A Spectral Estimation of Tempered Stable Stochastic Volatility Models and Option Pricing

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Abstract

A characteristic function-based method is proposed to estimate the time-changed Lévy models, which take into account both stochastic volatility and infinite-activity jumps. The method facilitates computation and overcomes problems related to the discretization error and to the non-tractable probability density. Estimation results and option pricing performance indicate that the infinite-activity model performs better than the finite-activity one. By introducing a jump component in the volatility process, a double-jump model is also investigated.

Keywords: Empirical characteristic function; Stochastic volatility; Infinite-activity jumps; Option pricing; Continuous GMM.

1. Introduction

Stochastic volatility and jumps in the stock price process are well documented. On the one hand, they are inherent components of the stock price dynamics (Bollerslev et al., 1994; Merton, 1976; Cont and Mancini, 2007; Ait-Sahalia and Jacod, 2009). On the other hand, they play important roles in explanation of distributional characteristics of returns and of the implied volatility smile/skew of options. Much work has been done on jump-diffusion stochastic volatility models (Bakshi et al., 1997; Bates, 1996, 2000; Pan, 2002; Andersen et al., 2002). Even though these models perform acceptably in fitting the stock price process and in pricing options, they feature a counter-factual assumption that jumps are rare events. By observing time-series evolution of stock prices, we find that the stock price process is accompanied not only by large jumps, but also by a lot of small jumps. Jumps are empirically not rare events. Based on this observation, alternative models are

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being developed. These models use infinite-activity Lévy processes to capture both large jumps and small jumps in the stock price dynamics (Madan et al., 1998; Carr et al., 2002; Carr and Wu, 2003).

In this paper, we introduce the tempered stable process, based on which a flexible option pricing model is built. Depending on different values of its stable index, the tempered stable process can be an infinite-activity process that can generate an infinite number of (large and small) jumps within any finite time interval, or be a finite-activity process that generates only a finite number of large jumps. Therefore, this process unifies the jump-diffusion model and the infinite activity model into one framework with a rich structure. Financial applications of the tempered stable process have also been investigated by Carr et al. (2002) and Wu (2006).

A major difficulty in continuous-time financial modeling is the lack of efficient tools for estimating and making inference with discretely observed samples, especially when models have latent factors and jumps. This is particularly striking for the tempered stable model studied in this paper. The maximum likelihood estimation is usually inapplicable because the transition density is rarely in closed-form, the frequently used simulation-based methods are difficult to implement since the model is hard to simulate, and the traditional GMM is computationally demanding since high order derivatives need to be calculated. Fortunately, for most of the Lévy models, the analytical characteristic functions are obtainable. The characteristic function is equivalent to and contains the same information as the probability density function, and we can thus directly use it for estimation.

We apply the time-change approach to introduce stochastic volatility in our models (Clark, 1973; Carr et al., 2003; Carr and Wu, 2004). Given a nonnegative right continuous with left limit stochastic process v_t, we define a stopping time as

$$T_t = \int_0^t v_{s-} ds, \tag{1}$$

which is finite almost surely. Intuitively, we could think of t as calender time and T_t as business time. The variable v_t reflects the intensity of economic activity. Stochastic volatility is generated by replacing calender time t with business time T_t. For a stochastic process $X(t)$, its time-changed counterpart is defined by $X_{T_t} = X(T_t)$. If we assume independence between v_t and $X(t)$, through

iterated expectation we can obtain the conditional characteristic function of X_{T_t}

$$
\begin{aligned}
\phi_X(u,\tau) &= E[e^{iuX_{T_\tau}}|\mathcal{F}_t] = E[E[e^{iuX_{T_\tau}}|T_\tau]|\mathcal{F}_t] \\
&= E[e^{-cT_\tau}|\mathcal{F}_t] = E[e^{-c\int_t^{t+\tau} v_s ds}|\mathcal{F}_t],
\end{aligned}
\tag{2}
$$

and the joint conditional characteristic function of X_{T_τ} and $v_{t+\tau}$

$$
\begin{aligned}
\phi_{X,v}(u_1,u_2,\tau) &= E[e^{iu_1 X_{T_\tau}+iu_2 v_{t+\tau}}|\mathcal{F}_t] = E[e^{-cT_\tau}e^{iu_2 v_{t+\tau}}|\mathcal{F}_t] \\
&= E[e^{-c\int_t^{t+\tau} v_s ds}e^{iu_2 v_{t+\tau}}|\mathcal{F}_t],
\end{aligned}
\tag{3}
$$

where, as we shall see in the text, c is related to the characteristic function of $X(t)$.

Since our models contain the latent volatility factor v_t, we propose a characteristic function-based implied-state method to jointly estimate models by using information contained in both stock and options markets. Given parameters, we can back out the unobserved volatility from options, and with these backed-out volatility, we implement a characteristic function-based GMM. With this method, we not only obtain the consistent estimates of model parameters, but also identify market prices of risks as well as filter out a sequence of state variables which should be the best proxy for the true ones (Pastorello et al., 2003).

Estimation using the characteristic function has been investigated since 1970's (Feuerverger and Mureika, 1977; Feuerverger and McDunnough, 1981a, 1981b; Feuerverger, 1990). The method is recently redeveloped for estimation of continuous-time financial models by Singleton (2001), Jiang and Knight (2002) and Chacko and Viceira (2003). Carrasco et al. (2007) extend the method by using a continuum of moment conditions, which makes this method obtain the MLE efficiency. This approach is very useful for estimating models that do not contain unobserved state variables such as stochastic volatility. We overcome the problem of non-observability of stochastic volatility jointly using stock prices and options. Pan (2002) advocates an implied-state GMM to focus directly on joint dynamics of stock returns and near-the-money short maturity options. Pastorello et al. (2003) propose a general iterative and recursive method on estimating structural nonadaptive models. Our method is similar to those of Pan (2002) and Pastorello et al. (2003) except that we directly use the characteristic function. Direct use of the characteristic function is not trivial. It makes estimation

of many Lévy models feasible and avoids model discretization errors since characteristic functions of these models are exact.

Monte Carlo and empirical studies show that the method is computationally less costly than the other methods and can be easily adapted to different Lévy models. Estimation results and option pricing performance indicate that the infinite-activity model performs better than the finite-activity one, which results in higher mean of moments and option pricing errors. We also provide an extension to investigate the double-jump model by introducing a jump component in the volatility process. Our empirical study points to an only marginal improvement of this double-jump model.

The remainder of this paper is organized as follows. Section 2 builds a time-changed Lévy model. Section 3 describes the characteristic function-based estimation method and provides a Monte Carlo study. Section 4 discusses empirical results and evaluates models. Section 5 extends the model by introducing a jump component in the volatility process. Lastly, section 6 concludes the paper. Proofs are provided in the appendix.

2. A Time-changed Lévy model

2.1. Risk-neutral stock price and volatility dynamics

Under a given probability space $(\Omega, \mathcal{F}, Q)$ and the complete filtration $\{\mathcal{F}_t\}_{t \geq 0}$, we introduce the tempered stable process X_t, which is a Lévy process on R with the Lévy density defined as:

$$V(x) = c \frac{e^{-\lambda_+ x}}{x^{1+\alpha}} 1_{x>0} + c \frac{e^{-\lambda_- |x|}}{|x|^{1+\alpha}} 1_{x<0}, \tag{4}$$

where $c > 0$ and $\lambda_+, \lambda_- > 0$. To guarantee the finite quadratic variation, the stable index α should be less than 2. The Lévy density $V(x)$ measures the arrival rate of jumps with size x defined on R^0 (real line without zero). Its characteristic function has the form:

$$
\begin{aligned}
\phi_X(u) &= E[e^{iuX(t)}] = e^{-t\psi_X(u)}, \\
\psi_X(u) &= -c\Gamma(-\alpha)\Big[(\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha\Big],
\end{aligned}
\tag{5}
$$

where $\alpha \neq 0, 1$, $\psi(u)$ is called the characteristic exponent, $u \subseteq R$ is the characteristic index, $\Gamma(\cdot)$ is a gamma function, and when $\alpha = 0$ or 1, the characteristic exponent has a different form (Carr et

al., 2002; Wu, 2006; Cont and Tankov, 2004).

