mathbf{q}}_2^\phi)$. Comparing these payoffs yields the two thresholds

$$\bar{q}_1^A = \max\left\{ \frac{1}{2} - \frac{1}{2w}(f_A - f_B + Z(\bar{\mathbf{q}}_2^A) - Z(\bar{\mathbf{q}}_2^B)), \frac{1}{w}\left(w_2 - f_A + Z(\bar{\mathbf{q}}_2^A) - Z(\bar{\mathbf{q}}_2^\phi) \right) \right\}$$

and

$$\bar{q}_1^B = \min\left\{ \frac{1}{2} - \frac{1}{2w}(f_A - f_B + Z(\bar{\mathbf{q}}_2^A) - Z(\bar{\mathbf{q}}_2^B)), \frac{1}{w}\left(w_1 + f_B + Z(\bar{\mathbf{q}}_2^B) - Z(\bar{\mathbf{q}}_2^\phi) \right) \right\},$$

such that the advisor prefers $m_1 = A$ if $q_1 \geq \bar{q}_1^A$, $m_2 = \phi$ if $\bar{q}_1^B \leq q_1 \leq \bar{q}_1^A$, and $m_2 = B$ otherwise. The subscript 1 in both thresholds stands for the advice cutoff applied to the first customer. The superscript A in $\bar{q}_1^A$ indicates that the advisor compares $m_1 = A$, while the one in $\bar{q}_1^B$ that he compares $m_1 = B$. Note that the inequality $\bar{q}_1^B \leq \bar{q}_1^A$ always holds true by definition.[7] We can now characterize the pattern of advice for any given compensation (f_n, b_n).

Lemma 1 *Suppose that $w_h \geq w_0 > w_l$. If customers follow his recommendation, the advisor's optimal recommendation is characterized as follows:*

$$m_1 = \begin{cases} A & \text{if } q_1 \in [\bar{q}_1^A, 1], \\ \phi & \text{if } q_1 \in [\bar{q}_1^B, \bar{q}_1^A], \\ B & \text{if } q_1 \in [0, \bar{q}_1^B], \end{cases}$$

[7] To deal with corner solutions, we redefine $\bar{q}_1^A = 0$ if $w \leq f_A - f_B + Z(\bar{\mathbf{q}}_2^A) - Z(\bar{\mathbf{q}}_2^B)$ or $w_2 \leq f_A + Z(\bar{\mathbf{q}}_2^A) - Z(\bar{\mathbf{q}}_2^\phi)$, and $\bar{q}_1^A = 1$ if $w \leq -(f_A - f_B + Z(\bar{\mathbf{q}}_2^A) - Z(\bar{\mathbf{q}}_2^B))$ or $w_1 \leq -(f_A + Z(\bar{\mathbf{q}}_2^A) - Z(\bar{\mathbf{q}}_2^\phi))$. Similarly, we redefine $\bar{q}_1^B = 0$ if $w \leq f_A - f_B + Z(\bar{\mathbf{q}}_2^A) - Z(\bar{\mathbf{q}}_2^B)$ or $w_1 \leq -(f_B + Z(\bar{\mathbf{q}}_2^B) - Z(\bar{\mathbf{q}}_2^\phi))$, and $\bar{q}_1^B = 1$ if $w \leq -(f_A - f_B + Z(\bar{\mathbf{q}}_2^A) - Z(\bar{\mathbf{q}}_2^B))$ or $w_2 \leq f_B + Z(\bar{\mathbf{q}}_2^B) - Z(\bar{\mathbf{q}}_2^\phi)$.

and for any given $k \in \{A, B, \phi\}$,

$$m_2 = \begin{cases} A & \text{if } q_2 \in [\bar{q}_2^{kA}, 1], \\ \phi & \text{if } q_2 \in [\bar{q}_2^{kB}, \bar{q}_2^{kA}], \\ B & \text{if } q_2 \in [0, \bar{q}_2^{kB}]. \end{cases}$$

To illustrate Lemma 1, we restrict attention to symmetric compensation (f, b). In the case of no bonus ($b = 0$), both $\bar{q}_1^A = \bar{q}_2^{kA} \equiv \bar{q}^A$ and $\bar{q}_1^B = \bar{q}_2^{kB} \equiv \bar{q}^B$ hold for any given $k \in \{A, B, \phi\}$, where $\bar{q}^A \geq \bar{q}^B$. Whether $\bar{q}^A = \bar{q}^B$ holds true or not depends on the parameters (w_1, w_2) and the instrument f. We can indeed show below that for given (f_A, f_B), $\bar{q}^A > \bar{q}^B$ holds true if $w_2 - w_1 > f_A + f_B$, and $\bar{q}^A = \bar{q}^B$ otherwise. So if $\bar{q}^A > \bar{q}^B$, the advisor recommends product A to the i-th customer (sends message $m_i = A$) if $q_i \geq \bar{q}^A$, neither product ($m_i = \phi$) if $q_i \in [\bar{q}^B, \bar{q}^A]$, and B otherwise, irrespective of the order of their arrivals. Figure 1 illustrates this pattern of advice

In contrast, with $b > 0$ advice cutoffs $\bar{q}_1^n$ and $\bar{q}_2^{kn}$ for all $k \in \{A, B, \phi\}$ are no longer equal for any given $n = A, B$. Figure 2 depicts a pattern of advice with a strictly positive bonus, where we take the case with $\bar{q}_2^{BA} = \bar{q}_2^{\phi A}$ and $\bar{q}_2^{AB} = \bar{q}_2^{\phi B}$. Hence, in the depicted case the advisor also recommends neither product with some probability.

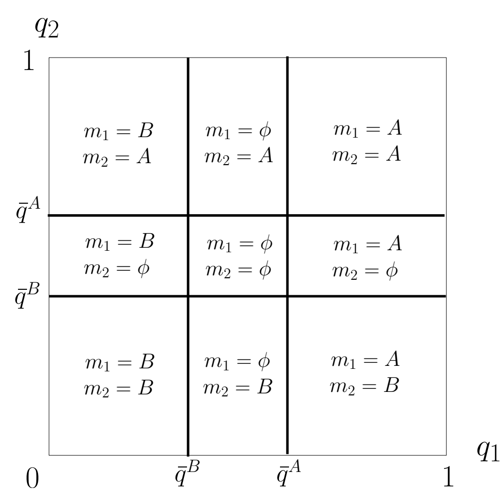

Figure 1: Pattern of advice without bonus, conditional on $\bar{q}^A > \bar{q}^B$.

Firm profits For any given $k \in \{A, B, \phi\}$, $\bar{\mathbf{q}}_1 = (\bar{q}_1^A, \bar{q}_1^B)$, and $\bar{\mathbf{q}}_2^k = (\bar{q}_2^{kB}, \bar{q}_2^{kA})$, define by

$$\Pr_A(1) = G(\bar{q}_1^B)(1 - G(\bar{q}_2^{BA})) + (G(\bar{q}_1^A) - G(\bar{q}_1^B))(1 - G(\bar{q}_2^{\phi A})) + (1 - G(\bar{q}_1^A))G(\bar{q}_2^{AA})$$

the probability that the advisor recommends product A to either customer, and similarly by

$$\Pr_B(1) = G(\bar{q}_1^B)(1 - G(\bar{q}_2^{BB})) + (G(\bar{q}_1^A) - G(\bar{q}_1^B))G(\bar{q}_2^{\phi B}) + (1 - G(\bar{q}_1^A))G(\bar{q}_2^{AB})$$

the probability that the advisor recommends product B to either customer. Also, define by

$$\Pr_A(2) = (1 - G(\bar{q}_1^A))(1 - G(\bar{q}_2^{AA}))$$

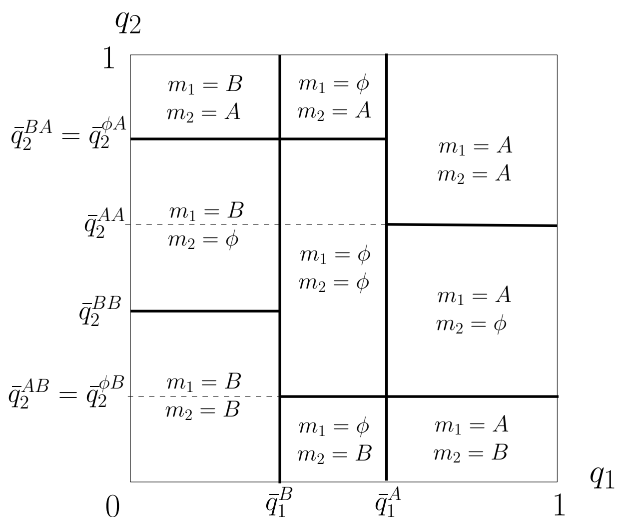

Figure 2: Pattern of advice with a bonus, conditional on $\bar{q}_2^{BA} = \bar{q}_2^{\phi A}$ and $\bar{q}_2^{AB} = \bar{q}_2^{\phi B}$.

the probability that the advisor recommends product A to both customers, and by

$$\Pr_B(2) = G(\bar{q}_1^B)G(\bar{q}_2^{BB})$$

the probability that the advisor recommends product B to both customers. We denote by

$$S_n = Pr_n(1) + 2Pr_n(2)$$

expected sales of firm n. For given compensation (f_n, b_n) and product price p_n, for $n = A, B$, expected profits are then written as

$$\pi_n = \mathrm{S}_n(p_n - c_n - f_n) - \Pr_n(2)b_n.$$

The Impossibility of Unbiased Advice in a Unregulated Equilbrium Together with the result in the main text, we can show the optimality of nonlinear incentives in the fully or partially covered market.