The parameters in the Lévy density (4) play different roles: c measures the overall and relative frequency of jumps; λ_+ and λ_- govern how fast the tails decay and lead to a skewed distribution when they are not the same; and the stable index α governs how the process evolves between big jumps. Specifically, if $\alpha < 0$, the tempered stable process becomes a compound poisson type finite activity process, while if $\alpha \geq 0$, it is an infinite activity process (in particular, when $\alpha = 0$, the tempered stable process becomes the well-know variance gamma process.).

The stock price process under the risk-neutral measure Q is modeled by an exponential time-changed Brownian motion and tempered stable process:

$$
\begin{aligned}
S_t &= S_0 \exp \left\{ (r-q)t + \left[W_{T_t} - k_W(1)T_t \right] + \left[X_{T_t} - k_X(1)T_t \right] \right\}, &(6) \\
T_t &\equiv \int_0^t v_{s-}ds, &(7)
\end{aligned}
$$

where r is a constant risk-free rate, q the dividend yield, W_t a standard Brownian motion, X_t the tempered stable process, T_t the stochastic business time, v_t is called the variance rate and reflects the intensity of economic activity, and $k_W(1)$ and $k_X(1)$ the convexity adjustments. For any stochastic process Y_t, the convexity adjustment can be derived from its cumulant exponent $k(s)$, which is defined as

$$
k(s) \equiv \frac{1}{t} log(E[e^{sY_t}]) \equiv -\psi_Y(-is), \tag{8}
$$

where $\psi_Y(\cdot)$ is the characteristic exponent of the process Y_t.

The time-change approach is a standard technique to generate stochastic volatility (Clark, 1973; Carr et al., 2003). Randomly changed time can be regarded as business time or trading time. The randomness in business time generates the stochastic volatility. In fact, with the time-changing approach, we introduce not only stochastic volatility, but also stochastic higher moments such as skewness and kurtosis because the higher moments of the jump part depend on the variance rate. The function $t :\mapsto T_t$ should be nonnegative and nondecreasing, requiring that the variance rate v_t be a nonnegative process.

A well-known nonnegative process we can use for v_t is the square-root process of Cox et al. (CIR process, 1985). Under the risk-neutral measure, this process has the following stochastic differential

equation (SDE):

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dZ_t, \tag{9}$$

where if $\kappa > 0$, κ is the rate of mean-reversion, θ the long-run mean of the variance rate, σ a variation parameter, and Z_t a standard Brownian motion.

We allow W_t in (6) and Z_t in (9) to be correlated with the instantaneous correlation $[dW_t dZ_t] = \rho dt$, where $\rho \in [-1, 1]$. This is to accommodate the so-called leverage effect of the diffusion part. The leverage effect of jump is actually inherent in the time-changed model because during a time of high variance rate, business time flows faster and price jumps occur at an increased rate.

Under these specifications, we obtain a model of the stock price process capturing both jumps and stochastic volatility. In the following, we refer to this model as LTS-SV. This general model can flexibly explain the negative skewness and leptokurtosis in the distribution of stock returns. Negative skewness can arise either from the difference in tail parameters of the tempered stable process or from negative correlation between the variance rate and return process. The positive excess kurtosis can arise either from a high jump frequency induced by the tempered stable process or from a volatile variance rate.

Since the return process (6) is correlated with the variance rate process (9), we firstly internalize this correlation with the approach proposed by Carr and Wu (2004) and then derive the conditional characteristic function of log return with the transform approach of Duffie et al. (2000).

PROPOSITION 1: *Define a new filtration $\mathcal{G}_t$ generated by the business time sigma algebra $\mathcal{F}_{T_t}$. The conditional characteristic function of log return $R_{t+\tau} = \ln(S_{t+\tau}/S_t)$ in LTS-SV model with the variance rate process (9) under the risk-neutral measure Q is*

$$
\begin{aligned}
\phi_R(u; \tau, v_t) &\equiv E^Q[e^{iuR_{t+\tau}}|\mathcal{G}_t] \\
&= e^{iu(r-q)\tau + A(u,\tau) + B(u,\tau)v_t}, \tag{10}
\end{aligned}
$$

where

$$
\begin{aligned}
A(u,\tau) &= -\frac{\kappa\theta}{\sigma^2}\left[2\log\left(1 - \frac{(\gamma - \kappa^*)(1 - e^{-\gamma\tau})}{2\gamma}\right) + (\gamma - \kappa^*)\tau\right], \\
B(u,\tau) &= \frac{2[\varphi_W(u) + \varphi_X(u)](1 - e^{-\gamma\tau})}{(\gamma - \kappa^*)(1 - e^{-\gamma\tau}) - 2\gamma},
\end{aligned}
$$

$$\varphi_W(u) = \frac{1}{2}(iu + u^2), \quad \varphi_X(u) = \psi_X(u) + iuk_X(1),$$

$$\gamma = \sqrt{(\kappa^*)^2 + 2\sigma^2[\varphi_W(u) + \varphi_X(u)]},$$

$$\kappa^* = \kappa - iu\rho\sigma.$$

Note that the characteristic function (10) depends on the unobserved variance rate v_t. We also note that the information flow is now modeled by the complete filtration $(\mathcal{G}_t)_{t \geq 0}$ generated by the business time sigma algebra $\mathcal{F}_{T_t}$. With the above characteristic functions, we can use the fast Fourier transform (FFT) to numerically compute option prices if we can observe the variance rate. Option pricing with FFT is proposed by Carr and Madan (1999). Chourdakis (2005) advocates the fractional Fourier transform (FRFT) in pricing options. It is demonstrated that FRFT is more efficient than FFT in the sense of computational precision by careful selection of the integration upper bound and grid sizes of the characteristic index and log strike. In this paper, we apply FRFT to pricing options.

2.2. Market prices of risks and objective joint CCF

By introducing stochastic volatility and jumps to the stock price process, the market is no longer complete. There may exist many equivalent martingale measures that can guarantee absence of arbitrage. This feature may produce extra difficulty and complexity in change of measure since the objective dynamics could be extremely different from the risk-neutral one. We are interested in the structure-preserving change of measure because it preserves tractability and the same structure under both measures.

Under the objective measure P, which is assumed to be absolutely continuous with respect to Q, we propose the following stock price and variance rate dynamics,

$$S_t = S_0 \exp\left\{(r-q)t + \pi_W T_t + \left[k_X^P(1) - k_X(1)\right]T_t \right.$$
$$\left. + \left[W_{T_t}^P - \frac{1}{2}T_t\right] + \left[X_{T_t}^P - k_X^P(1)T_t\right]\right\}, \tag{11}$$

and

$$dv_t = [\kappa(\theta - v_t) + \pi_v v_t]dt + \sigma\sqrt{v_t}dZ_t^P \tag{12}$$

with $T_t = \int_0^t v_{s-} ds$. Define $\kappa^P \equiv \kappa - \pi_v$. In equations (11) and (12), the term $\pi_W T_t$ denotes the risk premium for the diffusion, the term $\pi_X \equiv (k_X^P(1) - k_X(1))T_t$ represents the risk premium for the jump, and $\pi_v v_t$ is the risk premium for volatility.

Under change of measure, W_t^P and Z_t^P are still Brownian motions. To guarantee the absolute continuity between X_t and X_t^P, coefficients α and c should remain unchanged and only tail parameters could be different (Sato, 1999; Cont and Tankov, 2004). Thus, under the objective measure, the tempered stable process has the Lévy density with the same structure as that under the risk-neutral measure, but with different tail parameters. Furthermore, we assume that the risk-neutral measure is simply an exponential tilting of the objective measure. This change of measure is justified by the well-known Esscher transform. The Esscher transform is a minimum entropy change of measure method (Chan, 1999), which indicates that there exists a constant ξ such that the objective Lévy density is related to the risk-neutral one by $V^P(x) = e^{\xi x} V(x)$. We thus have the following objective Lévy density of the tempered stable process X_t^P

$$V^P(x) = c\frac{e^{-(\lambda_+ - \xi)x}}{x^{1+\alpha}}1_{x>0} + c\frac{e^{-(\lambda_- + \xi)|x|}}{|x|^{1+\alpha}}1_{x<0}. \tag{13}$$

The intuition behind this measure change is consistent with our understanding of financial market movements. Large jumps play very important roles in option pricing and risk management since they determine the tail behavior of the return distribution.

To implement estimation, we need the joint conditional characteristic function of return and variance rate under the objective measure. The following proposition gives the tractable joint CCF of log return and variance rate using the same method as before.

PROPOSITION 2: *The joint conditional characteristic function of log return and variance rate with specifications of (11) and (12) under the objective measure equipped with the filtration $\mathcal{G}_t$ is given by*

$$
\begin{aligned}
\phi_{R,v}(u_1, u_2; \tau, v_t) &\equiv E^P[e^{iu_1 R_{t+\tau} + iu_2 v_{t+\tau}} | \mathcal{G}_t] \\
&= e^{iu_1(r-q)\tau + A(u_1, u_2, \tau) + B(u_1, u_2, \tau)v_t},
\end{aligned} \tag{14}
$$

8

where $A(u_1, u_2, \tau) = A(\tau)$, $B(u_1, u_2, \tau) = B(\tau)$ and

$$
\begin{aligned}
A(\tau) &= \frac{\kappa\theta(ac - d)}{bcd}\log\left(\frac{c + de^{b\tau}}{c + d}\right) + \frac{\kappa\theta}{c}\tau, \qquad B(\tau) = \frac{1 + ae^{b\tau}}{c + de^{b\tau}}, \\
a &= iu_2(d + c) - 1, \\
b &= \frac{d\left[-(\kappa^P - iu_1\rho\sigma) - 2uc\right] + a(-(\kappa^P - iu_1\rho\sigma)c + \sigma^2)}{ac - d}, \\
c &= -\frac{(\kappa^P - iu_1\rho\sigma) + \sqrt{(\kappa^P - iu_1\rho\sigma)^2 + 2\sigma^2 u}}{2u}, \\
d &= (1 - iu_2 c)\frac{-(\kappa^P - iu_1\rho\sigma) + iu_2\sigma^2 + \sqrt{(\kappa^P - iu_1\rho\sigma)^2 + 2\sigma^2 u}}{-2iu_2(\kappa^P - iu_1\rho\sigma) + (iu_2\sigma)^2 - 2u}, \\
u &= \varphi_W^P(u_1) + \varphi_X^P(u_1) - iu_1(\pi_W + \pi_X), \\
\varphi_W^P &= \frac{1}{2}(iu_1 + u_1^2), \qquad \varphi_X^P = \psi_X^P(u_1) + iu_1 k_X^P(1).
\end{aligned}
$$

This general model nests a number of specific models, obtained by imposing appropriate restrictions on parameters. For example, the jump-diffusion stochastic volatility model can be obtained by imposing α to be negative. Therefore, estimation will naturally select the best-fitting specification.

3. Econometric methodology

We assume that stock and options markets are fully integrated. It is well-known that the information content in the stock market differs from that in the options market. The stock market contains the historical information regarding the stock price evolution, whereas options are forward-looking. Parameter estimates should reflect both sources of information. We propose a characteristic function-based estimation method which makes full use of information contained in both markets and exploits tractability of the characteristic function of the model.

It is shown (Feuerverger and McDunnough, 1981b) that a GMM estimation can be carried out by using the following moment conditions based on the conditional characteristic function when the process Y_t is stationary and Markovian,

$$
E\left[z(u, Y_t)\left(\exp\{iuY_{t+\tau}\} - \phi(u; Y_t, \Theta)\right)\right] = 0, \tag{15}
$$

where $u \in R$ and Θ is a vector of parameter, $\exp\{iuY_{t+\tau}\}$ is the empirical characteristic function (ECF), and $\phi(u; Y_t, \Theta) = E[e^{iuY_{t+\tau}}|Y_t]$ is the conditional characteristic function (CCF) of $Y_{t+\tau}$. This

estimation is equivalent to MLE and globally efficient if the instrument $z(u, Y_t)$ is selected as

$$z(u, Y_t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\partial \log f(Y_{t+\tau}|Y_t, \Theta)}{\partial \Theta} \exp\{-iuY_{t+\tau}\} dY_{t+\tau}, \tag{16}$$

which requires *a priori* knowledge of the probability density function. For most of Lévy models, it is impossible to have this information.

Recently, Singleton (2001), Jiang and Knight (2002) and Chacko and Viceira (2005) reinvestigate this approach. The basic idea behind these endeavors is to construct moment conditions by firstly dividing the domain of the characteristic index into a finite number of grids and then approximating the instrument. One critical problem here is that when grids become finer and finer in order to improve efficiency, the associated covariance matrix becomes singular. Carrasco et al. (2007) develop a characteristic function based GMM with a continuum of moment conditions (CCF-CGMM) that is computationally less demanding than the commonly used simulation-based method and traditional GMM and solves the problems of singularity and instability induced by the discrete moment condition methods.

In our model if we can observe the variance rate v_t, the full system $Y_t = (R_t, v_t)'$ is a Markov process and stationary. Section 2.2 has already given the joint conditional characteristic function of return and volatility, which depends only on v_t. Therefore, the moment condition (15) implies

$$E\Big[Z(z, v_t)\Big(\exp\{iuY_{t+\tau}\} - \phi_{R,v}(u; \tau, v_t)\Big)\Big] = 0, \tag{17}$$

where $u = (u_1, u_2)$, for some set of instruments $Z(z, v_t), z \in R$.

Define a Hilbert space of complex-valued functions

$$\mathcal{L}^2(p) = \Big\{ f : \mathbf{R}^d \to \mathbf{C}; \int |f(u)|^2 p(u) du < \infty \Big\}, \tag{18}$$

where p is the reference pdf of a distribution and $p(u) > 0$ for all $u \in \mathbf{R}^d$. p dampens off all the oscillating behavior of integrands in estimation and as long as $p > 0$, the choice of p does not affect the estimation efficiency in large sample. In our implementation, we choose the standard normal distribution as the reference pdf. The inner product on $\mathcal{L}^2(p)$ is defined as $\langle f, g \rangle = \int f(u)\overline{g(u)}p(u)du$, where the overline denotes complex conjugate. In this setting, Carrasco et al. (2007) propose to

construct the optimal instruments with the exponential functions

$$Z(z, v_t) = e^{izv_t}. \tag{19}$$

The basic idea of this choice of instruments is that although we can not construct the optimal instruments directly, they can be spanned by a set of basis functions (19). Accordingly, we have moment functions

$$H(u^*; Y_{t+\tau}, v_t, \Theta) = e^{izv_t}\Big(\exp\{iuY_{t+\tau}\} - \phi_{R,v}(u; \tau, v_t)\Big), \tag{20}$$

where $u^* = (u, z)$, and Θ is a vector of model parameters which we need to estimate. It is clear that the moment functions (20) form a martingale difference sequence and thus are not autocorrelated.