Proposition 1 *Nonlinear incentives are part of any equilibrium, i.e., there is no equilibrium in which $b_n = b = 0$, regardless of whether the "full coverage" assumption, as invoked in the main text, applies or not.*

In the remainder of this section, we prove this result by arguing to a contradiction, thus supposing initially that $b_n = b = 0$. In this case, both $\bar{q}_2^{kA}$ and $\bar{q}_2^{kB}$ are independent of $k \in \{A, B, \phi\}$ as $\bar{q}_2^{kn}$ $(n = A, B)$ are all equal for any given $k \in \{A, B, \phi\}$, and thereby advice cutoff $\bar{q}_1^A$ is equal to $\bar{q}_2^{kA}$ for all $k \in \{A, B, \phi\}$ and similarly $\bar{q}_1^B$ is equal to $\bar{q}_2^{kB}$ as $Z(\bar{\mathbf{q}}_2^k)$ are all equal for $k \in \{A, B, \phi\}$. With this, denote $\bar{q}^A = \bar{q}_1^A = \bar{q}_2^{kA}$ and $\bar{q}^B = \bar{q}_1^B = \bar{q}_2^{kB}$, which are written as

$$\bar{q}^A = \max\left\{\frac{1}{2} - \frac{1}{2w}(f_A - f_B), \frac{1}{w}(w_2 - f_A)\right\}$$

and

$$\bar{q}^B = \min\left\{\frac{1}{2} - \frac{1}{2w}(f_A - f_B), \frac{1}{w}\left(w_1 + f_B\right)\right\},$$

respectively. If $w_2 - w_1 \leq f_A + f_B$ holds, then

$$\bar{q}^A = \frac{1}{2} - \frac{1}{2w}(f_A - f_B) = \bar{q}^B,$$

while if $w_2 - w_1 > f_A + f_B$ holds, then

$$\bar{q}^A = \frac{1}{w}\left(w_2 - f_A\right) \quad \text{and} \quad \bar{q}^B = \frac{1}{w}(w_1 + f_B).$$

For these two cases, we suppose in our argument to a contradiction that the respecitve choices f_n and $b_n = 0$ are mutually optimal. As we only consider a marginal deviation, the subsequent case distinction applies for the given (equilbrium) compensation.

Consider first the case of $w_2 - w_1 \leq f_A + f_B$, in which $\bar{q}^A$ is equal to $\bar{q}^B$. In this case, advice cutoffs are all equal, and therefore the analysis reduces to the one in the main text.

Next, consider the case of $w_2 - w_1 > f_A + f_B$, in which $\bar{q}^A$ and $\bar{q}^B$ are no longer equal. We can show that both $\bar{q}^A$ and $\bar{q}^B$ lie in the open interval $(0, 1)$ and hence $\bar{q}^A > \bar{q}^B$ holds.[8] From this, the pattern of advice when $b_n = b = 0$ is described as follows. For any given $i(= 1, 2)$-th customer, the advisor sends message A if $q_i \geq \bar{q}^A$, ϕ if $q_i \in [\bar{q}^B, \bar{q}^A]$, and B otherwise (see Figure 1).

Following closely the analysis in the main text, we first examine the effects of the marginal increases in f_n and b_n on the advice cutoffs $(\bar{q}_1^n, \bar{q}_2^{kn}) \in (0, 1)^2$, for any given $n \in \{A, B\}$ and $k \in \{A, B, \phi\}$. When $w_2 - w_1 > f_A + f_B$ with taking b_n as an arbitrarily small value, all advice cutoffs are written as follows:

$$\bar{q}_2^{AA} = \frac{1}{w}\left(w_2 - (f_A + b_A)\right), \quad \bar{q}_2^{AB} = \bar{q}_2^{\phi B} = \frac{1}{w}(w_1 + f_B),$$

$$\bar{q}_2^{BA} = \bar{q}_2^{\phi A} = \frac{1}{w}(w_2 - f_A), \quad \bar{q}_2^{BB} = \frac{1}{w}(w_1 + f_B + b_B),$$

$$\bar{q}_1^A = \frac{1}{w}\left(w_2 - f_A + Z(\bar{\mathbf{q}}_2^A) - Z(\bar{\mathbf{q}}_2^\phi)\right), \quad \bar{q}_1^B = \frac{1}{w}\left(w_1 + f_B + Z(\bar{\mathbf{q}}_2^B) - Z(\bar{\mathbf{q}}_2^\phi)\right).$$

Evaluated at $b_n = b = 0$, it follows that $\bar{q}_2^{AA} = \bar{q}_2^{BA} = \bar{q}_2^{\phi A}$ and $\bar{q}_2^{AB} = \bar{q}_2^{BB} = \bar{q}_2^{\phi B}$, so that $Z(\bar{\mathbf{q}}_2^A) = Z(\bar{\mathbf{q}}_2^B) = Z(\bar{\mathbf{q}}_2^\phi)$, $\bar{q}_1^A = \bar{q}_2^{kA} \equiv \bar{q}^A = (1/w)(w_2 - f_A) \in (0, 1)$, and $\bar{q}_1^B = \bar{q}_2^{kB} \equiv \bar{q}^B = (1/w)(w_1 + f_B) \in (0, 1)$ for any given $k \in \{A, B, \phi\}$.

Lemma 2 *Take the supposed equilibrium compensation with $b_n = 0$ and given values f_n, where $w_2 - w_1 > f_A + f_B$. At $\bar{q}^n = \bar{q}_1^n = \bar{q}_2^{kn} \in (0, 1)$ we have*

$$\forall k \in \{A, B, \phi\}, \quad \frac{\partial \bar{q}_2^{kA}}{\partial f_n} = \frac{\partial \bar{q}_1^A}{\partial f_n} = \begin{cases} -\frac{1}{w}, & \text{if } n = A, \\ 0, & \text{if } n = B, \end{cases} \tag{1}$$

[8] One can first observe that $\bar{q}^A$ is above zero as $w_2 > w_1 + f_A + f_B > f_A + f_B \geq f_A$, where $w_1 > 0$ and $f_B \geq 0$ due to limited liability, and also that $\bar{q}^B$ is above zero as $w_1 > 0$ and $f_B \geq 0$. Similarly, $\bar{q}^A$ is below one as otherwise $w_1 \leq -f_A \leq 0$, which contradicts the assumption of $w_1 > 0$, and also $\bar{q}^B$ is below one as otherwise $w_2 \leq f_B$, which contradicts $w_2 > w_1 + f_A + f_B > f_B$ as $w_1 > 0$ and $f_A \geq 0$.