Under certain regularity conditions, the CCF-CGMM estimator is

$$\hat{\Theta}_T = arg \min_{\Theta \in \Xi} \left|\left| \mathcal{W}_T \bar{H}_T(\Theta) \right|\right|^2, \tag{21}$$

where $\bar{H}_T(\Theta) = \frac{1}{T}\sum_{t=1}^{T} H(u^*; Y_{t+\tau}, v_t, \Theta)$ is the sample counterpart of moment conditions, Ξ is a compact parameter space, and $\mathcal{W}_T$ is the weighting covariance operator defined on $\mathcal{L}^2(p)$. The estimator $\hat{\Theta}_T$ is asymptotically normal,

$$\sqrt{T}(\hat{\Theta}_T - \Theta_0) \xrightarrow{d} \mathcal{N}(0, \mathcal{V}), \tag{22}$$

$$\mathcal{V} = \langle \mathcal{W}d_0, \mathcal{W}d_0 \rangle^{-1} \langle \mathcal{W}d_0, (\mathcal{W}\mathcal{K}\mathcal{W}^{-1})\mathcal{W}d_0 \rangle \langle \mathcal{W}d_0, \mathcal{W}d_0 \rangle^{-1}, \tag{23}$$

where Θ_0 is the true parameter, $d_0 = E\left(\frac{\partial H(\Theta_0)}{\partial \Theta_0}\right)$, and $\mathcal{K}$ is the covariance operator with the kernel $k(u_1^*, u_2^*) = E[H(u_1^*, \Theta_0)\overline{H(u_2^*, \Theta_0)}]$ such that

$$(\mathcal{K}f)(u_1^*) = \int k(u_1^*, u_2^*)f(u_2^*)p(du_2^*). \tag{24}$$

Consistent estimates $\mathcal{V}_T$, $\mathcal{K}_T$, and d_T of $\mathcal{V}$, $\mathcal{K}$, and d_0 can be obtained from the sample moment conditions. Since the inverse of $\mathcal{K}$ is not bounded, it needs to be stabilized by a regularization term α: $(\mathcal{K}^\alpha)^{-1} = (\mathcal{K}^2 + \alpha I)^{-1}\mathcal{K}$. See Carrasco and Florens (2000) and Carrasco et al. (2007).

In their original paper, Carrasco et al. (2007) have proved that the optimal weighting covariance operator is $\mathcal{W} = \mathcal{K}^{-1/2}$. Here we simply use the identity matrix as the weighting covariance operator in order to alleviate numerical instability. In GMM, Cochrane (2005) advocates to use the identity matrix as the weighting matrix, which produces more robust and stable estimates. In Appendix B, we conduct a Monte Carlo study, which shows that the loss of efficiency in using the identity matrix is tiny and that the algorithm is more stable and faster than that based on the optimal weighting operator.

For our model, the state variable v_t is unobservable. We overcome this problem by jointly using stock prices and options. The idea is that since options contain rich information on the volatility dynamics, we can back out volatilities from options and use them in estimation. This approach is in essence similar to the so-called implied-state method of Pan (2002) except that we directly use the characteristic function. We assume that there exists an one-to-one relationship between volatility and option. According to our model specification, the risk-neutral parameters can be fully identified using the option price data alone. By using both stock price and option data, we can not only identify the risk-premium parameters but also improve the estimation efficiency of the risk-neutral parameters since they appear both in the objective and the risk-neutral models (see a Monte Carlo study in Appendix B).

The CCF-CGMM can produce the MLE efficient estimator if the variance rate v_t is really observable and if the score belongs to the span of moment conditions. The use of an infinite number of moment conditions can theoretically guarantee this latter condition. When v_t is unobservable, the implied-state method can still have consistent estimator. In this case, we should take into account the fact that the variance rate is backed out from the option and it is parameter-dependent. The moment conditions in the objective function (21) now had better be explicitly written as

$$\bar{H}_T(\Theta) = \frac{1}{T} \sum_{t=1}^{T} H(u^*; R_{t+\tau}, V_t(O_t, \Theta^{RN}), \Theta), \tag{25}$$

where $V_t = [v_{t+\tau}, v_t]$, which only depend on the risk-neutral parameters. The calculation of the covariance matrix $\mathcal{V}_T$ needs to be corrected through d_T: $d_T = \frac{1}{T} \sum d(\Theta, R_{t+\tau}, V_t)$, and

$$d(\Theta, R_{t+\tau}, V_t) = \frac{\partial}{\partial \Theta} H(R_{t+\tau}, V_t, \Theta) + \frac{\partial}{\partial v_{t+\tau}} H(R_{t+\tau}, V_t, \Theta) \frac{\partial v_{t+\tau}}{\partial \Theta}$$

$$+\frac{\partial}{\partial v_t}H(R_{t+\tau}, V_t, \Theta)\frac{\partial v_t}{\partial \Theta}. \tag{26}$$

We refer readers to Pan (2002) and Pastorello et al. (2003) for further discussion of the consistency and root-T asymptotic normality of the implied-state estimates.

4. Empirical results and discussions

In this section, we present estimation results and discuss their implications. Subsection 4.1 presents the data, subsection 4.2 reports the parameter estimates, and subsection 4.3 implements model performance analysis.

4.1. Data

The data used in this paper are S&P 500 index and index options traded in Chicago Board Options Exchange (CBOE) during the period from January, 1996 to October, 2009. They are obtained from OptionMetrics. The data are in weekly frequency and are those traded on Wednesday. If Wednesday is a holiday, we select Thursday options. There are 721 weeks in total. The dataset contains the following series on option Trading Date, Expiration Date, Spot price, Strike Price, Best Bid and Ask Prices, Trading Volume, Open Interest, BS implied volatility and other Greeks. The interest rate used by *OptionMetrics* is calculated from a collection of continuously-compounded zero-coupon interest rates at various maturity, collectively referred to as zero-curve, which is derived from LIBOR rates and settlements prices of Eurodollar futures. For each option, we simply select the corresponding zero-curve rate that has the closest maturity to this option.

The upper panel of Figure 1 plots the time-series of S&P 500 index returns, from which the characteristics of "jumps" and "time-varying/stochastic volatility" are clearly observable. For the purpose of model estimation, we use S&P 500 index prices and index at-the-money short maturity call options. The at-the-money short maturity (ATM-SM) calls are constructed as follows: among all call options, we choose those with moneyness (S/K) larger than 0.97 and less than 1.03 and with maturity greater than 15 days and less than 45 days. When there are more than one call option available at each time instant, we select that with moneyness closest to 1. The constructed ATM-SM calls have the mean moneyness 1.000 with the standard deviation 0.003, and the mean maturity approximately 25 days with the standard deviation 10 days. Table 1 gives the descriptive
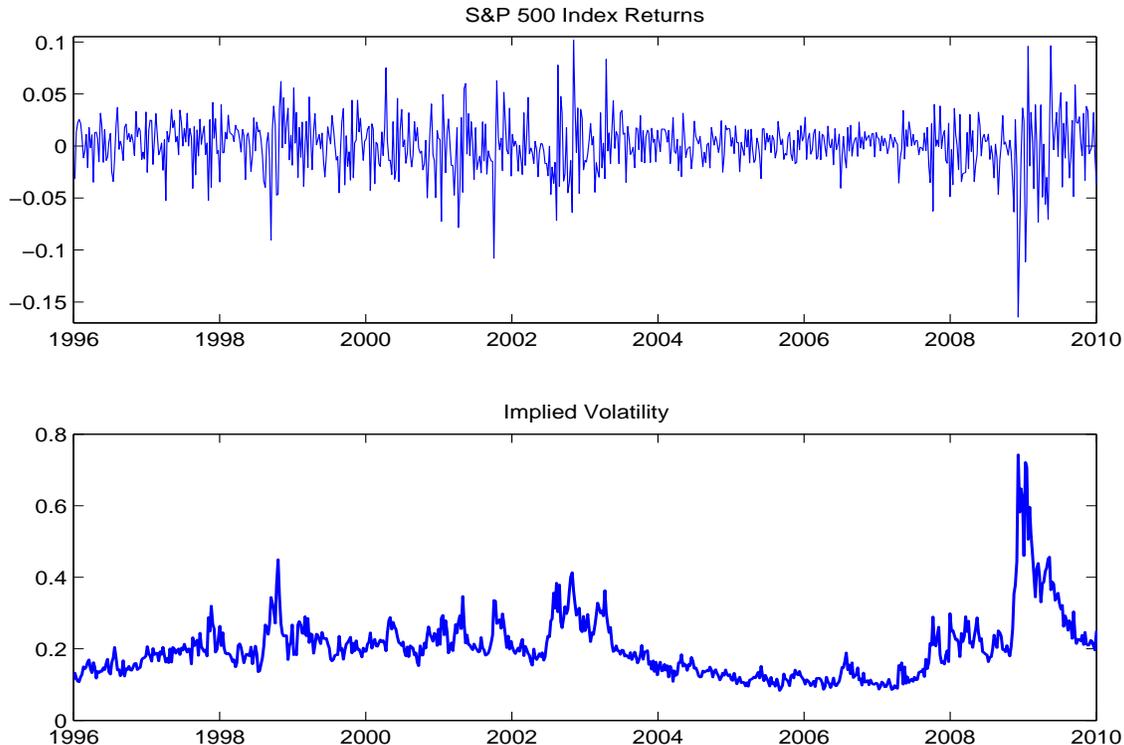
Figure 1: S&P 500 Index Returns and Implied Volatility

Note: The upper panel plots the time series of S&P 500 index returns and the lower panel plots the Black-Scholes implied volatility of the constructed at-the-money short maturity call options. The data are over the period from January 1996 to October 2009 in weekly frequency.

statistics of S&P index returns and the constructed ATM-SM call options. JB test easily rejects the normality hypothesis of the return distribution with very small p-value.

We also use other options to test our models. The following filters are applied to the dataset: first, we only consider call options. Second, in order to ensure that options are liquid enough, we select call options with maturity less than 1.5 years (365-day/52-week year) and with moneyness greater than 0.85 and less than 1.10. Furthermore, we rule out options with zero trading volume and with open interest less than 100 contracts. Third, we exclude call options with maturity less than 10 days and best bid prices less than 3/8 dollar to mitigate market microstructure problems. Finally, we check for the no-arbitrage conditions in option prices. By doing so, we obtain a cross-sectional call options with 37,320 observations in our sample, averagely 52 options each day.

Table 1: **Descriptive Statistics of Data**

A. S&P 500 Index Returns						
	T	Mean	St. Dev.	Skewness	Kurtosis	JB Test
Weekly	721	0.038	0.183	-0.600	7.208	1(< 0.001)
B. ATM-SM Calls						
	Mean Mn.	Std Mn.	Mean Mt.	Std Mt.	Mean IV.	Std IV.
ATM-SM	1.000	0.003	25.01	9.855	0.202	0.086

Note: Table presents the descriptive statistics of data we use for model estimation. Data are from January, 1996 to October, 2009 in weekly frequency. There are 721 weeks in total. In panel A, JB Test is the Jarque-Bera normality test, where the value 1 (with p-value in brackets) indicates rejection of the null hypothesis of normality. In panel B, Mn stands for moneyness, Mt maturity (in days), and IV BS implied volatility.

4.2. Joint CCF-CGMM estimators

Table 2 reports estimation results including parameter estimates and standard deviations. Models are estimated by the Joint CCF-CGMM described in Section 3. The standard errors, which are computed using the formula (23) combined with (25), (26), and $\mathcal{W} = I$, are presented in brackets.

We first estimate the general model (LTS-SV). Looking at the jump-related parameters, we find that the tempered stable process in this model acts as an infinite activity process with finite variation since the estimate of α is 0.856 (positive and less than one). The estimates of the risk-neutral tail parameters λ_+ and λ_- are respectively 198.2 and 3.40, indicating fast right tail dampening and left-skewed distribution. We have discussed in Section 2 that under the change of measure, only tail parameters change, and the other two (c and α) remain constant. The objective tail parameters are related to the risk-neutral ones through $\lambda_+^P = \lambda_+ - \xi$ and $\lambda_-^P = \lambda_- + \xi$. ξ is a risk-premium parameter. The positive estimate of ξ (9.189) implies that the risk-neutral distribution is more left-skewed than the objective one, consistent to the empirically observed fact.

Turning to the variance rate parameters, we notice that the estimate of the long-term mean is around 22.5%-squared, a little larger than the historical one (18.6%-squared), and the estimate of the volatility of volatility parameter is about 0.46. The estimate of κ (3.44) is a little bit large with comparison to previous empirical studies on the jump-diffusion stochastic volatility (JDSV) models (Bakshi et al., 1997; Bates, 1996, 2000; Andersen et al., 2002). The reason is that in the JDSV models the stochastic volatility is only from the diffusion part, whereas in our model we use the same variance rate process to time change both the diffusion part and the jump part. The estimated

Table 2: **Iterative Joint CCF-CGMM Estimation of Models**

Model	Risk-Neutral Parameters								Risk-Premium Parameters		
	κ	θ	σ	ρ	c	λ_+	λ_-	α	π_w	ξ	π_v
LTS-SV	3.439	0.051	0.460	-0.521	0.476	1.98e2	3.399	0.856	0.437	9.189	-2.242
	(0.328)	(0.004)	(0.045)	(0.038)	(0.972)	(1.58e2)	(0.376)	(0.585)	(0.171)	(1.241)	(0.571)
LTS-SVVG	5.434	0.048	0.507	-0.485	5.755	0.70e2	4.749	0.000	0.365	6.284	-3.532
	(0.417)	(0.003)	(0.049)	(0.033)	(2.761)	(0.89e2)	(0.399)	(—)	(0.162)	(1.206)	(0.837)
LTS-SVJD	5.122	0.046	0.497	-0.598	7.784	0.70e2	3.190	-1.000	0.429	6.187	-2.746
	(0.411)	(0.003)	(0.051)	(0.049)	(3.860)	(4.11e2)	(0.241)	(—)	(0.159)	(1.118)	(0.603)

Note: Models are estimated with the impled-state Joint CCF-CGMM method discussed in Section 3. Standard deviations are reported in brackets. LTS-SV is the general model without any restrictions, LTS-SVVG is the model with the stable index being 0 and LTS-SVJD is the model with the stable index being -1.

κ should reflect both the persistent diffusion volatility and the transient jump effect.

We have a negative risk premium of volatility π_v (-2.24), which is consistent with the negative correlation between the return process and variance rate process ($\hat{\rho}$ = -0.52). We observe that the diffusion risk premium is small (0.44), indicating that the market does not price this risk factor significantly. The negative premium on the stochastic variance risk and the relative importance of the jump and volatility risk premia over the diffusion risk premium are also reported in Bates (2000), Pan (2002), Carr and Wu (2008) and others and in Pan (2002), respectively. Our estimate of the volatility risk premium is also significant. This is because we time-change both the diffusion and jump parts with the same business time and the inherent feature of jumps in stock returns makes difference when estimating the volatility risk premium.

The estimate of α in the LTS-SV model, even though positive, is insignificant. We then study other two special cases where α takes values -1 and 0. Taking a negative value of α, we obtain a compound Poisson type jump-diffusion stochastic volatility model with an enriched structure. We refer to this model as LTS-SVJD, with which we can compare the infinite activity stochastic volatility model and the finite activity jump-diffusion stochastic volatility model. The parameter estimates of the variance rate process do not change much in comparison to those of the LTS-SV model, but the jump-related parameter estimates are different. The estimate of c, which reflects jump frequency, has a larger value (7.78 vs 0.48). The negative value of α forces c to capture both large and small jumps and in this case, large jumps and small jumps are indistinguishable.