$$\forall k \in \{A, B, \phi\}, \quad \frac{\partial \bar{q}_2^{kB}}{\partial f_n} = \frac{\partial \bar{q}_1^B}{\partial f_n} = \begin{cases} 0, & \textit{if } n = A, \\ \frac{1}{w}, & \textit{if } n = B, \end{cases} \tag{2}$$

$$\left(\frac{\partial \bar{q}_2^{AA}}{\partial b_n}, \frac{\partial \bar{q}_2^{BA}}{\partial b_n}, \frac{\partial \bar{q}_2^{\phi A}}{\partial b_n}, \frac{\partial \bar{q}_1^{A}}{\partial b_n} \right) = \begin{cases} \frac{1}{w} \left(-1, 0, 0, -\left(1 - G(\bar{q}^A)\right)\right), & \textit{if } n = A, \\ (0, 0, 0, 0), & \textit{if } n = B, \end{cases} \tag{3}$$

and

$$\left(\frac{\partial \bar{q}_2^{BA}}{\partial b_n}, \frac{\partial \bar{q}_2^{BB}}{\partial b_n}, \frac{\partial \bar{q}_2^{\phi B}}{\partial b_n}, \frac{\partial \bar{q}_1^{B}}{\partial b_n} \right) = \begin{cases} (0, 0, 0, 0), & \textit{if } n = A, \\ \frac{1}{w} \left(0, 1, 0, G(\bar{q}^B)\right), & \textit{if } n = B. \end{cases} \tag{4}$$

Proof. We focus on firm A and examine the effects of the marginal increases in f_A and b_A on the advice cutoffs. The same argument applies to the case of $n = B$, leading to the remaining part of (1)–(4). Consider first the effect of a marginal increase in f_A on the advice cutoffs $\bar{q}_2^{kn}$ for any given $n \in \{A, B\}$ and $k \in \{A, B, \phi\}$. Differentiating with respect to f_A leads to a downward shift of the advice cutoffs $\bar{q}_2^{kA}$, while $\bar{q}_2^{kB}$ remains unchanged, precisely

$$\frac{\partial \bar{q}_2^{kA}}{\partial f_A} = -\frac{1}{w} \quad \text{and} \quad \frac{\partial \bar{q}_2^{kB}}{\partial f_A} = 0. \tag{5}$$

Next, consider the effect of a marginal increase in f_A on $\bar{q}_1^n$:

$$\frac{\partial \bar{q}_1^A}{\partial f_A} = \frac{1}{w} \left(-1 + \frac{\partial Z(\bar{\mathbf{q}}_2^A)}{\partial f_A} - \frac{\partial Z(\bar{\mathbf{q}}_2^\phi)}{\partial f_A} \right) = -\frac{1}{w},$$

where we apply condition (5) to $(\partial Z(\bar{\mathbf{q}}_2^k))/(\partial f_A)$ with $\bar{q}^n = \bar{q}_1^n = \bar{q}_2^{kn}$ to obtain $(\partial Z(\bar{\mathbf{q}}_2^A))/(\partial f_A) = (\partial Z(\bar{\mathbf{q}}_2^B))/(\partial f_A) = (\partial Z(\bar{\mathbf{q}}_2^\phi))/(\partial f_A)$, which is given by

$$\left(w_0 - (f_A + \bar{q}^A w + w_l)\right) \left(\frac{\partial \bar{q}_2^{kA}}{\partial f_A} \right) g(\bar{q}^A) + 1 - G(\bar{q}^A) = 1 - G(\bar{q}^A),$$

where we use $\bar{q}^A = (1/w)(w_2 - f_A)$ with $w_2 = w_0 - w_l$. Similarly, differentiating $\bar{q}_1^B$ with respect to f_A, evaluated at $\bar{q}^B = \bar{q}_2^{kB} \in (0, 1)$, yields

$$\frac{\partial \bar{q}_1^B}{\partial f_A} = \frac{1}{w} \left(\frac{\partial Z(\bar{\mathbf{q}}_2^B)}{\partial f_A} - \frac{\partial Z(\bar{\mathbf{q}}_2^\phi)}{\partial f_A} \right) = 0,$$

where $(\partial Z(\bar{\mathbf{q}}_2^B))/(\partial f_A) = (\partial Z(\bar{\mathbf{q}}_2^\phi))/(\partial f_A)$.

As with a marginal increase in f_A, consider now a marginal increase in b_A. We now have

$$\frac{\partial \bar{q}_2^{AA}}{\partial b_A} = -\frac{1}{w} \quad \text{and} \quad \frac{\partial \bar{q}_2^{BA}}{\partial b_A} = \frac{\partial \bar{q}_2^{\phi A}}{\partial b_A} = \frac{\partial \bar{q}_2^{kB}}{\partial b_A} = 0 \tag{6}$$

and

$$\frac{\partial \bar{q}_1^A}{\partial b_A} = \frac{1}{w} \left(\frac{\partial Z(\bar{\mathbf{q}}_2^A)}{\partial b_A} - \frac{\partial Z(\bar{\mathbf{q}}_2^\phi)}{\partial b_A} \right) = -\frac{1}{w} \left(1 - G(\bar{q}^A)\right),$$

where we apply condition (6) to $(\partial Z(\bar{\mathbf{q}}_2^k))/(\partial b_A)$, so as to obtain

$$\frac{\partial Z(\bar{\mathbf{q}}_2^A)}{\partial b_A} = \left(w_0 - (f_A + \bar{q}^A w + w_l)\right)\left(\frac{\partial \bar{q}_2^{kA}}{\partial f_A}\right) g(\bar{q}^A) + 1 - G(\bar{q}^A) = 1 - G(\bar{q}^A),$$

as well as $(\partial Z(\bar{\mathbf{q}}_2^B))/(\partial b_A) = (\partial Z(\bar{\mathbf{q}}_2^\phi))/(\partial b_A) = 0$. Similarly, differentiating $\bar{q}_1^B$ with respect to b_A, evaluated at $\bar{q}^B = \bar{q}_2^{kB} \in (0,1)$, yields

$$\frac{\partial \bar{q}_1^B}{\partial b_A} = \frac{1}{w}\left(\frac{\partial Z(\bar{\mathbf{q}}_2^B)}{\partial b_A} - \frac{\partial Z(\bar{\mathbf{q}}_2^\phi)}{\partial b_A}\right) = 0,$$

where we use $(\partial Z(\bar{\mathbf{q}}_2^B))/(\partial b_A) = (\partial Z(\bar{\mathbf{q}}_2^\phi))/(\partial b_A) = 0$. **Q.E.D.**

Using (1)–(4), we can evaluate the effect of a marginal increase in $x_n \in \{f_n, b_n\}$ on expected sales, $\mathrm{S}_n^x = \mathrm{Pr}^x(1) + 2\mathrm{Pr}_n^x(2)$ (where we now suppress the assumption that $b_n = 0$).

Lemma 3 *At $\bar{q}^n = \bar{q}_1^n = \bar{q}_2^{kn} \in (0,1)$, for any given $n \in \{A,B\}$ and $k \in \{A,B,\phi\}$,*

$$S_n^b = \begin{cases} (1 - G(\bar{q}^A))S_A^f, & \text{if } n = A, \\ G(\bar{q}^B)S_B^f, & \text{if } n = B. \end{cases} \tag{7}$$

Proof. Focusing on firm A, first note that

$$\begin{aligned} \mathrm{S}_A &= \mathrm{Pr}_A(1) + 2\mathrm{Pr}_A(2) \\ &= G(\bar{q}_1^B)(1 - G(\bar{q}_2^{BA})) + (G(\bar{q}_1^A) - G(\bar{q}_1^B))(1 - G(\bar{q}_2^{\phi A})) + (1 - G(\bar{q}_1^A))G(\bar{q}_2^{AA}) \\ &\quad + 2(1 - G(\bar{q}_1^A))(1 - G(\bar{q}_2^{AA})). \end{aligned}$$

Differentiating S_A with respect to $x_A \in \{f_A, b_A\}$, evaluated at $\bar{q}^n = \bar{q}_1^n = \bar{q}_2^{kn} \in (0,1)$ for any given $n \in \{A,B\}$ and $k \in \{A,B,\phi\}$, yields the marginal increase in sales by the marginal increase in x_A, that is, $\mathrm{S}_A^x = \mathrm{Pr}_A^x(1) + 2\mathrm{Pr}_A^x(2)$, which can be written as

$$\begin{aligned} \mathrm{S}_A^x &= \left(1 - G(\bar{q}_2^{\phi A}) - G(\bar{q}_2^{AA}) - 2(1 - G(\bar{q}_2^{AA}))\right) g(\bar{q}_1^A)\frac{\partial \bar{q}_1^A}{\partial x_A} \\ &\quad - G(\bar{q}_1^B)g(\bar{q}_2^{BA})\frac{\partial \bar{q}_2^{BA}}{\partial x_A} - (G(\bar{q}_1^A) - G(\bar{q}_1^B))g(\bar{q}_2^{\phi A})\frac{\partial \bar{q}_2^{\phi A}}{\partial x_A} - \left(1 - G(\bar{q}_1^A)\right) g(\bar{q}_2^{AA})\frac{\partial \bar{q}_2^{AA}}{\partial x_A} \\ &= -g(\bar{q}^A)\left(\frac{\partial \bar{q}_1^A}{\partial x_A} + G(\bar{q}^B)\frac{\partial \bar{q}_2^{BA}}{\partial x_A} + \left(G(\bar{q}^A) - G(\bar{q}^B)\right)\frac{\partial \bar{q}_2^{\phi A}}{\partial x_A} + \left(1 - G(\bar{q}^A)\right)\frac{\partial \bar{q}_2^{AA}}{\partial x_A}\right) \\ &= \begin{cases} \frac{2}{w}g(\bar{q}^A), & \text{if } x_A = f_A, \\ \frac{2}{w}g(\bar{q}^A)\left(1 - G(\bar{q}^A)\right), & \text{if } x_A = b_A, \end{cases} \end{aligned}$$

where the first equality follows from (1)–(4), the second from $\bar{q}^n = \bar{q}_1^n = \bar{q}_2^{kn} \in (0,1)^2$ for any given $n = A, B$ and $k \in \{A,B,\phi\}$, and the third again from (1)–(4). This leads to

$$\mathrm{S}_A^b = \left(1 - G(\bar{q}^A)\right)\mathrm{S}_A^f,$$

which corresponds to (7) in case of $n = A$. The same argument applies to $n = B$, yielding the remaining part of (7). **Q.E.D.**

Consider now marginal adjustments $(df_n, db_n) \in \mathbb{R}^2$ such that total expected sales remain unchanged, that is,

$$\mathrm{S}_n^f df_n + \mathrm{S}_n^b db_n = 0. \tag{8}$$

Applying (7) to (8), we can derive a relationship between these marginal adjustments (df_n, db_n) in the absence of bonuses ($b_n = b = 0$).

Lemma 4 *For any given $n = A, B$, consider marginal adjustments $(df_n, db_n) \in \mathbb{R}^2$ defined by (8). If $b_n = b = 0$, (df_n, db_n) must satisfy*

$$df_n = \begin{cases} -(1 - G(\bar{q}^A))db_A, & \text{if } n = A, \\ -G(\bar{q}^B)db_B, & \text{if } n = B. \end{cases} \tag{9}$$

Consider next the total derivative of firm profits

$$d\pi_n = -\mathrm{S}_n df_n - \mathrm{Pr}_n(2)db_n, \tag{10}$$

where we omitted the term $-(\mathrm{Pr}_n^f(2)df_n + \mathrm{Pr}_n^b(2)db_n)b_n$, which equals zero as $b_n = b = 0$. Since $\bar{q}^n = \bar{q}_1^n = \bar{q}_2^{kn} \in (0,1)^2$ for any given $n = A, B$ and $k \in \{A, B, \phi\}$, we have $\mathrm{Pr}_A(1) = 2G(\bar{q}^A)(1 - G(\bar{q}^A))$, $\mathrm{Pr}_B(1) = 2G(\bar{q}^B)(1 - G(\bar{q}^B))$, $\mathrm{Pr}_A(2) = (1 - G(\bar{q}^A))^2$, and $\mathrm{Pr}_B(2) = G(\bar{q}^B)^2$, so that $\mathrm{S}_A = 2(1 - G(\bar{q}^A))$ and $\mathrm{S}_B = 2G(\bar{q}^B)$. Evaluating the term when (df_n, db_n) satisfies (9), expression (10) finally becomes $d\pi_n = db_n \mathrm{Pr}_n(2)$. In other words, when firm n starts paying a bonus and reduces its commission so that total expected sales remain unchanged, this strictly increases profits with derivative $d\pi_n/db_n = \mathrm{Pr}_n(2)$, which is the likelihood with which the advisor will recommend firm n's product to both customers. This completes the proof.

3 Optimality of Non-Linear Incentives for an Arbitrary Number of Customers

Preliminary Results. Denote the total number of customers by I. We also define by $\mathbf{1}_{\mathbb{N}} : S \to \{0,1\}$ an indicator function, where $\mathbb{N} = \{1,2,\dots\}$ is the set of natural numbers. Also, we simply write s_n as s without subscript n.

Advice. Take any incentives $(f_n, (b_n^s)_{s=1}^{I-1})$ for $n = A, B$ as given. To begin with, consider the pattern of advice for the last $I(\geq 2)$-th arriving customer. For any given $s = 0, 1, \dots, I-1$, having sold s units of product A and $(I-1-s)$ units of product B to $(I-1)$ customers, the advisor anticipates that he receives an expected payoff equal to $f_A + \mathbf{1}_{\mathbb{N}}(s)b_A^s + q_I w + w_l$ from recommending product A (through sending message $m_I = A$) to the I-th customer whose observed product suitability is $q_I \in [0,1]$, and $f_B + \mathbf{1}_{\mathbb{N}}(I-1-s)b_B^{I-1-s} + (1-q_I)w + w_l$ from recommending product B (through sending message $m_I = B$). Comparing these payoffs yields the threshold

$$\bar{q}_I^s = \frac{1}{2} - \frac{1}{2w}(f_A - f_B + \mathbf{1}_{\mathbb{N}}(s-1)b_A^{s-1} - \mathbf{1}_{\mathbb{N}}(I-1-s)b_B^{I-1-s}),$$

such that the advisor prefers $m_I = A$ if $q_I \geq \bar{q}_I^s$ and $m_I = B$ otherwise. The subscript I in $\bar{q}_I^s$ stands for the advice cutoff applied to the I-th customer, and the superscript s indicates that the advisor has already sold s units of product A and $(I-1-s)$ units of product B to $(I-1)$ customers.[9]

Consider any other i-th arriving customer for any given $i = 1, \dots, I-1$. We recursively define the associated threshold $\bar{q}_i^s$ for any given $s = 0, \dots, i-1$ from $i = I-2$ to the first $(i = 1)$ customer. The subscript i in $\bar{q}_i^s$ stands for the advice cutoff applied to the i-th customer, and the superscript s indicates that the advisor has already sold s units of product A and $(i-1-s)$ units of product B to $(i-1)$ customers (if there is any). Suppose that for any given $i = 1, \dots, I-1$ the threshold of $\bar{q}_{i+1}^s$ is well-defined and let

$$\begin{aligned} Z(\bar{q}_{i+1}^{\tilde{s}}) &= \int_0^{\bar{q}_{i+1}^{\tilde{s}}} (f_B + \mathbf{1}_{\mathbb{N}}(i-\tilde{s})b_B^{i-\tilde{s}} + (1-q)w + w_l)g(q)dq \\ &\quad + \int_{\bar{q}_{i+1}^{\tilde{s}}}^1 (f_A + \mathbf{1}_{\mathbb{N}}(\tilde{s}-1)b_A^{\tilde{s}-1} + qw + w_l)g(q)dq. \end{aligned}$$

Having sold s units of product A and $(i-1-s)$ units of product B to $(i-1)$, the advisor anticipates that he receives an expected payoff equal to

$$f_A + \mathbf{1}_{\mathbb{N}}(s-1)b_A^{s-1} + q_i w + w_l + Z(\bar{q}_{i+1}^{s+1})$$

from recommending product A (through sending message $m_i = A$), and

$$f_B + \mathbf{1}_{\mathbb{N}}(i-1-s)b_B^{i-1-s} + (1-q_i)w + w_l + Z(\bar{q}_{i+1}^{s})$$

[9] Note that we restrict the exposition to interior thresholds. To deal with corner solutions, we redefine $\bar{q}_I^s = 0$ if $w \leq f_A - f_B + \mathbf{1}_{\mathbb{N}}(s)b_A^s - \mathbf{1}_{\mathbb{N}}(I-1-s)b_B^{I-1-s}$ and $\bar{q}_I^s = 1$ if $w \leq -(f_A - f_B + \mathbf{1}_{\mathbb{N}}(s)b_A^s - \mathbf{1}_{\mathbb{N}}(I-1-s)b_B^{I-1-s})$.