When α is equal to 0, the tempered stable process becomes the variance gamma process studied by Madan et al. (1998), which is infinitely active and of finite variation. We refer to this model as LTS-SVVG. We notice again that the variance rate parameter estimates are not very different, but jump parameter estimates are different. The estimate c is larger than that in the LTS-SV model but less than that in the LTS-SVJD model, consistent with change of the stable index α.

4.3. Model performance analysis

In the implementation of CCF-CGMM estimation, we approximate the (double) integral with the weighted sums in the objective function (21) by selecting 16 equally spaced values of each characteristic index (u_1 and u_2) from the range $[-\pi, \pi]$ (outside this range, the integrand is almost zero.), and thus have 256 moment conditions in total. By studying these moment conditions, we

Table 3: **Statistics of Moment Conditions**

	LTS-SV	LTS-SVVG	LTS-SVJD	LTS-SVDJ
Mean Mom	3.429	3.462	3.491	3.440
γ_1	-0.087	-0.098	-0.105	-0.091
γ_2	-0.010	0.022	0.032	0.007
γ_3	0.007	0.015	-0.013	0.037
$\bar{\gamma}_1$	0.131	0.135	0.153	0.137
$\bar{\gamma}_2$	0.034	0.035	0.041	0.043
$\bar{\gamma}_3$	0.021	0.019	0.031	0.025

Note: Table presents descriptive statistics of the estimated discretized moment conditions. There are 256 moment conditions in total. Mean Mom stands for the mean of all the absolute moment conditions with scale 10^{-2}; γ_i represents mean of the ith autocorrelation of the moment functions and $\bar{\gamma}_i$ is mean of the absolute values of the ith autocorrelation of the moment functions.

can make a qualitative test of the goodness-of-fit of models. Table 3 presents descriptive statistics of the estimated moment conditions. We find that the mean of moment conditions for the LTS-SV model is the smallest. As mentioned in Section 3, the full system $(\ln S_t, v_t)'$ is Markovian, so the sequence of moment functions $\{H_t\}_{t=1}^{T}$ forms a martingale difference sequence with respect to $\mathcal{G}_t$ and hence is theoretically non-autocorrelated. Table 3 empirically investigates autocorrelations of these moment functions. γ_i is mean of the autocorrelations of all moment conditions and $\bar{\gamma}_i$ is mean of absolute values of the autocorrelations of moment conditions. Putting together, we find that the LTS-SV model is the best.

We begin to compute option prices with parameters and variance rates estimated in the previous subsection. In practice, this is more interesting. We test models according to their capacity in pricing options. The relative pricing error, which is defined as follows

$$Rerr = \frac{1}{N} \sum_{t=1}^{T} \sum_{i=1}^{n_t} \frac{|P_{it}^{md} - P_{it}^{mk}|}{P_{it}^{mk}}, \tag{27}$$

is used to measure the performance. In (27), N is the total number of options we consider, T the number of weeks, n_t the number of options at date t, P_{it}^{md} the model-implied option price, and P_{it}^{mk} the market-quoted option price of the ith option at date t.

Table 4 presents option pricing errors. We find that among all groups, the LTS-SV model performs the best except in only 3 groups, that with respect to maturity, the LTS-SV performs better than all the other models, and that with respect to moneyness, the LTS-SV model performs

Table 4: **Option Pricing Errors**

| Maturity | Model | Moneyness (S/K) | | | | | Sub-Total |
		< 0.94	0.94-0.97	0.97-1.00	1.00-1.03	>1.03	
< 60	LTS-SV	0.109	0.092	0.062	0.071	0.046	0.067
	LTS-SVVG	0.120	0.108	0.100	0.063	0.033	0.076
	LTS-SVJD	0.231	0.095	0.084	0.064	0.041	0.074
	LTS-SVDJ	0.128	0.107	0.071	0.077	0.049	0.074
		[0.203]	[0.179]	[0.168]	[0.178]	[0.233]	[0.188]
		(617)	(3125)	(7085)	(6293)	(5231)	(22351)
60 < 180	LTS-SV	0.097	0.077	0.076	0.063	0.048	0.070
	LTS-SVVG	0.128	0.107	0.125	0.095	0.051	0.102
	LTS-SVJD	0.127	0.084	0.107	0.084	0.067	0.090
	LTS-SVDJ	0.107	0.083	0.085	0.070	0.053	0.078
		[0.204]	[0.197]	[0.181]	[0.184]	[0.206]	[0.192]
		(1188)	(1505)	(2916)	(2273)	(1744)	(9626)
> 180	LTS-SV	0.097	0.080	0.083	0.101	0.076	0.088
	LTS-SVVG	0.141	0.117	0.114	0.145	0.104	0.126
	LTS-SVJD	0.097	0.083	0.092	0.116	0.087	0.097
	LTS-SVDJ	0.091	0.082	0.088	0.102	0.078	0.089
		[0.209]	[0.210]	[0.209]	[0.193]	[0.201]	[0.204]
		(1229)	(730)	(1190)	(1204)	(990)	(5343)
Sub-Total	LTS-SV	0.099	0.086	0.068	0.073	0.050	0.071
	LTS-SVVG	0.132	0.109	0.108	0.080	0.046	0.090
	LTS-SVJD	0.136	0.091	0.091	0.075	0.049	0.082
	LTS-SVDJ	0.105	0.097	0.076	0.078	0.051	0.077
		[0.206]	[0.189]	[0.175]	[0.181]	[0.223]	[0.191]
		(3034)	(5360)	(11191)	(9770)	(7965)	(37320)

Note: The option pricing error is measured by the relative pricing error, defined in (27). In total, we consider 15 groups of options, divided according to maturity and moneyness. For each group, we report the option pricing error of each model, the average BS implied volatility (in square brackets), and the number of options (in brackets).

the best except for the deep in-the-money options. We also find that all these models perform relatively better in pricing in-the-money options and worse in pricing deep out-of-money options. Putting all together, the infinite activity infinite variation LTS-SV is preferable to the finite activity LTS-SVJD model. The average pricing error is 0.071 for the LTS-SV model, whereas it is 0.082 for the LTS-SVJD model.

5. Extension: A double-jump model

Recently, some empirical investigations with jump-diffusion stochastic volatility models argue that a jump component is also necessary in the volatility process (Eraker et al., 2003; Broadie et al., 2007). Thus, a natural extension of our model is to introduce a jump component in the variance

rate process. To this end, we model the variance rate process under the risk-neutral measure with the following SDE,

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dZ_t + dJ_t, \tag{28}$$

where Z_t is a Brownian motion, which is correlated with W_t and independent of X_t. The new process J_t is a compound Poisson pure jump process that is independent of W_t, Z_t and X_t, whose jump sizes are independent and exponentially distributed with mean μ_J, and whose jump times follow a Poisson process with jump intensity λ_J. The characteristic function of this jump process J_t is

$$\phi_J(u) = \exp\left\{-t\lambda_J \frac{iu\mu_J}{iu\mu_J - 1}\right\}, \tag{29}$$

where $u \in R$ is the characteristic index.

The variance rate process (28) is the so-called basic affine process (Duffie and Garleanu, 2001). With this specification, we now have jump components both in the return process and in the variance rate process. We assume that under change of measure, the jump process J_t does not change its parameters, that is, the risk premium for the risk factor J_t is zero. We thus have the objective model as follows,

$$
\begin{aligned}
S_t &= S_0 \exp\left\{(r-q)t + \pi_W T_t + \left[k_X^P(1) - k_X(1)\right]T_t \right. \\
&\quad \left. + \left[W_{T_t}^P - \frac{1}{2}T_t\right] + \left[X_{T_t}^P - k^P(1)T_t\right]\right\},
\end{aligned} \tag{30}
$$

$$dv_t = [\kappa(\theta - v_t) + \pi_v v_t]dt + \sigma\sqrt{v_t}dZ_t^P + dJ_t, \tag{31}$$

with $T_t = \int_0^t v_{s-}ds$. Define $\kappa^P \equiv \kappa - \pi_v$ and $\pi_X \equiv (k_X^P(1) - k_X(1))T_t$.