from recommending product B (through sending message $m_i = B$). Comparing these payoffs yields the threshold

$$\bar{q}_i^s = \frac{1}{2} - \frac{1}{2w}\left(f_A - f_B + \mathbf{1}_{\mathbb{N}}(s-1)b_A^{s-1} - \mathbf{1}_{\mathbb{N}}(i-1-s)b_B^{i-1-s} + Z(\bar{q}_{i+1}^{s+1}) - Z(\bar{q}_{i+1}^{s})\right), \quad (11)$$

such that the advisor prefers $m_i = A$ if $q_i \geq \bar{q}_i^s$ and $m_i = B$ otherwise.[10] We can now characterize the pattern of advice across all customers for any given compensation (f_n, b_n).[11]

Lemma 5 *When customers follow his recommendation, the advisor's optimal recommendation is characterized as follows: For any given integers (i,s) with $0 \leq s < i \leq I$,*

$$m_i = \begin{cases} A & \text{if } q_i \in [\bar{q}_i^s, 1], \\ B & \text{if } q_i \in [0, \bar{q}_i^s]. \end{cases}$$

We next relate two consecutive cutoffs as follows:

Lemma 6 *For any given integers (i,s) with $0 \leq s < i \leq I-1$, if $(\bar{q}_{i+1}^s, \bar{q}_{i+1}^{s+1}) \in (0,1)^2$ holds true, the advice cutoff $\bar{q}_i^s$ satisfies*

$$\bar{q}_i^s \;=\; \bar{q}_{i+1}^{s+1} + \int_{\bar{q}_{i+1}^{s+1}}^{\bar{q}_{i+1}^{s}} G(q)dq. \quad (12)$$

Proof. Taking two integers (i,s) with $0 \leq s < i \leq I-1$ as given, we show that the advice cutoff $\bar{q}_i^s$ defined by (11) reduces to (12) if both $\bar{q}_{i+1}^s$ and $\bar{q}_{i+1}^{s+1}$ lie in the open interval $(0,1)$. Suppose thus that $(\bar{q}_{i+1}^s, \bar{q}_{i+1}^{s+1}) \in (0,1)^2$. Then, we can rewrite $Z(\bar{q}_{i+1}^{s+1}) - Z(\bar{q}_i^{s+1})$ as

$$\mathbf{1}_{\mathbb{N}}(s)b_A^s - \mathbf{1}_{\mathbb{N}}(s-1)b_A^{s-1} + Z(\bar{q}_{i+2}^{s+2}) - Z(\bar{q}_{i+2}^{s+1}) + 2w\int_{\bar{q}_{i+1}^{s+1}}^{\bar{q}_{i+1}^{s}} G(q)dq,$$

where we use equation (11). With this, the advice cutoff $\bar{q}_i^s$ can be written as

$$\begin{aligned} & \frac{1}{2} - \frac{f_A - f_B + \mathbf{1}_{\mathbb{N}}(s)b_A^s - \mathbf{1}_{\mathbb{N}}(i-1-s)b_B^{i-1-s} + Z(\bar{q}_{i+2}^{s+2}) - Z(\bar{q}_{i+2}^{s+1})}{2w} + \int_{\bar{q}_{i+1}^{s+1}}^{\bar{q}_{i+1}^{s}} G(q)dq \\ = \;& \bar{q}_{i+1}^{s+1} + \int_{\bar{q}_{i+1}^{s+1}}^{\bar{q}_{i+1}^{s}} G(q)dq, \end{aligned}$$

where the first two terms reduce to $\bar{q}_{i+1}^{s+1}$, which yields equation (12). **Q.E.D.**

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[10] To deal with corner solutions, we redefine $\bar{q}_i^s = 0$ if $w \leq f_A - f_B + \mathbf{1}_{\mathbb{N}}(s)b_A^s - \mathbf{1}_{\mathbb{N}}(i-1-s)b_B^{i-1-s} + Z(\bar{q}_{i+1}^{s+1}) - Z(\bar{q}_{i+1}^{s})$ and $\bar{q}_i^s = 1$ if $w \leq -(f_A - f_B + \mathbf{1}_{\mathbb{N}}(s)b_A^s - \mathbf{1}_{\mathbb{N}}(i-1-s)b_B^{i-1-s} + Z(\bar{q}_{i+1}^{s+1}) - Z(\bar{q}_{i+1}^{s}))$.

[11] In what follows, it will be inconsequential how we resolve cases of indifference as these are zero-probability events.

Firm Profits. For any given $s \in S$ and compensation $(f_n, (b_n^s)_{s=1}^{I-1})$ for $n = A, B$, we denote by $\mathrm{Pr}_n(s)$ the probability that the advisor makes s-unit sales of product n. Following the pattern of advice given by Lemma 1, the probability $\mathrm{Pr}_n(s)$ is essentially expressed as a function consisting of advice cutoffs $\bar{q}_i^s$ for all integers (i, s) with $0 \leq s < i \leq I$. Formally, we can define those probabilities as follows.

Taking $s = 1, \ldots, I$ as given, consider any given sequence of integers $(i_0, \ldots, i_s)$ with $0 \leq i_0 < \cdots < i_s \leq I$. Let

$$\tilde{G}(i_k, i_{k+1}) = \begin{cases} \Pi_{i=i_k+1}^{i_{k+1}-1} G(\bar{q}_i^k), & \text{if } i_k + 1 < i_{k+1}, \\ 1, & \text{if } i_k + 1 = i_{k+1}, \end{cases}$$

for any given integer $k = 0, \ldots, s-1$. Then we define the recursive function

$$H_{k+1}(i_{k+1}) = \left(1 - G(\bar{q}_{i_{k+1}}^k)\right) \left[\sum_{i_k=k}^{i_{k+1}-1} H_k(i_k) \tilde{G}(i_k, i_{k+1}) \right],$$

where $i_k + 1 \leq i_{k+1} \leq i_{k+2} - 1$ if $k \neq s-1$ and $i_k + 1 \leq i_{k+1} \leq I$ if $k = s-1$. We set an initial value of the function as $H_0(i_0) = 1$ for any given $i_0 \in \{0, \ldots, i_1 - 1\}$ with $1 \leq i_1 \leq I - (s-1)$. Now, with function $H_k(i_k)$ as $k = s$, the probability that the advisor sells s units of product A (and $(I - s)$ units of product B) to all I customers can be expressed as

$$\mathrm{Pr}_A(s) = \sum_{i_s=s}^{I} H_s(i_s) \tilde{G}(i_s, I+1),$$

which must be equal to $\mathrm{Pr}_B(I - s)$ by definition. The probability that the advisor sells s units of product B for I customers is also defined in the same way as $\mathrm{Pr}_B(s) = \mathrm{Pr}_A(I - s)$. We denote by

$$\mathrm{S}_n = \sum_{s=1}^{I} s \mathrm{Pr}_n(s)$$

the expected total sales (volume) for firm n. For given compensation (f_n, b_n) and product price p_n, for $n = A, B$, expected profits are written as

$$\pi_n = \mathrm{S}_n(p_n - c_n - f_n) - \sum_{s=2}^{I} \mathrm{Pr}_n(s) b_n^{s-1}, \tag{13}$$

where $I \geq 2$ is the total number of customers.

Main Result. We can extend the optimality of nonlinear incentives as follows:

Proposition 2 *Suppose that there are any given finite number of customers. Nonlinear incentives are part of any equilibrium, i.e., there is no equilibrium in which* $b_n^1 = \cdots = b_n^{I-1} = b = 0$.

We proceed as in the main text. Suppose thus that firms set their linear incentives with $b_n^{\tilde{s}-1} = b = 0$ for any given sales-unit $\tilde{s} = 2, \ldots, I$. In this case, advice cutoffs $\bar{q}_i^s$ for all (i, s) with $0 \leq s < i \leq I$ should all be equal, which we denote by $\bar{q} \in (0, 1)$.[12]

Lemma 7 *Suppose that $\bar{q}_i^s = \bar{q} \in (0,1)$ holds true for any given integers (i, s) with $0 \leq s < i \leq I$. Then,*

$$\frac{\partial \bar{q}_i^s}{\partial f_n} = \begin{cases} -\frac{1}{2w} & \text{if } n = A, \\ \frac{1}{2w} & \text{if } n = B, \end{cases}$$

$$\frac{\partial \bar{q}_i^s}{\partial b_A^{I-1}} = \begin{cases} 0 & \text{if } s \neq i-1, \\ -\frac{1}{2w}\left(1 - G(\bar{q})\right)^{I-i} & \text{if } s = i-1, \end{cases}$$

and

$$\frac{\partial \bar{q}_i^s}{\partial b_B^{I-1}} = \begin{cases} 0 & \text{if } s \neq 0, \\ \frac{1}{2w}\left(G(\bar{q})\right)^{I-i} & \text{if } s = 0. \end{cases}$$

Proof. Since $\bar{q}_i^s$ are all equal for any given integers (i, s) with $0 \leq s < i \leq I$ and it lies in the open interval $(0, 1)$, we can apply Lemma 6. Differentiating equation (12) with resepct to instrument $x_n \in \{f_n, b_n\}$, evaluated at $\bar{q}_i^s = \bar{q} \in (0, 1)$, yields

$$\begin{aligned} \frac{\partial \bar{q}_i^s}{\partial x_n} &= G(\bar{q})\frac{\partial \bar{q}_{i+1}^s}{\partial x_n} + \left(1 - G(\bar{q})\right)\frac{\partial \bar{q}_{i+1}^{s+1}}{\partial x_n} \\ &= \sum_{l=0}^{I-i} \binom{I-i}{l} \left(G(\bar{q})\right)^{I-i-l}\left(1 - G(\bar{q})\right)^l \frac{\partial \bar{q}_I^{s+l}}{\partial x_n}, \end{aligned} \tag{14}$$

where $0 \leq s < i \leq I - 1$.

We now focus on firm A. The same argumet applies to firm B, too. Applying both

$$\frac{\partial \bar{q}_I^s}{\partial f_A} = -\frac{1}{2w} \quad (\forall s = 0, \ldots, I-1) \quad \text{and} \quad \frac{\partial \bar{q}_I^s}{\partial b_A^{I-1}} = \begin{cases} 0 & \text{if } s \neq I-1, \\ -\frac{1}{2w} & \text{if } s = I-1 \end{cases}$$

to equation (14) yields both $\partial \bar{q}_i^s / \partial f_A$ and $\partial \bar{q}_i^s / \partial b_A^{I-1}$. **Q.E.D.**

Next consider marginal profits, evaluated at $b_n^{s-1} = b = 0$ for any given $s = 2, \ldots, I$:

$$\frac{\partial \pi_n}{\partial f_n} = \mathrm{S}_n^f(p_n - c_n - f_n) - \mathrm{S}_n \tag{15}$$

and

$$\frac{\partial \pi_n}{\partial b_n^{s-1}} = \mathrm{S}_n^{b^{s-1}}(p_n - c_n - f_n) - \mathrm{Pr}_n(s), \tag{16}$$

where we denote by $\mathrm{Pr}_n^x(s)$ the respective partial derivative of the s-unit sale $\mathrm{Pr}_n(s)$ with respect to $x_n \in \{f_n, b_n^{s-1}\}$, and by $\mathrm{S}_n^x = \sum_{s=1}^{I} s\mathrm{Pr}_n^x(s)$ the partial derivative of total sales S_n.

[12] The advice cutoff $\bar{q}$ should lie in the open interval $(0, 1)$, as otherwise the advisor would always recommend a particular firm's product to customers, which contradicts the assumption that advice is essential.

Lemma 8 *For any given $I \geq 2$, suppose that advice cutoffs $\bar{q}_i^s \in (0,1)$ are all equal for any given integers (i,s) with $0 \leq s < i \leq I$. Then,*

$$S_n^{b^{I-1}} = \begin{cases} (1-G(\bar{q}))^{I-1} S_A^f & \text{if } n = A, \\ (G(\bar{q}))^{I-1} S_B^f & \text{if } n = B. \end{cases} \tag{17}$$

Proof. Consider firm A. We show that (17) holds for $n = A$. The same argument applies to equation (17) for $n = B$. We first focus on the respective partial derivatives $\Pr_A^f(s)$ and $\Pr_A^{b^{I-1}}(s)$, evaluated at $\bar{q} = \bar{q}_i^s \in (0,1)$ for any given integers (i,s) with $0 \leq s < i \leq I$. Note that the probability of the s-unit sales at $\bar{q}_i^s = \bar{q}$ for any given $s = 1, \ldots, I$ is simply written as

$$\Pr_A(s) = \binom{I}{s} (1-G(\bar{q}))^s (G(\bar{q}))^{I-s},$$

where $\binom{I}{s}$ indicates a binomial coefficient. Consider the case of $s = I$ to begin with. In this case,

$$\Pr_A^f(I) = \sum_{s=1}^{I} (1-G(\bar{q}))^{I-1} \left(-g(\bar{q}) \frac{\partial \bar{q}_s^{s-1}}{\partial f_A}\right),$$

and similarly

$$\Pr_A^{b^{I-1}}(I) = \sum_{s=1}^{I} (1-G(\bar{q}))^{I-1} \left(-g(\bar{q}) \frac{\partial \bar{q}_s^{s-1}}{\partial b_A^{I-1}}\right).$$

Next we proceed to the case of $s = 1, \ldots, I-1$. When taking $s = 1, \ldots, I-1$ as given, using the fact that $\bar{q} = \bar{q}_i^s$ holds for any given integers (i,s) with $0 \leq s < i \leq I$, we are able to explicitly write

$$\begin{aligned} \Pr_A^f(s) = & \sum_{k=1}^{I-1} \sum_{l=1}^{\min\{s,k\}} \binom{k-1}{l-1} (1-G(\bar{q}))^{l-1} (G(\bar{q}))^{k-l} \left(g(\bar{q}) \frac{\partial \bar{q}_k^{l-1}}{\partial f_A}\right) \\ & (1-G(\bar{q}))^{s-l} (G(\bar{q}))^{I-k-(s-l)-1} \left(\binom{I-k}{s-l+1}(1-G(\bar{q})) - \binom{I-k}{s-l} G(\bar{q})\right) \\ & + \binom{I-1}{s-1} (1-G(\bar{q}))^{s-1} (G(\bar{q}))^{I-s} \left(-g(\bar{q}) \frac{\partial \bar{q}_I^{s-1}}{\partial f_A}\right) \\ & + \sum_{k=s+1}^{I} \binom{k-1}{s} (1-G(\bar{q}))^{s} (G(\bar{q}))^{k-1-s} \left(g(\bar{q}) \frac{\partial \bar{q}_k^{s}}{\partial f_A}\right) (G(\bar{q}))^{I-k}, \end{aligned}$$

where $s < I$. Using both $\partial \bar{q}_i^s / \partial f_A$ and $\partial \bar{q}_i^s / \partial b_A^{I-1}$ derived in Lemma 7, as with $\Pr_A^f(s)$, we can also write

$$\begin{aligned} \Pr_A^{b^{I-1}}(s) = & \sum_{k=1}^{s} (1-G(\bar{q}))^{k-1} \left(g(\bar{q}) \frac{\partial \bar{q}_k^{k-1}}{\partial b_A^{I-1}}\right) \\ & (1-G(\bar{q}))^{s-k} (G(\bar{q}))^{I-s-1} \left(\binom{I-k}{s-k+1}(1-G(\bar{q})) - \binom{I-k}{s-k} G(\bar{q})\right) \\ & + (1-G(\bar{q}))^{s} \left(g(\bar{q}) \frac{\partial \bar{q}_{s+1}^{s}}{\partial b_A^{I-1}}\right) (G(\bar{q}))^{I-s-1}. \end{aligned}$$

Taking the sums of both $\Pr_A^f(s)$ and $\Pr_A^{b^{I-1}}(s)$ over $s(=1,\ldots,I)$ leads to

$$\begin{aligned}
\sum_{s=1}^{I} s\Pr_A^f(s) &= \sum_{s=1}^{I}\left(\sum_{k=1}^{s}\binom{s-1}{k-1}(1-G(\bar q))^{k-1}(G(\bar q))^{s-k}\left(-g(\bar q)\frac{\partial \bar q_s^{k-1}}{\partial f_A}\right)\right)\\
&\qquad\left(\underbrace{\sum_{l=1}^{I-s+1}\binom{I-s}{l-1}(1-G(\bar q))^{l-1}(G(\bar q))^{I-s-(l-1)}}_{=(1-G(\bar q)+G(\bar q))^{I-s}=1}\right)\\
&= \left(\frac{g(\bar q)}{2w}\right)\sum_{s=1}^{I}\left(\underbrace{\sum_{k=1}^{s}\binom{s-1}{k-1}(1-G(\bar q))^{k-1}(G(\bar q))^{s-k}}_{=(1-G(\bar q)+G(\bar q))^{s-1}=1}\right) = I\left(\frac{g(\bar q)}{2w}\right),
\end{aligned}$$

where the second equality follows from Lemma 7, and finally to

$$\begin{aligned}
\sum_{s=1}^{I} s\Pr_A^{b^{I-1}}(s) &= \sum_{s=1}^{I}(1-G(\bar q))^{s-1}\left(-g(\bar q)\frac{\partial \bar q_s^{s-1}}{\partial b_A^{I-1}}\right)\\
&\qquad\left(\underbrace{\sum_{l=1}^{I-s+1}\binom{I-s}{l-1}(1-G(\bar q))^{l-1}(G(\bar q))^{I-s-(l-1)}}_{=(1-G(\bar q)+G(\bar q))^{I-s}=1}\right)\\
&= \left(\frac{g(\bar q)}{2w}\right)\sum_{s=1}^{I}(1-G(\bar q))^{s-1}(1-G(\bar q))^{I-s}\\
&= I\left(\frac{g(\bar q)}{2w}\right)(1-G(\bar q))^{I-1},
\end{aligned}$$

where the second equality follows again from Lemma 7. Taken together, we can thus derive equation (17) for $n = A$. **Q.E.D.**

With Lemma 8, we can show the optimality of nonlinear incentives for any given finite number of customers.