Following the same approach as in Proposition 2, we can derive the analytical joint conditional characteristic function of return and variance rate under the objective measure as follows due to Duffie and Garleanu (2001).

PROPOSITION 3: *The joint conditional characteristic function of log return and variance rate with specifications (30) and (31) under the objective measure equipped with the augmented filtration $\mathcal{G}_t$ is*

$$\phi_{R,v}(u_1, u_2; \tau, v_t) \equiv E^P[e^{iu_1 R_{t+\tau} + iu_2 v_{t+\tau}}|\mathcal{G}_t]$$

Table 5: **Parameter Estimates of Double-Jump Model**

VR CIR		VR Jump		Return Jump		Risk Premia	
κ	4.367	μ_J	0.038	c	0.641	π_w	0.436
	(0.322)		(0.012)		(1.512)		(0.139)
θ	0.038	λ_J	0.525	λ_+	1.11e2	ξ	5.694
	(0.003)		(0.051)		(0.93e2)		(1.147)
σ	0.368			λ_-	3.285	π_v	-2.230
	(0.030)				(0.904)		(0.628)
ρ	-0.508			α	0.741		
	(0.038)				(0.602)		

Note: The table presents parameter estimates of the double-jump model. VR CIR represents the parameters related to the CIR part of the variance rate process (28). VR Jump represents the parameters in the jump process J_t. Return Jump represents the parameters in the tempered stable process and Risk Premia represents the risk-premium parameters. Standard deviations are in brackets.

$$= e^{iu_1(r-q)\tau + A(u_1,u_2,\tau) + B(u_1,u_2,\tau)v_t}, \tag{32}$$

where $A(u_1, u_2, \tau) = A_1(u_1, u_2, \tau) + A_2(u_1, u_2, \tau)$,

$$A_1(u_1,u_2,\tau) = \frac{\kappa\theta(a_1c_1-d_1)}{b_1c_1d_1}\log\left(\frac{c_1+d_1e^{b_1\tau}}{c_1+d_1}\right) + \frac{\kappa\theta}{c_1}\tau,$$

$$A_2(u_1,u_2,\tau) = \frac{\lambda_J(a_2c_2-d_2)}{b_2c_2d_2}\log\left(\frac{c_2+d_2e^{b_2\tau}}{c_2+d_2}\right) + \frac{\lambda_J(1-c_2)}{c_2}\tau,$$

$$B(u_1,u_2,\tau) = \frac{1+a_1e^{b_1\tau}}{c_1+d_1e^{b_1\tau}},$$

and other parameters (a_i, b_i, c_i, d_i, $i = 1, 2$) are given in Appendix A.

Our econometric method can be easily applied to estimate this double-jump model. Table 5 presents the parameter estimates of the double-jump model. Focusing on jump parameters of the variance rate process, the mean of jump size μ_J is 0.038 and the jump intensity λ_J is 0.525. To compare with the return jump size and intensity estimates in the previous studies (Pan, 2002, Andersen et al., 2002, Eraker et al., 2003 and others), these values indicate that the variance rate does not undergo large and frequent jumps. Implications of other parameter estimates are nearly the same as in LTS-SV model.

Looking at moment conditions and autocorrelations of the moment functions in Table 3 under the name LTS-SVDJ, we find that the absolute mean of moment conditions is 3.44, smaller than those of the LTS-SVVG model and the LTS-SVJD model, but larger than that of the LTS-SV model

and that the result from autocorrelations is undetermined. As for option pricing errors, Table 4 reports the relative pricing errors of this double-jump model (LTS-SVDJ). The improvement in the option pricing error over the LTS-SV model is observable only in few cases and negligible. We can thus conclude that Poisson jumps in the stochastic volatility are not critically important in modeling stock price dynamics and pricing options when the infinite activity jump component is introduced in the stock price process.

6. Conclusion

We have studied stock price dynamics by taking into account both stochastic volatility and jumps. Jumps are captured by the tempered stable process, and stochastic volatility is introduced by time changing the stochastic processes. For model estimation, we propose a characteristic function based estimation method. The empirical study indicates that the infinite activity stochastic volatility model in general is preferable to the jump-diffusion stochastic volatility model. We also make an extension to study a double-jump model. Our empirical study points to an only marginal improvement of this double-jump model.

In this paper, we estimate the models using stock price data and only at-the-money short maturity call options. All other call and put options are discarded in model estimation. Even though at-the-money short maturity options are the most liquid financial securities, there are many other options which are also very liquid and contain rich information about financial market movement. Therefore, it may be interesting to study models and their implications using not only at-the-money options, but also out-of-the-money and in-the-money options.

tuarial Sciences and Finance (2008) for helpful comments.

Appendix

Appendix A. Proofs of Propositions

We first prove Proposition 2. Since Brownian motions W_t^P and Z_t^P are correlated, we use the approach proposed by Carr and Wu (2004) to implement a change of measure in order to internalize this correlation. Define a new measure M, which is absolutely continuous with respect to the objective measure P

$$\frac{dM}{dP}\Big|_{\mathcal{G}_t} = \exp\left\{\left[iu(W_{T_t}^P - \frac{1}{2}T_t) + \varphi_W^P T_t\right] + \left[iu(X_{T_t}^P - k_X^P(1)T_t) + \varphi_X^P T_t\right]\right\}. \tag{A.1}$$

Under this new measure M, the variance rate process becomes,

$$dv_t = [\kappa\theta - (\kappa^P - iu\rho\sigma)v_t]dt + \sigma\sqrt{v_t}dZ_t^M, \tag{A.2}$$

where $\kappa^{P*} = \kappa^P - iu\rho\sigma$ and Z_t^M is now independent of W_t^P. The joint conditional characteristic function of $R_{t+\tau}$ and $v_{t+\tau}$ can then be calculated as follows

$$
\begin{aligned}
\phi_{R,v}(u_1, u_2; \tau, v_t) &\equiv E^P[e^{iu_1 R_{t+\tau} + iu_2 v_{t+\tau}}|\mathcal{G}_t] \\
&= e^{iu_1(r-q)\tau} E^P\left[e^{iu_1[(\pi_W + \pi_X)T_\tau + (W_{T_\tau}^P - \frac{1}{2}T_\tau) + (X_{T_\tau}^P - k_X^P(1)T_\tau)] + iu_2 v_{t+\tau}}\Big|\mathcal{G}_t\right] \\
&= e^{iu_1(r-q)\tau} E^P[M(\tau)e^{-[\varphi_W^P + \varphi_X^P - iu_1(\pi_W + \pi_X)]T_\tau + iu_2 v_{t+\tau}}|\mathcal{G}_t] \\
&= e^{iu_1(r-q)\tau} E^M[e^{-\int_t^{t+\tau} uv_s ds} e^{iu_2 v_{t+\tau}}|\mathcal{G}_t] \\
&= e^{iu_1(r-q)\tau + A(u_1, u_2, \tau) + B(u_1, u_2, \tau)v_t}, \tag{A.3}
\end{aligned}
$$

where $u = \varphi_W^P + \varphi_X^P - iu_1(\pi_W + \pi_X)$ and E^M is the expectation operator under the new measure M. $A(u_1, u_2, \tau) = A(t)$ and $B(u_1, u_2, \tau) = B(t)$ solve the following ODEs

$$\dot{B}(t) = u + \kappa^{P*}B(t) - \frac{1}{2}\sigma^2 B^2(t), \tag{A.4}$$

$$\dot{A}(t) = -\kappa\theta B(t), \tag{A.5}$$

with the boundary conditions $B(t + \tau) = iu_2$ and $A(t + \tau) = 0$. By solving these two ODEs, we obtain the result of Proposition 2. Under the change of measure defined in the text, Proposition 1 can be easily proved by setting $u_2 = 0$ and suppressing the risk-premium parameters.