Proof of Proposition 2. Suppose that $b_n^{\tilde s-1} = b = 0$ for any given sales unit $\tilde s \in \{2,\ldots,I\}$. We now focus attention on firm A to show that it is profitable for firm A to set a bonus $b_A^{I-1} > 0$. Analogously, we can show that the same is true for firm B.

The marginal profits with respect to f_A and b_A^{I-1}, evaluated at $b_n^{\tilde s-1} = b = 0$, are given by (15) and (16) for $n = A$ and $s = I$. We show that the value of (16) is positive if commission f_A is optimal with setting (15) equal to zero. Since all the advice cutoffs are the same as $\bar{q}_i^s = \bar{q} \in (0,1)$ for any given integers (i,s) with $0 \le s < i \le I$, we can derive the simple relationships:

$$\begin{aligned}
\mathrm{S}_A &= \sum_{s=1}^{I} s\mathrm{Pr}_A(s) \\
&= \sum_{s=1}^{I} s\binom{I}{s}(1-G(\bar{q}))^s(G(\bar{q}))^{I-s} \\
&= I(1-G(\bar{q}))\left(\sum_{s=1}^{I}\binom{I-1}{s-1}(1-G(\bar{q}))^{s-1}(G(\bar{q}))^{I-1-(s-1)}\right) \\
&= I(1-G(\bar{q}))\left(\underbrace{\sum_{s=0}^{I-1}\binom{I-1}{s}(1-G(\bar{q}))^{s}(G(\bar{q}))^{I-1-s}}_{=(G(\bar{q})+(1-G(\bar{q}))^{I-1}=1}\right) = I(1-G(\bar{q})),
\end{aligned}$$

and

$$\mathrm{Pr}_A(I) = (1-G(\bar{q}))^I,$$

which leads to

$$\mathrm{Pr}_A(I) = \left(\frac{(1-G(\bar{q}))^{I-1}}{I}\right)\mathrm{S}_A.$$

From (15), we have

$$p_A - c_A - f_A = \frac{\mathrm{S}_A}{\mathrm{S}_A^f},$$

where $\mathrm{S}_A^f > 0$. Together with the equation for $\mathrm{Pr}_A(I)$ derived above, this reduces (16) to

$$\left(\frac{\mathrm{S}_A^{b^{I-1}}}{\mathrm{S}_A^f} - \frac{(1-G(\bar{q}))^{I-1}}{I}\right)\mathrm{S}_A = (1-G(\bar{q}))^{I-1}\left(\frac{I-1}{I}\right)\mathrm{S}_A > 0,$$

where the equality follows from (17) for $n = A$. Thus, we can show that the marginal profit with respect to b_A^{I-1} is positive. **Q.E.D.**

4 Backfiring of Regulation with Asymmetric Costs

To examine the impact of nonlinear incentives on advice, we have focused on the case where firms are equally cost efficient. With regulation under which firms are forced to set linear compensation (see Section 5), suitability and welfare are always maximized with unbiased advice. This is, however, no longer true when one firm is more cost efficient than the other. Without loss of generality, we now assume that $c_A < c_B$, and let $\Delta_c = c_B - c_A > 0$. For any given advice cutoffs $(\bar{q}_1, \bar{q}_2^A, \bar{q}_2^B) \in [0,1]^3$, the gross utility is written as

$$U = (1/2)\left(\mathbb{E}[v \mid \bar{q}_1] + G(\bar{q}_1)\mathbb{E}[v \mid \bar{q}_2^B] + (1 - G(\bar{q}_1))\,\mathbb{E}[v \mid \bar{q}_2^A]\right)$$

where

$$\mathbb{E}[v \mid \bar{q}] = \int_0^{\bar{q}} v_B(q)g(q)dq + \int_{\bar{q}}^1 v_A(q)g(q)dq,$$

which simplifies to $v_h + \Delta_v(G(\bar{q})(1-2\bar{q}) + \int_0^{\bar{q}} G(q)dq - \int_{\bar{q}}^1 G(q)dq)$, where $\Delta_v = v_h - v_l > 0$. The gross utility is maximized at $\bar{q}_1 = \bar{q}_2^A = \bar{q}_2^B = 1/2$ with unbiased advice. Also, the expected production cost is written as

$$C = c_A + \frac{1}{2}\left(G(\bar{q}_1)\left(1 + G(\bar{q}_2^B) - G(\bar{q}_2^A)\right) + G(\bar{q}_2^A)\right)\Delta_c,$$

which strictly increases proportional to the size of any given advice cutoff. As liability represents a transfer, welfare equals $W = U - C$, which is strictly maximized when

$$\bar{q}_1 = \bar{q}_2^0 = \bar{q}_2^1 = q^{FB} := \frac{1}{2} - \frac{\Delta_c}{2\Delta_v}.$$

Here, the term $\frac{\Delta_c}{2\Delta_v}$ captures the "bias" towards product A that is necessary to achieve efficiency. In what follows, we illustrate when the imposition of linear compensation leads to a reduction in welfare. For this we proceed in two steps. We first provide a general example, where we consider a particular range of w. We then no longer apply this restriction and consider instead a particular functional specification of the distribution of q_i.

Illustration 1. The intuition for the subsequent illustration is the following. Suppose that as w is sufficiently high and as regulation dampens the cost-effectiviness of incentives, in a regulated equilibrium no firm provides positive incentives. In this case, with $\bar{q}_1 = \bar{q}_2^A = \bar{q}_2^B = 1/2$, advice is unbiased from customers' perspective, but the market share of the less cost-efficient firm is clearly too high. Suppose now that without regulation, for the particular range of w, only the more cost-efficient firm would provide sales incentives (through a positive bonus). We show below that in this case, the resulting shift in welfare is always strictly positive.

Proposition 3 *Suppose that $0 < c_A < c_B$. There exists a threshold for the liability*

$$\bar{w} = 2g(1/2)\left(2\int_{1/2}^1 v_A(q)g(q)dq - c_A\right),$$

so that when w is sufficiently close to $\bar{w}$, but below it, a regulation that forces firms to use only linear incentives, backfires and strictly reduces welfare, even though the resulting advice becomes less biased from customers' perspective.

Now we comment more formally on the respective construction. In Proposition 3, we define the threshold $\bar{w}$ such that without regulation it is optimal for both firms to set $f_n = b_n = 0$ if $w \geq \bar{w}$. Precisely, to define the threshold, consider the marginal profit for firm $n(= A, B)$ with respect to instrument $x_n \in \{f_n, b_n\}$, evaluated at $\bar{q}_1 = \bar{q}_2^A = \bar{q}_2^B = 1/2$, which is given by $\mathrm{S}_n^f(p_n - c_n) - \mathrm{S}_n$ for $x_n = f_n$ and by $\mathrm{S}_n^b(p_n - c_n) - \mathrm{Pr}_n(2)$ for $x_n = b_n$, where advice cutoffs are all equal, $S_n^f = 2S_n^b = g(1/2)/w$, and prices are (cf. Section 4 in the main text)[13]

$$p_n = p = 2\int_{1/2}^{1} v_A(q)G(q)dq = 2\int_{0}^{1/2} v_B(q)G(q)dq.$$

We know that the product margin is higher for firm A than for firm B and also that the bonus is more cost effective than the commission due to the non-optimality of linear incentives (Proposition 1 in the main text). So, the lowest value of w, such that neither firm sets any instrument in equilibrium, is defined when the marginal profit for firm A with respect to the bonus is equal to zero at $\bar{q}_1 = \bar{q}_2^A = \bar{q}_2^B = 1/2$, which leads to threshold $\bar{w}$ defined in Proposition 3. With regulation, we can similarly define the threshold of w^* such that neither firm sets any commission denoted by f_n^R in equilibrium if $w \geq w^*$, which is exactly half of the threshold $\bar{w}$, and therefore $f_n^R = 0$ under regulation if w is sufficiently close to $\bar{w}$. Precisely, by the first-order-condition with respect to f_A, evaluated at $f_n^R = 0$ (as well as $b_n = 0$), the threshold of w^* is given by

$$w^* = g(1/2)\left(2\int_{1/2}^{1} v_A(q)g(q)dq - c_A\right),$$

where we use $S_n^f = 2S_n^b$.