The proof of Proposition 3 is similar to that of Proposition 2 except that we now solve

$$
\begin{aligned}
\dot{B}(t) &= u + \kappa^{P*}B(t) - \frac{1}{2}\sigma^2 B^2(t), \\
\dot{A}(t) &= -\kappa\theta B(t) - \lambda_J \frac{\mu_J B(t)}{1 - \mu_J B(t)},
\end{aligned}
$$

with $A(t) = A(u_1, u_2, \tau)$ and $B(t) = B(u_1, u_2, \tau)$ as well as boundary conditions $B(t+\tau) = iu_2$ and $A(t + \tau) = 0$. The solution is

$$
\begin{aligned}
A_1(u_1, u_2, \tau) &= \frac{\kappa\theta(a_1 c_1 - d_1)}{b_1 c_1 d_1} \log\left(\frac{c_1 + d_1 e^{b_1\tau}}{c_1 + d_1}\right) + \frac{\kappa\theta}{c_1}\tau, \\
A_2(u_1, u_2, \tau) &= \frac{\lambda_J(a_2 c_2 - d_2)}{b_2 c_2 d_2} \log\left(\frac{c_2 + d_2 e^{b_2\tau}}{c_2 + d_2}\right) + \frac{\lambda_J(1 - c_2)}{c_2}\tau, \\
B(u_1, u_2, \tau) &= \frac{1 + a_1 e^{b_1\tau}}{c_1 + d_1 e^{b_1\tau}}, \\
a_1 &= iu_2(d_1 + c_1) - 1, \\
b_1 &= \frac{d_1(-\kappa^{P*} - 2uc_1) + a_1(-\kappa^{P*}c_1 + \sigma^2)}{a_1 c_1 - d_1}, \\
c_1 &= -\frac{\kappa^{P*} + \sqrt{(\kappa^{P*})^2 + 2\sigma^2 u}}{2u}, \\
d_1 &= (1 - iu_2 c_1)\frac{-\kappa^{P*} + iu_2\sigma^2 + \sqrt{(\kappa^{P*})^2 + 2\sigma^2 u}}{-2iu_2\kappa^{P*} + (iu_2\sigma)^2 - 2u}, \\
a_2 &= \frac{d_1}{c_1}, \quad b_2 = b_1, \quad c_2 = 1 - \frac{\mu_J}{c_1}, \quad d_2 = \frac{d_1 - \mu_J a_1}{c_1}, \\
u &= \varphi_W^P(u_1) + \varphi_X^P(u_1) - iu_1(\pi_W + \pi_X), \\
\varphi_W^P &= \frac{1}{2}(iu_1 + u_1^2), \quad \varphi_X^P = \psi_X^P(u_1) + iu_1 k_X^P(1), \\
\kappa^{P*} &= \kappa^P - iu_1\rho\sigma.
\end{aligned}
$$

By setting $u_2 = 0$ and suppressing the risk-premium parameters, we could obtain the risk-neutral conditional characteristic function, which can be used for option pricing.

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	μ	κ	θ	σ	ρ
True Value	0.150	6.000	0.025	0.300	-0.600
A. Identity Weighting Matrix					
Mean	0.149	6.522	0.027	0.291	-0.623
Median	0.150	6.465	0.025	0.295	-0.590
RMSE	0.047	1.881	0.008	0.057	0.144
B. Optimal Weighting Matrix					
Mean	0.153	6.195	0.028	0.312	-0.591
Median	0.151	5.827	0.026	0.290	-0.593
RMSE	0.047	1.214	0.010	0.069	0.123

Note: The Monte Carlo study is implemented using Heston stochastic volatility model. The number of simulations is 100 with the sample size 500 in weekly frequency. There are in total 11 exploding estimation results when using the optimal weighting matrix. We delete these results when computing statistics in Panel B.

Appendix B. Monte Carlo Studies

In this appendix, we implement two Monte Carlo studies using Heston stochastic volatility model. The first is to manifest the estimation efficacy of the Joint CCF-CGMM, and the second is to show the efficiency gain in estimation when jointly using the stock price and options data.

Heston stochastic volatility model can be obtained from our general model by simply suppressing the tempered stable process and related components. Under the objective measure, it has the form

$$S_t = S_0 \exp\left\{\mu t + \left(W_{T_t} - \frac{1}{2}T_t\right)\right\}, \tag{B.1}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dZ_t, \tag{B.2}$$

with $T_t \equiv \int_0^t v_s ds$. The equivalence between the above time-changed form and the original (Heston, 1993) can be seen from

$$W_{T_t} \overset{d}{=} \int_0^t \sqrt{v_s}dW_s, \tag{B.3}$$

where d indicates the equivalence in distribution. Under the change of measure discussed in the text, we can obtain the risk-neutral model. Following the same procedure as described in Propositions, we can derive the objective joint conditional characteristic function of return and variance rate and the risk-neutral conditional characteristic function of return.

The first Monte Carlo study is based on 100 simulations with sample size 500 in weekly frequency. We simulate Heston stochastic volatility model with an efficient scheme proposed by Ander-

Table B.7: **Monte Carlo Study II**

	κ	θ	σ	ρ
True Value	6.000	0.025	0.300	-0.600
A. Options Alone				
Mean	6.275 [5.522]	0.025 [0.025]	0.282 [0.281]	-0.238 [-0.181]
Median	6.276 [5.580]	0.025 [0.025]	0.281 [0.279]	-0.209 [-0.184]
RMSE	1.147 [0.699]	0.002 [0.001]	0.024 [0.023]	0.524 [0.530]
B. Stock Prices and Options				
Mean	5.757 [5.751]	0.025 [0.025]	0.278 [0.288]	-0.598 [-0.603]
Median	5.778 [5.816]	0.025 [0.025]	0.277 [0.281]	-0.603 [-0.601]
RMSE	0.840 [0.627]	0.002 [0.001]	0.025 [0.023]	0.027 [0.019]

Note: The Monte Carlo study is implemented using Heston stochastic volatility model. The number of simulations is 100 with sample size 500 and 1000 (in square brackets) in weekly frequency. The simulated options are those of at-the-money short maturity call options with maturity being one month and moneyness being 1.

sen (2008). In this model, we have five parameters $\Theta = (\mu, \kappa, \theta, \sigma, \rho)$, and the true values are given by $\Theta_0 = (0.15, \ 6.00, \ 0.025, \ 0.30, \ -0.60)$. The model is estimated by the Joint CCF-CGMM using the simulated stock prices and volatility, and both the identity and optimal weighting matrices are used. We find that the optimization is less sensitive to initial values when using the identity weighting matrix since using the optimal weighting matrix results in 11 exploding estimates among 100 simulations/estimations. Table B.6 presents the Monte Carlo study results, which indicate that the loss of efficiency is tiny.

In the second Monte Carlo study, we assume zero equity risk premium and volatility risk premium. Then the risk-neutral model is the same as the objective one. We also assume the interest rate known with the value 5%, and only estimate $\Theta = (\kappa, \theta, \sigma, \rho)$. We use the same true values as in the first Monte Carlo study. In each simulation, we generate weekly stock prices and at-the-money short maturity call options with maturity being one month and moneyness being 1. We estimate the model using options data alone and using both stock prices and options data, respectively. In total, 100 simulations are implemented. Table B.7 clearly shows the efficiency gain in estimation with different sample sizes when using both stock prices and options data, especially for parameters κ and ρ. More importantly, as indicated in the text, the risk-premium parameters can be identified when using both stock prices and options data in estimation.

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