With these facts, consider now that w is sufficiently close to $\bar{w}$ but below it. Then, only firm A is willing to set $b_A > 0$ alone, which must be sufficiently small but positive in an unregulated equilibrium, whereas with regulation neither firm sets any compensation as $w > w^*$. In this case, the advice cutoffs without regulation would satisfy the inequalities $q^{FB} < \bar{q}_2^A < \bar{q}_1 < 1/2 = \bar{q}_2^B$ while the advice cutoff $\bar{q}$ commonly applied to both customers with regulation is equal to $\bar{q} = 1/2$. This implies that welfare should be lower with regulation than without.

Illustration 2. For the second illustration, we consider a uniform distribution as in Example 1 in the main text, while no longer imposing a restriction on w as in Illustration 1. It is now further instructive to illustrate the shift in welfare as follows. As with U and C, denote by U^R and C^R the respective gross utility and expected production cost with

[13] We suppose here that profits are strictly concave in the two instruments.

regulation. Define

$$\begin{aligned} \Delta U &= U^R - U \\ &= \frac{1}{2}\left[H(\bar{q}) - H(\bar{q}_1) + G(\bar{q}_1)\left(H(\bar{q}) - H(\bar{q}_2^B)\right) + (1 - G(\bar{q}_1))\left(H(\bar{q}) - H(\bar{q}_2^A)\right)\right]\Delta_v, \end{aligned}$$

where

$$H(\tilde{q}) = G(\tilde{q})(1 - 2\tilde{q}) + \int_0^{\tilde{q}} G(q)dq - \int_{\tilde{q}}^1 G(q)dq$$

for any given $\tilde{q} \in [0, 1]$,[14] and

$$\begin{aligned} \Delta C &= C^R - C \\ &= \frac{1}{2}\left[G(\bar{q}) - G(\bar{q}_1)\left(1 - G(\bar{q}_2^A) + G(\bar{q}_2^B)\right) + \left(G(\bar{q}) - G(\bar{q}_2^A)\right)\right]\Delta_c. \end{aligned}$$

Hence, we can decompose $\Delta W = W^R - W = \Delta U - \Delta C$, and now consider these changes separately in the subsequent figures. There, we fix the parameters $(c_A, c_B, v_h, v_l) = (0.55, 0.65, 1, 0)$, so that $\Delta_c = 0.1$, $\Delta_v = 1$, $q^{FB} = 0.45$, and $\bar{w} = 2w^* = 0.4$. The top panel in Figure 3 depicts how compensation without regulation (f_n, b_n) and compensation with regulation f_n^R change in w. The middle panel depicts the respective equilibrium advice cutoffs $(\bar{q}_1, \bar{q}_2^A, \bar{q}_2^B)$ without regulation and the cutoff $\bar{q}$ with regulation. Finally, the bottom panel depicts the differences of $(\Delta U, \Delta C, \Delta W)$.

Consider low values of w where both $f_n > 0$ and $f_n^R > 0$ in the first place. Interestingly, an increase in w then has only a small effect on advice cutoffs,[15] while the imposition of regulation has a large effect. In addition, regulation has a positive effect as it reduces the bias of advice and as it increases cost-efficiency. The latter effect reverses and dominates when considering high values of w, which is only the case where regulation again has a negative effect on total welfare.

[14] This function is easily shown to be a bell-shaped function with the peak when $\tilde{q} = 1/2$, that is, $H(\tilde{q})$ is increasing in $\tilde{q}$ if $\tilde{q} \in (0, 1/2)$ while decreasing if $\tilde{q} \in (1/2, 1)$.

[15] Recall that with symmetric costs, the effect was zero.

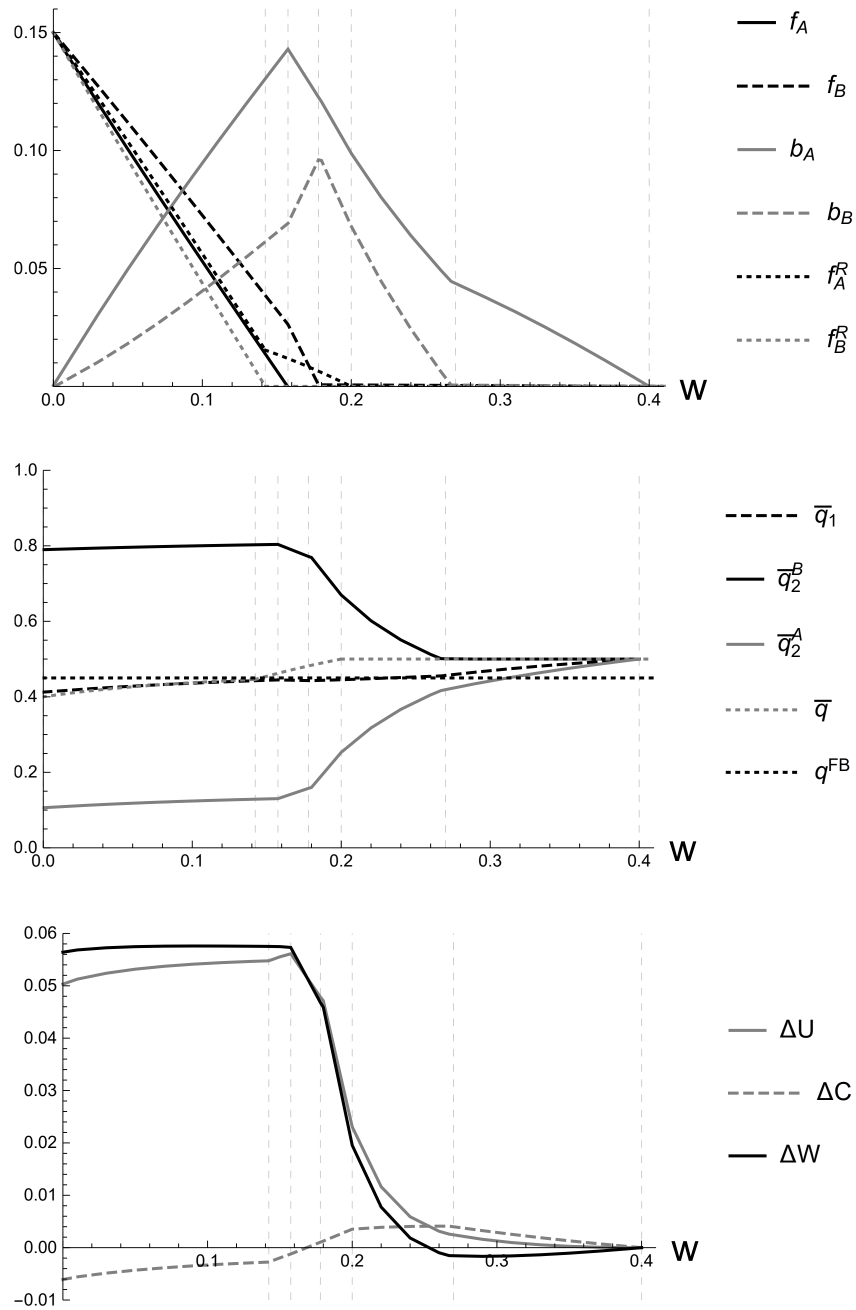

Figure 3: (Top panel) Compensation with and without regulation; (Middle panel) The associated equilibrium advice cutoffs; (Bottom panel) The differences in suitability of advice, production cost, and welfare, provided that G is a uniform distribution and $(c_A, c_B, v_h, v_l) = (0.55, 0.65, 1, 0)$.